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Private ownership economies with externalities and existence of competitive equilibria: A differentiable approach [1]

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Abstract

We consider a general equilibrium model of a private ownership economy with consumption and production externalities. Utility functions and production technologies may be affected by the consumption and production activities of all other agents in the economy. We use differential techniques to show that the set of competitive equilibria is non-empty and compact. Fixing the externalities, the assumptions on utility functions and production technologies are standard in a differentiable framework. Competitive equilibria are written in terms first-order conditions associated with agents' behavior and market clearing conditions, following the seminal work by Smale (1974). Adapting differential techniques to economies with externalities is non trivial and it requires some ingenious adjustments, because the production technologies are not required to be convex with respect to the consumption and production activities of all agents.

JEL classification: C62, D51, D62.

Key words: externalities, private ownership economy, competitive equilibrium, differentiable approach.

[1] This paper dates back to 2011 and it has been available on the website: http://www.parisschoolofeconomics.eu/fr/del-mercato-elena/publications. We are delighted that this paper has been quoted in Balasko (2014). Furthermore, we have recently found a similar contribution by Ericson and Kung (2015). The 2011's version has been presented at the Public Economic Theory (PET 10) and Public Goods, Public Projects, Externalities (PGPPE) Closing Conference, Bogazici University, 2010, and the Fifth Economic Behavior and Interaction Models (EBIM) Doctoral Workshop on Economic Theory, Bielefeld University, 2010. We are grateful to participants of these conferences for useful comments.

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1 Introduction

We consider a general equilibrium model of a private ownership economy with consumption and production externalities. We use differential techniques to show that the set of competitive equilibria is non-empty and compact.

The general equilibrium model of Arrow and Debreu has been extended to economies with consumption and production externalities. Our model of externalities is based on the seminal models given in Arrow and Hahn (1971), Laffont and Laroque (1972), and Laffont (1988), where individual preferences and production technologies are affected by the consumption and production activities of all other agents. Consumption externalities have been widely recognized in the literature on network externalities and other-regarding preferences. Polluting production activities are a classical example of production externalities induced on individual preferences and technological processes of other firms.

We consider a private ownership economy with a finite number of commodities, households and firms. Each household's preferences are represented by a utility function. Each firm's production technology is represented by a *transformation function.*[3] Utility and transformation functions may be affected by the consumption and production activities of all other agents. Each agent (household or firm) maximizes his goal by taking as given both commodity prices and choices of every other agent in the economy. The associated concept of competitive equilibrium is an equilibrium *à la Nash*, the resulting allocation being feasible with the initial resources of the economy. This notion is given, for instance, in Arrow and Hahn (1971), and Laffont (1988), and it includes as a particular case the classical equilibrium definition without externalities.

Fixing the externalities, the assumptions on utility and transformation functions are standard in a differentiable framework. In particular, preferences and production technologies are convex. However, we do not require preferences and production technologies to be convex with respect to the externalities. Our main theorem (Theorem 8) states that the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. We prove Theorem 8 following the seminal work by Smale (1974), and more recent contributions by Villanacci and Zenginobuz (2005), del Mercato (2006) and

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[3] This is a convenient way to represent a production set using an inequality on a function called the *transformation function*, see for instance Mas-Colell et al. (1995).

Bonnisseau and del Mercato (2008). That is, we use:

(1) Smale's approach.
(2) Homotopy techniques.
(3) The topological *degree modulo* 2.

In Smale's approach, equilibrium is written as a solution of an extended system of equations consisting of first-order conditions associated with agents' behavior and market clearing conditions. No externalities are taken into account in the seminal work of Smale (1974). However, this approach is used in many different settings such as incomplete markets, public goods and externalities. In the presence of externalities, Smale's approach overcomes the following difficulty: individual demand and supply depend on individual demand and supply of all other agents which, in turn, depend on individual demand and supply of all other agents, and so on. So, it is problematic to define an aggregate demand and an aggregate supply which depend only on prices and initial endowments.

The homotopy idea is that any economy is connected by an arc to some other economy that has an *unique regular* equilibrium. Along this arc, equilibrium moves in a continuous way without sliding off the boundary. It does not imply that at the end of the arc there is still an unique regular equilibrium. But, it implies that at the end of the arc the equilibrium set is non-empty and compact. This is a kind of standard technique used in many contribution in general equilibrium theory, see for instance Villanacci and Zenginobuz (2005), del Mercato (2006), Mandel (2008), Bonnisseau and del Mercato (2008), Kung (2008), and Ericson and Kung (2015). However, adapting this technique to economies with externalities is non trivial and it requires some ingenious adjustments, because the production technologies are not required to be convex with respect to the consumption and production activities of all agents (for more details, see Subsection 4.2). Furthermore, our proof covers the case in which the economy is a standard private ownership economy without externalities at all.[4]

The technique described above is based on the homotopy invariance of the topological degree. Our proof is based on the topological *degree modulo* 2.[5] The degree modulo 2 is simpler than the *Brouwer degree* which is based on the notion of *oriented manifold*.[6] The reader can find in Geanakoplos and Shafer (1990) a brief review of the degree theory. In Section 6, we recall the definition and fundamental properties of the degree modulo 2.

[4] In Chapter 9 of Villanacci et al. (2002), the reader can find a homotopy proof for a standard private ownership economies without externalities. Our proof is simpler than the latter one, because we do not homotopize the shares.
[5] See for instance, Chapter 4 in Milnor (1965), and Villanacci et al. (2002).
[6] See for instance, Chapter 5 in Milnor (1965), and Mas-Colell (1985).

We now compare our contribution with previous works. del Mercato (2006) and Balasko (2014) study pure exchange general equilibrium models with externalities. In Villanacci and Zenginobuz (2005), and Kung (2008), there are no externalities on the production side. In Ericson and Kung (2015), utility and transformation functions are also affected by the commodity price. In order to get their existence result, the authors need to perturb utility and transformation functions.[7] Our contribution highlights the fact that, in the presence of consumption and production externalities, there is no need for perturbing utility and transformation functions to get the existence of an equilibrium. Furthermore, the contribution of Ericson and Kung (2015) presents some technical problems.[8] The existence results of Laffont and Laroque (1972), Bonnisseau and Médecin (2001), and Mandel (2008), are more general than ours. In Laffont and Laroque (1972), and Bonnisseau and Médecin (2001), the authors use fixed point arguments. The approach in Mandel (2008) differs from ours for two main reasons: the author uses the excess demand approach and the topological Brouwer degree. In order to use the excess demand approach, the author enlarges the commodity space treating externalities as additional variables. Moreover, in Bonnisseau and Médecin (2001), and Mandel (2008), production technologies are also non-convex with respect to the individual firm's choice. For that reason, their equilibrium notion involves the concept of *pricing rule* and their existence proofs consist of techniques more sophisticated than those we use. In our mild context, we provide an existence proof simpler than the ones provided in Bonnisseau and Médecin (2001), and Mandel (2008).

The paper is organized as follows. In Section 2, we present the model and the assumptions. In Section 3, we provide the notion of competitive equilibrium and equilibrium function. The equilibrium function is built on first-order conditions associated with households and firms maximization problems, and market clearing conditions. In Section 4, we first present our main result (Theorem 8) which states that the equilibrium set is non-empty and compact. Second, we provide the homotopy theorem (Theorem 9) which is used to prove Theorem 8. In order to apply Theorem 9, in Subsection 4.1 we build an appropriate private ownership economy that has a unique regular equilibrium. In Subsection 4.2, we provide our homotopy and its properties. All the lemmas are proved in Section 5. Finally, in Section 6, the reader can find a brief review on the topological *degree modulo* 2.

[7] See pages 53–55 in Ericson and Kung (2015).

[8] Some basic assumptions for the existence of an equilibrium are missing, i.e., the possibility of inaction and the compactness of the set of feasible allocations, or any related assumptions. See for instance, Assumptions 1(2) and 3 in our paper, or Assumptions P(2) and UB in Bonnisseau and Médecin (2001).

2 The model and the assumptions

There is a finite number C of physical commodities labeled by the superscript $c \in \mathcal{C} := \{1, \dots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \dots, J\}$ and a finite number H of households labeled by the subscript $h \in \mathcal{H} := \{1, \dots, H\}$. Each firm is owned by households and it is characterized by a technology described by a transformation function. Each household is characterized by preferences described by a utility function, the shares on firms' profits and an endowment of commodities. Utility and transformation functions may be affected by the consumption and production activities of all other agents. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j, as usual if $y_j^c > 0$ then commodity c is produced as an output, if $y_j^\ell < 0$ then commodity ℓ is used as an input, $y_{-j} := (y_f)_{f \neq j}$ denotes the production plan of firms other than j, $y := (y_j)_{j \in \mathcal{J}}$.
- $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption, $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than h, $x := (x_h)_{h \in \mathcal{H}}$.
- The production set of firm j is described by a *transformation function*. This is a convenient way to represent a production set using an inequality on a function called the transformation function, see for instance Mas-Colell et al. (1995). In the case of a single-output technology, the production set is commonly described by a *production function*. The transformation function is the counterpart of the production function in the case of production processes which involve several outputs.

  The main innovation of this paper comes from the dependency of the transformation function t_j on the production and consumption activities of all other agents. So, t_j describes both the technology of firm j and the way in which firm j's technology is affected by the activities of the other agents. More precisely, for a given (y_{-j}, x), the production set of the firm j is given by the following set,

$$Y_j(y_{-j}, x) := \Big\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \leq 0 \Big\}$$

  where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$, $t := (t_j)_{j \in \mathcal{J}}$.
- Household h has preferences described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

  $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). So, u_h also describes the way in which household h's preferences are affected by the actions of the other agents. Denote $u := (u_h)_{h \in \mathcal{H}}$.

- $s_{jh} \in [0,1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j\in\mathcal{J}} \in [0,1]^J$ denotes the vector of the shares owed by household h; $s := (s_h)_{h\in\mathcal{H}} \in [0,1]^{JH}$.
- $e_h := (e_h^1, .., e_h^c, .., e_h^C) \in \mathbb{R}_{++}^C$ denotes household h's endowment, $e := (e_h)_{h\in\mathcal{H}} \in \mathbb{R}_{++}^{CH}$ and $r := \sum_{h\in\mathcal{H}} e_h \in \mathbb{R}_{++}^C$.
- $E := ((u, e, s), t)$ is a private ownership economy with externalities.
- p^c is the price of one unit of commodity c, $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}_{++}^C$.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote $w^{\setminus} := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

We make the following assumptions on the transformation functions.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is continuous on its domain. For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiable and $D_{y_j} t_j(\cdot,\cdot,\cdot)$ is continuous on $\mathbb{R}^{CJ} \times \mathbb{R}_{++}^{CH}$.
(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, $t_j(0, y_{-j}, x) = 0$.
(3) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly quasi-convex, i.e., it is a C^2 function and for all $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ is positive definite on $\operatorname{Ker} D_{y_j} t_j(y'_j, y_{-j}, x)$.[9]
(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, $D_{y_j} t_j(y'_j, y_{-j}, x) \gg 0$ for all $y'_j \in \mathbb{R}^C$.

Fixing the externalities, the assumptions on t_j are standard in a differentiable framework. From Points 1 and 4 of Assumption 1, the production set is a C^1 manifold of dimension C and its boundary is a C^1 manifold of dimension $C-1$. Point 2 of Assumption 1 states that inaction is possible. Consequently, using standard arguments from profit maximization, the individual wealth of household h derived from his endowment $e_h \in \mathbb{R}_{++}^C$ and his profit shares is strictly positive for every price $p \in \mathbb{R}_{++}^C$, from which one deduces the non-emptiness of the interior of the individual budget constraint. From Point 3 of Assumption 1, the production set is convex. Furthermore, if the profit maximization problem has a solution then it is unique, because the function $t_j(\cdot, y_{-j}, x)$ is continuous and strictly quasi-convex. We remark that t_j is not required to be quasi-convex with respect to all the variables, so we do not require the production set to be convex with respect to the externalities. From Point 4 of Assumption 1, the function $t_j(\cdot, y_{-j}, x)$ is strictly increasing and so the production set satisfies

[9] Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the *scalar product* of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the *matrix product* of A and B. Without loss of generality, vectors are treated as row matrices and A denotes both the matrix and the following linear mapping $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T denotes the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the scalar product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.

the classical "free disposal" property.

Remark 2 *All our analysis holds true if some commodities are not involved in the technological process of firm j. In this case, for every firm j, one defines the set $\mathcal{C}_j$ of all the commodities $c \in \mathcal{C}$ that are involved in the technological process of firm j, where C_j denotes the cardinality of the set $\mathcal{C}_j$ with $2 \leq C_j \leq C$. The production plan of firm j is then defined as $y_j := (y_j^c)_{c \in \mathcal{C}_j} \in \mathbb{R}^{C_j}$ and the transformation function t_j is a function from $\mathbb{R}^{C_j} \times \prod_{f \neq j} \mathbb{R}^{C_f} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. In this case, all the assumptions on the transformation functions are written just replacing $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ by $\mathbb{R}^{C_j} \times \prod_{f \neq j} \mathbb{R}^{C_f} \times \mathbb{R}^{CH}_{++}$. Furthermore, in the definition of a competitive equilibrium, one also adapts the market clearing condition for every commodity c by considering only the firms that use commodity c in their technological process. That is, for every commodity c, the sum over $j \in \mathcal{J}$ is replaced by the sum over $j \in \mathcal{J}(c) := \{j \in \mathcal{J} \colon c \in \mathcal{C}_j\}$.*

For any given externality $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, define the set of all the production plans which belong to the production sets, that is

$$Y(x, y) := \{y' \in \mathbb{R}^{CJ} \colon t_j(y'_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J}\} \tag{1}$$

We remind that $r = \sum_{h \in \mathcal{H}} e_h$ and we make the following assumption which is analogous to Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001), and Assumption P(3) in Mandel (2008).

Assumption 3 (Uniform Boundedness) *There exists a bounded set $C(r) \subseteq \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,*

$$Y(x, y) \cap \{y' \in \mathbb{R}^{CJ} : \sum_{j \in \mathcal{J}} y'_j + r \gg 0\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 3.

Lemma 4 *There exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, the following set is included in $K(r)$.*

$$A(x, y; r) := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \colon y' \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \leq r\}$$

It is well known that the boundedness of the set of feasible allocations is a crucial condition for the non-emptiness and the compactness of the equilibrium set. Fixing the externalities, from Assumption 3 one deduces that the set of feasible allocations is bounded. So, in this sense, Assumption 3 is standard. Furthermore, Assumption 3 guarantees also that the set of feasible allocations $A(x, y; r)$ is *uniformly bounded* with respect to any possible externalities (x, y). In particular, it implies that the set $\mathcal{F}(r) = \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} | \; t_j(y_j, y_{-j}, x) \leq 0, \forall j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r\}$ is bounded. However, for the non-emptiness of the equilibrium set it would not be sufficient to assume only the boundedness of the set $\mathcal{F}(r)$.[10] Lemma 4 is used to prove compactness properties of the homotopy, see Steps 1.1 and 2.1 in the proof of Proposition 15, see Section 5.

We make the following assumptions on the utility functions.

Assumption 5 *For all $h \in \mathcal{H}$,*

(1) The function u_h is continuous on its domain. For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiable and $D_{x_h} u_h(\cdot,\cdot,\cdot)$ is continuous on $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$.

(2) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiably strictly increasing, i.e., $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ for all $x'_h \in \mathbb{R}^C_{++}$.

(3) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ it is differentiably strictly quasi-concave, i.e., it is C^2 and for all $x'_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ is negative definite on $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$.

(4) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ and for every $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\mathrm{cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}^C_{++}$$

Fixing the externalities, the assumptions on u_h are standard in a differentiable framework. From Point 3 of Assumption 5, the upper contour sets are convex, and if the utility maximization problem has a solution then it is unique. Notice that u_h is not required to be quasi-concave with respect to all the variables, so we do not require preferences to be convex with respect to the externalities. Point 4 of Assumption 5 is the classical Boundary Condition (BC) which means that the closure of the upper counter sets is included in $\mathbb{R}^C_{++}$. We notice that in Points 1 and 4 of Assumption 5, we consider consumption externalities x_{-h} on the boundary of the set $\mathbb{R}^{C(H-1)}_{++}$ in order to look at the limit of the behavior of u_h with respect to the consumption externalities. It means that BC is still valid whenever consumption externalities converge to zero for some commodities.[11]

[10] See Bonnisseau and Médecin (2001) and Mandel (2008), where the authors also need uniform boundedness assumptions in order to prove the non-emptiness of the equilibrium set.

[11] See Step 2.2 in the proof of Proposition 15, Section 5.

3 Competitive equilibrium and equilibrium function

In this section, we provide the definition of competitive equilibrium à la Nash and the notion of equilibrium function.

Without loss of generality, commodity C is the "numeraire good". So, given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation, we denote $p := (p^{\setminus}, 1) \in \mathbb{R}^{C}_{++}$.

Definition 6 (Competitive equilibrium) $(x^*, y^*, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for the economy* E *if for all* $j \in \mathcal{J}$, y^*_j *solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y^*_{-j}, x^*) \leq 0 \end{aligned} \tag{2}$$

for all $h \in \mathcal{H}$, x^*_h *solves the following problem*

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^*_{-h}, y^*) \\ &\text{subject to } \; p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j) \end{aligned} \tag{3}$$

and (x^*, y^*) *satisfies market clearing conditions, that is*

$$\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y^*_j \tag{4}$$

Using first-order conditions, one easily characterizes the solutions of firms and households maximization problems. The proof of the following proposition is standard, because in problems (2) and (3) each agent takes as given both the price and the actions of the other agents.

Proposition 7

(1) *From Assumption 1, if* y^*_j *is a solution to problem (2), then it is unique and it is completely characterized by KKT conditions.*[12]

(2) *From Point 2 of Assumption 1 and Assumption 5, there exists a unique solution* x^*_h *to problem (3) and it is completely characterized by KKT conditions.*

(3) *As usual, from Point 2 of Assumption 5, household* h*'s budget constraint holds with an equality. Thus, at equilibrium, due to the Walras law, the market clearing condition for commodity* C *is "redundant". So, one replaces condition (4) with* $\sum_{h \in \mathcal{H}} x^{*\setminus}_h = \sum_{h \in \mathcal{H}} e^{\setminus}_h + \sum_{j \in \mathcal{J}} y^{*\setminus}_j$.

[12] "KKT conditions" means Karush–Kuhn–Tucker conditions.

Let $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}^{C-1}_{++}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^\backslash) := ((x_h, \lambda_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}}, p^\backslash)$ where λ_h denotes the Lagrange multiplier associated with household h's budget constraint, and α_j denotes the Lagrange multiplier associated with firm j's technological constraint. We can now describe the competitive equilibria associated with the economy E using the *equilibrium function* $F : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F(\xi) := ((F^{h.1}(\xi), F^{h.2}(\xi))_{h\in\mathcal{H}}, (F^{j.1}(\xi), F^{j.2}(\xi))_{j\in\mathcal{J}}, F^M(\xi)) \tag{5}$$

where $F^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F^{h.2}(\xi) := -p \cdot (x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh} y_j)$, $F^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F^M(\xi) := \sum_{h\in\mathcal{H}} x^\backslash_h - \sum_{j\in\mathcal{J}} y^\backslash_j - \sum_{h\in\mathcal{H}} e^\backslash_h$.

$\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$ is an *extended equilibrium* for the E if and only if $F(\xi^*) = 0$. By Proposition 7, $(x^*, y^*, p^{*\backslash})$ is a competitive equilibrium for E if and only if there exists (λ^*, α^*) such that ξ^* is an *extended equilibrium* for E. We simply call ξ^* an equilibrium.

4 Existence and compactness

In this section, we show that the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. The result is provided by the following theorem.

Theorem 8 *The equilibrium set $F^{-1}(0)$ is non-empty and compact.*

In order to prove Theorem 8, we use a homotopy approach following the seminal paper by Smale (1974). The following theorem is a consequence of the homotopy invariance of the topological degree. Following Chapter 4 in Milnor (1965), and more recent contributions by Villanacci and Zenginobuz (2005), del Mercato (2006) and Bonnisseau and del Mercato (2008), our homotopy approach is based on the theory of *degree modulo 2*. The degree modulo 2 is simpler than the *Brouwer degree* which requires the concepts of *oriented manifold*.[13] The reader can find a survey of the degree theory, for example, in Geanakoplos and Shafer (1990) and in Villanacci et al. (2002). In Section 6, we recall the definition and fundamental properties of the degree modulo 2.

Theorem 9 (Homotopy Theorem) *Let M and N be C^2 manifolds of the same dimension contained in euclidean spaces. Let $y \in N$ and $f, g : M \to N$ be two functions such that f is continuous, g is C^1, y is a regular value for g*

[13] See for instance, Chapters 5 in Milnor (1965), and Mas-Colell (1985).

and $\#g^{-1}(y)$ *is odd. Let* L *be a continuous homotopy from* g *to* f *such that* $L^{-1}(y)$ *is compact. Then,*

(1) $g^{-1}(y)$ *is compact and* $\deg_2(g, y) = 1$*,*
(2) $f^{-1}(y)$ *is compact and* $\deg_2(f, y) = 1$*.*

The equilibrium function F plays the role of the function f given in Theorem 9. In order to construct the function playing the role of the function g, we proceed as follows. In Subsection 4.1, we construct an appropriate economy called the "test economy". G is the equilibrium function associated with the test economy and it plays the role of g. The test economy has an unique regular equilibrium, i.e., $\#G^{-1}(0) = 1$ and 0 is a regular value of G, see Proposition 12. In Subsection 4.2, we provide the required homotopy H from G to F playing the role of the homotopy L. Proposition 15 states the compactness of $H^{-1}(0)$. Using Propositions 12 and 15, all the assumptions of Theorem 9 are satisfied, and so one gets the following lemma.

Lemma 10 $F^{-1}(0)$ *is compact and* $\deg_2(F, 0) = 1$*.*

Using Lemma 10 and the *non-triviality property* of the topological degree one gets $F^{-1}(0) \neq \emptyset$, and so Theorem 8 is completely proved.

Finally, we remark that if E is a *regular economy* (i.e., the equilibrium function F is C^1 and 0 is a regular value of F), then using Lemma 10 and the computation of the degree modulo 2, one obviously finds that, at a regular economy, the number of equilibria is finite and odd. [14] However, this paper does not address any regularity issues. In the presence of consumption and production externalities, the analysis of regular economies is a quite sensitive topics. For the case without any externalities on the production side, see for example Bonnisseau (2003), Kung (2008), and Bonnisseau and del Mercato (2010). In the presence of production externalities, the analysis of regular economies deserves a separate analysis, see del Mercato and Platino (2015).

4.1 The "test economy" and its properties

We construct the test economy in two steps. We first fix the externalities and we consider a Pareto optimal allocation of a standard production economy $\overline{E}$ without externalities. Second, using the Second Theorem of Welfare Economics, we construct an appropriate private ownership economy $\widetilde{E}$ that has an unique regular equilibrium. $\widetilde{E}$ is the test economy and it is an economy without externalities.

[14] The computation of the degree modulo 2 for C^1 functions and regular values is provided by Proposition 17 in Section 6.

Fix an allocation $(\overline{x}, \overline{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. Define $\overline{u}_h(x_h) := u_h(x_h, \overline{x}_{-h}, \overline{y})$ for all $h \in \mathcal{H}$, $\overline{t}_j(y_j) := t_j(y_j, \overline{y}_{-j}, \overline{x})$ for all $j \in \mathcal{J}$, and the production economy $\overline{E} := (\overline{u}, \overline{t}, r)$ where $r = \sum_{h \in \mathcal{H}} e_h$. It is well known that, under Assumptions 1, 3 and 5, there exists a Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ of the economy $\overline{E}$ and Lagrange multipliers $(\widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ is completely characterized by the first-order conditions for Pareto optimality. This result is stated in the following proposition.

Proposition 11 *There exists a Pareto optimal allocation* $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ *of the economy* $\overline{E}$ *and* $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) = ((\widetilde{\beta}_j)_{j \in \mathcal{J}}, (\widetilde{\theta}_h)_{h \neq 1}, \widetilde{\gamma}) \in \mathbb{R}^{J}_{++} \times \mathbb{R}^{H-1}_{++} \times \mathbb{R}^{C}_{++}$ *such that* $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ *is the unique solution to the following system.*

$$\begin{cases} (1)\ D_{x_1}\overline{u}_1(x_1) = \gamma,\ \forall\, h \neq 1:\ (2)\ \theta_h D_{x_h}\overline{u}_h(x_h) = \gamma,\ (3)\ \overline{u}_h(x_h) = \overline{u}_h(\widetilde{x}_h) \\ \forall\, j \in \mathcal{J}:\ (4)\ \gamma = \beta_j D_{y_j}\overline{t}_j(y_j),\ (5)\ -\overline{t}_j(y_j) = 0,\ (6)\ \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j = r \end{cases} \tag{6}$$

Also, it is well known that the Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ can be *supported* by some price system $\widetilde{p}$.[15] From system (6), one easily deduces a *supporting price* $\widetilde{p}$, a redistribution of initial endowments $\widetilde{e} = (\widetilde{e}_h)_{h \in \mathcal{H}}$ and the equilibrium equations satisfied by $(\widetilde{x}, \widetilde{y})$ for appropriate Lagrange multipliers $(\widetilde{\lambda}, \widetilde{\alpha}) \in \mathbb{R}^{H}_{++} \times \mathbb{R}^{J}_{++}$. More precisely, define

$$\widetilde{e}_h := \widetilde{x}_h - \sum_{j \in \mathcal{J}} s_{jh}\widetilde{y}_j \tag{7}$$

and the function $G : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$G(\xi) := ((G^{h.1}(\xi), G^{h.2}(\xi))_{h \in \mathcal{H}}, (G^{j.1}(\xi), G^{j.2}(\xi))_{j \in \mathcal{J}}, G^M(\xi)) \tag{8}$$

where $G^{h.1}(\xi) := D_{x_h}u_h(x_h, \overline{x}_{-h}, \overline{y}) - \lambda_h p$, $G^{h.2}(\xi) := -p \cdot (x_h - \widetilde{e}_h - \sum_{j \in \mathcal{J}} s_{jh}y_j)$, $G^{j.1}(\xi) := p - \alpha_j D_{y_j}t_j(y_j, \overline{y}_{-j}, \overline{x})$, $G^{j.2}(\xi) := -t_j(y_j, \overline{y}_{-j}, \overline{x})$ and $G^M(\xi) := \sum_{h \in \mathcal{H}} x_h^{\backslash} - \sum_{j \in \mathcal{J}} y_j^{\backslash} - \sum_{h \in \mathcal{H}} \widetilde{e}_h^{\backslash}$.

The function G is nothing else than the equilibrium function associated with the economy $\widetilde{E} := ((\overline{u}, \widetilde{e}, s), \overline{t})$ which is a private ownership economy without externalities. Now, define $\widetilde{\xi} := (\widetilde{x}, \widetilde{\lambda}, \widetilde{y}, \widetilde{\alpha}, \widetilde{p}^{\backslash})$ with $\widetilde{p} := \frac{\widetilde{\gamma}}{\widetilde{\gamma}^C}$, $\widetilde{\lambda}_1 := \widetilde{\gamma}^C$, $\widetilde{\lambda}_h := \frac{\widetilde{\gamma}^C}{\widetilde{\theta}_h}$ for all $h \neq 1$ and $\widetilde{\alpha}_j := \frac{\widetilde{\beta}_j}{\widetilde{\gamma}^C}$ for all $j \in \mathcal{J}$. Using system (6), it is an easy matter to check that $G(\widetilde{\xi}) = 0$. As stated in the following proposition, $\widetilde{\xi}$ is the unique regular equilibrium of the economy $\widetilde{E}$.

[15] Using Debreu's vocabulary, $(\widetilde{x}, \widetilde{y})$ is an *equilibrium relative to some price system* $\widetilde{p}$, see Section 6.4 in Debreu (1959).

Proposition 12 *$G^{-1}(0) = \{\widetilde{\xi}\}$, G is C^1 and 0 is a regular value for G.*

Remark 13 *The redistribution of endowments $\widetilde{e} = (\widetilde{e}_h)_{h\in\mathcal{H}}$ given in (7) is not necessarily strictly positive. However, at equilibrium, the individual wealth of household h is equal to $\widetilde{p} \cdot \widetilde{x}_h$ which is strictly positive. One might wish to consider a different redistribution that gives rise to positive endowments. But, such a redistribution involves also some redistributions of the shares.*[16] *Our redistribution of endowments $\widetilde{e} = (\widetilde{e}_h)_{h\in\mathcal{H}}$ does not involve any redistribution of the shares, so that we do not need to homotopize the shares, see the homotopy H in the next subsection.*

4.2 The homotopy and its properties

The basic idea is to homotopize the endowments and the externalities by an arc from the equilibrium conditions associated with the test economy $\widetilde{E}$ to the ones associated with our economy E. But, one finds the following difficulty. At equilibrium, the individual wealth is positive at the beginning as well as at the end of the homotopy arc.[17] Nevertheless, since the production sets are not required to be convex with respect to the choices of all agents, the individual wealth may not be positive along the homotopy arc. Consequently, the individual budget constraint may be empty along the homotopy arc. We illustrate the details below.

Homotopize first the endowments by a segment. Then, for every $\tau \in [0, 1]$ the individual wealth is given by $p \cdot [\tau e_h + (1-\tau)\widetilde{e}_h] + p \cdot \sum\limits_{j\in\mathcal{J}} s_{jh} y_j$ which is equal to

$$p \cdot [\tau e_h + (1-\tau)\widetilde{x}_h] + p \cdot \sum_{j\in\mathcal{J}} s_{jh}[y_j - (1-\tau)\widetilde{y}_j]$$

So, the individual wealth is positive if $p \cdot y_j \geq p \cdot (1-\tau)\widetilde{y}_j$ for all $j \in \mathcal{J}$. Using standard arguments from profit maximization, at equilibrium, this condition is satisfied if $(1-\tau)\widetilde{y}_j$ belongs to the production set of firm j. On the other hand, if at same time one homotopizes the externalities by a segment, the production set becomes the following set,

$$Y_j(\tau y_{-j} + (1-\tau)\overline{y}_{-j}, \tau x + (1-\tau)\overline{x})$$

But, one does not know whether or not the production plan $(1-\tau)\widetilde{y}_j$ belongs to

---

[16] For example, the redistribution $\widehat{e}_h := \widehat{s}_{jh} \sum_{h\in\mathcal{H}} e_h$ with $\widehat{s}_{jh} := \frac{\widetilde{p}\cdot\widetilde{x}_h}{\widetilde{p}\cdot\sum_{h\in\mathcal{H}} \widetilde{x}_h}$.

[17] Indeed, at the economy E, the equilibrium individual wealth $p \cdot (e_h + \sum\limits_{j\in\mathcal{J}} s_{jh} y_j^*)$ is strictly positive because of the possibility of inaction (Point 2 of Assumption 1) and standard arguments from profit maximization.

the production set given above, unless the transformation function t_j is quasi-convex with respect to all the variables. Indeed, $t_j(0, y_{-j}, x) = 0$ because of the possibility of inaction, and $t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0$ because $G\left(\widetilde{\xi}\right) = 0$. If t_j is quasi-convex with respect to all the variables, then $t_j((1-\tau)\widetilde{y}_j, \tau y_{-j} + (1-\tau)\overline{y}_{-j}, \tau x + (1-\tau)\overline{x}) \leq 0$, and so $(1-\tau)\widetilde{y}_j$ belongs to the production set given above. But, t_j is not required to be quasi-convex with respect to all the variables.

Thus, to overcome the difficulty described above, we define the homotopy H in two times using two homotopies Φ and Γ. Namely,

- in the first homotopy Φ, we homotopize the endowments **without** homotopizing the externalities,
- in the second homotopy Γ, we homotopize the externalities in preferences and production technologies **without** homotopizing the endowments.

Remark 14 *Notice that,*

(1) If the externalities are fixed, then only one homotopy is needed, namely the homotopy Φ. So, our homotopy proof covers the case in which the economy E is a standard private ownership economy without externalities.[18]

(2) If the production sets are convex with respect to the choices of all agents, then endowments and externalities can be obviously homotopized at the same time.

Formally, using $(\overline{x}, \overline{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ and $\widetilde{e}_h$ given by (7), define the following convex combinations

$$e_h(\tau) := \tau e_h + (1-\tau)\widetilde{e}_h,\ x(\tau) := \tau x + (1-\tau)\overline{x},\ y(\tau) := \tau y + (1-\tau)\overline{y} \quad (9)$$

and the two following homotopies $\Phi, \Gamma : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$,

$$\Phi(\xi,\tau) := ((\Phi^{h.1}(\xi,\tau), \Phi^{h.2}(\xi,\tau))_{h\in\mathcal{H}}, (\Phi^{j.1}(\xi,\tau), \Phi^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Phi^M(\xi,\tau)) \quad (10)$$

where $\Phi^{h.1}(\xi,\tau) = D_{x_h}u_h(x_h, \overline{y}_{-j}, \overline{x}) - \lambda_h p$, $\Phi^{h.2}(\xi,\tau) = -p\cdot[x_h - e_h(\tau) - \sum_{j\in\mathcal{J}} s_{jh}y_j]$, $\Phi^{j.1}(\xi,\tau) = p - \alpha_j D_{y_j}t_j(y_j, \overline{y}_{-j}, \overline{x})$, $\Phi^{j.2}(\xi,\tau) = -t_j(y_j, \overline{y}_{-j}, \overline{x})$, $\Phi^M(\xi,\tau) = \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h(\tau)^{\setminus}$.

$$\Gamma(\xi,\tau) := ((\Gamma^{h.1}(\xi,\tau), \Gamma^{h.2}(\xi,\tau))_{h\in\mathcal{H}}, (\Gamma^{j.2}(\xi,\tau), \Gamma^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Gamma^M(\xi,\tau)) \quad (11)$$

[18] In Chapter 9 of Villanacci et al. (2002), the reader finds a homotopy proof for a standard private ownership economies without externalities. Our proof is simpler than the latter one because we do not homotopize the shares.

where $\Gamma^{h.1}(\xi,\tau) = D_{x_h} u_h(x_h, x_{-h}(\tau), y(\tau)) - \lambda_h p$, $\Gamma^{h.2}(\xi,\tau) = -p \cdot [x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j]$, $\Gamma^{j.1}(\xi,\tau) = p - \alpha_j D_{y_j} t_j(y_j, y_{-j}(\tau), x(\tau))$, $\Gamma^{j.2}(\xi,\tau) = -t_j(y_j, y_{-j}(\tau), x(\tau))$, $\Gamma^{M}(\xi,\tau) = \sum_{h \in \mathcal{H}} x_h^{\setminus} - \sum_{j \in \mathcal{J}} y_j^{\setminus} - \sum_{h \in \mathcal{H}} e_h^{\setminus}$.

Finally, define the homotopy $H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$,

$$H(\xi,\psi) := \begin{cases} \Phi(\xi, 2\psi) \text{ if } 0 \le \psi \le \frac{1}{2} \\ \Gamma(\xi, 2\psi - 1) \text{ if } \frac{1}{2} \le \psi \le 1 \end{cases}$$

The homotopy H is continuous since Φ and Γ are composed by continuous functions. Importantly, $H\left(\xi, \frac{1}{2}\right)$ is well defined since $\Phi(\xi,1) = \Gamma(\xi,0)$. Furthermore, $H(\xi,0) = \Phi(\xi,0) = G(\xi)$ and $H(\xi,1) = \Gamma(\xi,1) = F(\xi)$. We conclude providing the following lemma.

Proposition 15 *$H^{-1}(0)$ is compact.*

5 Proofs

Proof of Lemma 4. Let $(x', y') \in A(x,y;r)$. Since $\sum_{h \in \mathcal{H}} x'_h \gg 0$, y' belongs to the bounded set $C(r)$ given by Assumption 3. Furthermore, for every $h \in \mathcal{H}$, $0 \ll x'_h \ll \sum_{h \in \mathcal{H}} x'_h \le \sum_{j \in \mathcal{J}} y'_j + r$. Thus, there exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $A(x,y;r) \subseteq K(r)$.

■

Proof of Proposition 11. Let $\overline{E}$ be the production economy defined in Section 4.1. We remind that $A(\overline{x}, \overline{y}; r) := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : \bar{t}_j(y'_j) \le 0, \ \forall j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \le r\}$. Consider the set $U(r) := \{(u'_h)_{h \in \mathcal{H}} \in \prod_{h \in \mathcal{H}} \operatorname{Im} \overline{u}_h \mid \exists (x', y') \in A(\overline{x}, \overline{y}; r) \colon \overline{u}_h(x'_h) \ge u'_h, \ \forall h \in \mathcal{H}\}$. By Point 2 of Assumption 1, the set U_r is non-empty. Fix $(u'_h)_{h \in \mathcal{H}} \in U(r)$ and consider the following maximization problem

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}} \quad & \overline{u}_1(x_1) \\ \text{subject to} \quad & \bar{t}_j(y_j) \le 0, \ \forall j \in \mathcal{J} \\ & \overline{u}_h(x_h) \ge u'_h, \ \forall h \in \mathcal{H} \\ & \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j \le r \end{aligned} \tag{12}$$

Step 1. *Problem (12) has at least a solution.* Let K be the set determined by the constraints of problem (12). K is non-empty since $(u'_h)_{h\in\mathcal{H}} \in U_r$. We claim that K is compact. Define the set $N := \{(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : \overline{u}_h(x_h) \geq u'_h,\ \forall h \in \mathcal{H}\}$ and notice that $K = N \cap A(\overline{x}, \overline{y}; r)$. So, K is bounded by Lemma 4. Furthermore, K is a closed set included in $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ by Point 4 of Assumption 5 and the continuity of the functions $\overline{u}_h$ and $\overline{t}_j$. By Weierstrass' Theorem, there exists a solution to problem (12).

Step 2. *Let $(\widetilde{x}, \widetilde{y})$ be a solution to problem (12). Then, $(\widetilde{x}, \widetilde{y})$ solves the following problem and it is a Pareto optimal allocation of the economy $\overline{E}$.*

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} \quad & \overline{u}_1(x_1) \\ \text{subject to} \quad & -\overline{t}_j(y_j) \geq 0,\ \forall j \in \mathcal{J} \\ & \overline{u}_h(x_h) - \overline{u}_h(\widetilde{x}_h) \geq 0,\ \forall h \neq 1 \\ & r - \sum_{h\in\mathcal{H}} x_h + \sum_{j\in\mathcal{J}} y_j \geq 0 \end{aligned} \tag{13}$$

Let $\widetilde{K}$ be the set determined by the constraints of problem (13), $(\widetilde{x}, \widetilde{y})$ obviously belongs to $\widetilde{K}$. Consider now $(x,y) \in \widetilde{K}$ and remind that K is the set determined by the constraints of problem (12). If $\overline{u}_1(x_1) \geq u'_1$, then $(x,y) \in K$ and so $\overline{u}_1(\widetilde{x}_1) \geq \overline{u}_1(x_1)$. If $\overline{u}_1(x_1) < u'_1$, then $\overline{u}_1(\widetilde{x}_1) > \overline{u}_1(x_1)$ since $\overline{u}_1(\widetilde{x}_1) \geq u'_1$. Thus, $(\widetilde{x}, \widetilde{y})$ solves problem (13). Now, suppose by contradiction that $(\widetilde{x}, \widetilde{y})$ is not a Pareto optimal allocation of $\overline{E}$. Then, there is another allocation $(\widehat{x}, \widehat{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $\overline{t}_j(\widehat{y}_j) \leq 0$ for all j, $\sum_{h\in\mathcal{H}} \widehat{x}_h \leq r + \sum_{j\in\mathcal{J}} \widehat{y}_j$, $\overline{u}_h(\widehat{x}_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all h, and $\overline{u}_k(\widehat{x}_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$. If $k = 1$, then we get a contradiction since $(\widetilde{x}, \widetilde{y})$ solves problem (13). If $k \neq 1$, using the continuity of $\overline{u}_k$, there exists $\varepsilon > 0$ such that $\overline{u}_k(\widehat{x}_k - \varepsilon\mathbf{1}^c) > \overline{u}_k(\widetilde{x}_k)$ where the vector $\mathbf{1}^c \in \mathbb{R}^C_+$ has all the components equal to 0 except the component c which is equal to 1. Consider the allocation (x,y) defined by $x_1 := \widehat{x}_1 + \varepsilon\mathbf{1}^c$, $x_k := \widehat{x}_k - \varepsilon\mathbf{1}^c$, $x_h := \widehat{x}_h$ for all $h \neq 1$, $h \neq k$, and $y_j := \widehat{y}_j$ for all j. $(x,y) \in \widetilde{K}$ and $\overline{u}_1(x_1) > \overline{u}_1(\widetilde{x}_1)$ since $\overline{u}_1$ is strictly increasing. So, once again we get a contradiction since $(\widetilde{x}, \widetilde{y})$ solves problem (13).

Step 3. *Let $(\widetilde{x}, \widetilde{y})$ be a solution of problem (12). Then, there exists $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) := ((\widetilde{\beta}_j)_{j\in\mathcal{J}}, (\widetilde{\theta}_h)_{h\neq 1}, \widetilde{\gamma}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{H-1}_{++} \times \mathbb{R}^C_{++}$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ is the unique solution to system (6).* We first prove the existence of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$, afterwards we show the uniqueness of $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$.

Existence of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) \gg 0$. By Step 2, $(\widetilde{x}, \widetilde{y})$ solves problem (13). The KKT conditions associated with problem (13) are given by

$$\begin{cases} D_{x_1}\overline{u}_1(x_1) = \gamma,\ \forall h \neq 1:\ \theta_h D_{x_h}\overline{u}_h(x_h) = \gamma,\ \theta_h(\overline{u}_h(x_h) - \overline{u}_h(\widetilde{x}_h)) = 0 \\ \forall j \in \mathcal{J}:\ \gamma = \beta_j D_{y_j}\overline{t}_j(y_j),\ \beta_j(-\overline{t}_j(y_j)) = 0,\ \forall c \in \mathcal{C}:\ \gamma^c(r^c - \sum_{h\in\mathcal{H}} x_h^c + \sum_{j\in\mathcal{J}} y_j^c) = 0 \end{cases} \tag{14}$$

where $(\beta, \theta, \gamma) := ((\beta_j)_{j\in\mathcal{J}}, (\theta_h)_{h\neq 1}, (\gamma^c)_{c\in\mathcal{C}}) \in \mathbb{R}_+^J \times \mathbb{R}_+^{H-1} \times \mathbb{R}_+^C$ are the Lagrange multipliers associated with the constraint functions of problem (13). We first claim that KKT conditions are necessary conditions to solve problem (13). It is enough to verify that the Jacobian matrix associated with the constraint functions of problem (13) has full row rank equal to $N := J + (H-1) + C$. For every firm j, fix a commodity $c(j) \in \mathcal{C}$. The matrix given below is the $N \times N$ square sub-matrix which is obtained considering the partial derivatives of the constraint functions with respect to $((y_j^{c(j)})_{j\in\mathcal{J}}, (x_h^1)_{h\neq 1}, x_1)$. Point 4 of Assumption 1 and Point 2 of Assumption 5 imply that the determinant of this square sub-matrix is different from zero, which complete the proof of the claim. [19]

$$\begin{bmatrix} -D_{y_1^{c(1)}}\overline{t}_1(y_1) & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -D_{y_J^{c(J)}}\overline{t}_J(y_J) & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & D_{x_2^1}\overline{u}_2(x_2) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & D_{x_H^1}\overline{u}_H(x_H) & 0 \\ \left[\mathbf{1}^{c(1)}\right]^T & \cdots & \left[\mathbf{1}^{c(J)}\right]^T & -\left[\mathbf{1}^1\right]^T & \cdots & -\left[\mathbf{1}^1\right]^T & -I_C \end{bmatrix}$$

Therefore, there exists $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) \geq 0$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ solves system (14). Furthermore, Point 4 of Assumption 1 and Point 2 of Assumption 5 imply that all the Lagrange multipliers $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ must be strictly positive. Consequently, all the constraints of problem (13) are binding, and so $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ is a solution to system (6).

Uniqueness of $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$. Define $\widetilde{\theta}_1 := 1$, by equations (1) and (2) of system (6), for all h one gets $D_{x_h}\overline{u}_h(\widetilde{x}_h) = \frac{\widetilde{\gamma}}{\widetilde{\theta}_h}$. So, for every h, $\widetilde{x}_h$ solves the maximization problem: $\max_{x_h\in\mathbb{R}_{++}^C} \overline{u}_h(x_h)$ subject to $\frac{\widetilde{\gamma}}{\widetilde{\theta}_h} \cdot x_h \leq \frac{\widetilde{\gamma}}{\widetilde{\theta}_h} \cdot \widetilde{x}_h$ because KKT are sufficient conditions to solve this problem. Thus, the uniqueness of $\widetilde{x}_h$ obviously follows from the strict quasi-concavity of $\overline{u}_h$. Analogously, by equations (4) and (5) of system (6), $\widetilde{y}_j$ solves the maximization problem: $\max_{y_j\in\mathbb{R}^C} \frac{\widetilde{\gamma}}{\widetilde{\beta}_j} \cdot y_j$ subject to $\overline{t}_j(y_j) \leq 0$ for every j. Thus, the uniqueness of $\widetilde{y}_j$ follows from the continuity and the strict quasi-convexity of $\overline{t}_j$. Therefore, $(\widetilde{x}, \widetilde{y})$ is unique and consequently, the uniqueness of $(\widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma})$ obviously follows by equations (1),

[19] We remind that for every commodity c, the vector $\mathbf{1}^c \in \mathbb{R}_+^C$ has all the components equal to 0 except the component c which is equal to 1.

(2) and (4) of system (6).

■

Proof of Proposition 12. For the proof we use the functions $\overline{u}_h$ and $\overline{t}_j$ defined in Subsection 4.1. We have already pointed out that $G(\widetilde{\xi}) = 0$. Let $\xi' = (x', \lambda', y', \alpha', p'^{\backslash}) \in \Xi$ be such that $G(\xi') = 0$, we show that $\widetilde{\xi} = \xi'$.

First, notice that

$$\sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j = \sum_{h \in \mathcal{H}} e_h \tag{15}$$

Indeed, summing $G^{h.2}(\xi') = 0$ over h, one gets $\sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j = \sum_{h \in \mathcal{H}} \widetilde{e}_h$ by $G^M(\xi') = 0$. Using the definition of $\widetilde{e}_h$ given in (7) and Proposition 11, one obviously deduces (15).

Second, we show that

$$\overline{u}_h(x'_h) = \overline{u}_h(\widetilde{x}_h), \forall h \in \mathcal{H} \tag{16}$$

Using the definition of $\widetilde{e}_h$ given in (7) and $G^{h.1}(\xi') = G^{h.2}(\xi') = 0$, x'_h solves the following maximization problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ \overline{u}_h(x_h) \\ &\text{subject to } \ p' \cdot x_h \leq p' \cdot \widetilde{x}_h + \sum_{j \in \mathcal{J}} s_{jh} \, p' \cdot (y'_j - \widetilde{y}_j) \end{aligned} \tag{17}$$

because KKT are sufficient conditions to solve this problem. Analogously, from $G^{j.1}(\xi') = G^{j.2}(\xi') = 0$, y'_j solves the maximization problem: $\max_{y_j \in \mathbb{R}^C} p' \cdot y_j$ subject to $\overline{t}_j(y_j) \leq 0$. Notice that $\widetilde{y}_j$ satisfies the constraint of this problem because $G^{j.2}(\widetilde{\xi}) = 0$. Thus, $p' \cdot (y'_j - \widetilde{y}_j) \geq 0$ for all j, and consequently $\widetilde{x}_h$ belongs to the budget constraint of problem (17). So, $\overline{u}_h(x'_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all h. Now, suppose that $\overline{u}_k(x'_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$. From (15) and $G^{j.2}(\xi') = 0$ for all j, one deduces that (x', y') is a feasible allocation of the production economy $\overline{E}$, and so one gets a contradiction since $(\widetilde{x}, \widetilde{y})$ is a Pareto optimal allocation of $\overline{E}$ by Proposition 11. Thus, (16) is completely proved.

Now, define $\beta' := (\beta'_j)_{j \in \mathcal{J}}$ where $\beta'_j := \lambda'_1 \alpha'_j$ for all j, $\theta' := (\theta'_h)_{h \neq 1}$ where $\theta'_h := \frac{\lambda'_1}{\lambda'_h}$ for all $h \neq 1$ and $\gamma' := \lambda'_1 p'$. From $G^{h.1}(\xi') = 0$ for all h, $G^{j.1}(\xi') = G^{j.2}(\xi') = 0$ for all j, (15) and (16), it is an easy matter to check that $(x', y', \beta', \theta', \gamma')$ solves system (6). So, Proposition 11 implies that $(\widetilde{x}, \widetilde{y}, \widetilde{\beta}, \widetilde{\theta}, \widetilde{\gamma}) = (x', y', \beta', \theta', \gamma')$, and consequently, one obviously deduces that $\widetilde{\xi} = \xi'$.

We remark that G is C^1 by Point 1 of Assumptions 1 and 5. Finally, in order to show that 0 is a regular value for G, one proves that $D_\xi G(\widetilde{\xi})$ has full row rank. In this regard, we show that if $\Delta D_\xi G(\widetilde{\xi}) = 0$, then $\Delta = 0$ where $\Delta := \left((\Delta x_h, \Delta \lambda_h)_{h \in \mathcal{H}}, (\Delta y_j, \Delta \alpha_j)_{j \in \mathcal{J}}, \Delta p^{\backslash}\right) \in \mathbb{R}^{\dim \Xi}$. The system $\Delta D_\xi G(\widetilde{\xi}) = 0$ is given below.

$$\begin{cases} (h.1) \;\; \Delta x_h D^2_{x_h}\overline{u}_h(\widetilde{x}_h) - \Delta\lambda_h \widetilde{p} + \Delta p^{\setminus}[I_{C-1}|0] = 0, \; \forall \; h \in \mathcal{H} \\ (h.2) \;\; -\Delta x_h \cdot \widetilde{p} = 0, \; \forall \; h \in \mathcal{H} \\ (j.1) \;\; \sum_{h \in \mathcal{H}} \Delta\lambda_h s_{jh}\widetilde{p} - \widetilde{\alpha}_j \Delta y_j D^2_{y_j}\overline{t}_j(\widetilde{y}_j) - \Delta\alpha_j D_{y_j}\overline{t}_j(\widetilde{y}_j) - \Delta p^{\setminus}[I_{C-1}|0] = 0, \; \forall \; j \in \mathcal{J} \\ (j.2) \;\; -\Delta y_j \cdot D_{y_j}\overline{t}_j(\widetilde{y}_j) = 0, \; \forall \; j \in \mathcal{J} \\ (M) \;\; -\sum_{h \in \mathcal{H}} \widetilde{\lambda}_h \Delta x_h^{\setminus} + \sum_{j \in \mathcal{J}} \Delta y_j^{\setminus} = 0 \end{cases}$$

We first prove that $\Delta x_h = 0$ for all $h \in \mathcal{H}$. Otherwise, suppose that there is $\overline{h} \in \mathcal{H}$ such that $\Delta x_{\overline{h}} \neq 0$. The proof goes through the two following claims that contradict each others.

We first claim that $\Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \widetilde{\lambda}_h \Delta x_h^{\setminus}) > 0$. Multiplying $(h.1)$ by $\widetilde{\lambda}_h \Delta x_h$ and summing over h, from $(h.2)$ we get $\sum_{h \in \mathcal{H}} \widetilde{\lambda}_h \Delta x_h D^2_{x_h}\overline{u}_h(\widetilde{x}_h)(\Delta x_h) = -\Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \widetilde{\lambda}_h \Delta x_h^{\setminus})$. Multiplying $G^{h.1}(\widetilde{\xi}) = 0$ by Δx_h and using $(h.2)$, we get $\Delta x_h \cdot D_{x_h}\overline{u}_h(\widetilde{x}_h) = 0$ for all h. Therefore, Point 3 of Assumption 5 completes the proof of the claim since $\widetilde{\lambda}_h > 0$ for all h and $\Delta x_{\overline{h}} \neq 0$.

Second, we claim that $\Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \widetilde{\lambda}_h \Delta x_h^{\setminus}) \leq 0$. Multiplying both sides of $G^{j.1}(\widetilde{\xi}) = 0$ by Δy_j and using $(j.2)$, we get $\Delta y_j \cdot \widetilde{p} = 0$. So, multiplying $(j.1)$ by Δy_j and summing over j, from $(j.2)$ we get $-\sum_{j \in \mathcal{J}} \widetilde{\alpha}_j \Delta y_j D^2_{y_j}\overline{t}_j(\widetilde{y}_j)(\Delta y_j) = \Delta p^{\setminus} \cdot \sum_{j \in \mathcal{J}} \Delta y_j^{\setminus}$. Since, $\widetilde{\alpha}_j > 0$ for all j, Point 3 of Assumption 1 and $(j.2)$ imply that $\Delta p^{\setminus} \cdot \sum_{j \in \mathcal{J}} \Delta y_j^{\setminus} \leq 0$. Using (M), the claim is completely proved.

Since $\widetilde{p}^C = 1$ and $\Delta x_h = 0$ for all $h \in \mathcal{H}$, from $(h.1)$ we get $\Delta\lambda_h = 0$ for all h, and so $\Delta p^{\setminus} = 0$. Thus, multiplying $(j.1)$ by Δy_j, Point 3 of Assumption 1 and $(j.2)$ imply that $\Delta y_j = 0$. So, using once again $(j.1)$, we get $\Delta\alpha_j = 0$ by Point 4 of Assumption 1. Therefore, $\Delta = 0$.

■

Proof of Proposition 15. Observe that $H^{-1}(0) = \Phi^{-1}(0) \cup \Gamma^{-1}(0)$. Since the union of a finite number of compact sets is compact, it is enough to show that $\Phi^{-1}(0)$ and $\Gamma^{-1}(0)$ are compact.

Claim 1. $\Phi^{-1}(0)$ *is compact.*

We prove that, up to a subsequence, every sequence $(\xi^\nu, \tau^\nu)_{\nu \in \mathbb{N}} \subseteq \Phi^{-1}(0)$

converges to an element of $\Phi^{-1}(0)$, where $\xi^\nu := (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^{\nu \backslash})_{\nu \in \mathbb{N}}$. Since $\{\tau^\nu : \nu \in \mathbb{N}\} \subseteq [0,1]$, up to a subsequence, $(\tau^\nu)_{\nu \in \mathbb{N}}$ converges to some $\tau^* \in [0,1]$. From Steps 1.1, 1.2, 1.3 and 1.4 below, up to a subsequence, $(\xi^\nu)_{\nu \in \mathbb{N}}$ converges to some $\xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^{* \backslash}) \in \Xi$. Since Φ is continuous, taking the limit, one gets $(\xi^*, \tau^*) \in \Phi^{-1}(0)$.

We remind that for every $\tau \in [0,1]$, $e_h(\tau)$ is given by (9).

Step 1.1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.* We show that for $r = \sum_{h \in \mathcal{H}} e_h$, the sequence $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ is included in the bounded set $K(r)$ given by Lemma 4. By $\Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j we get

$$t_j(y_j^\nu, \overline{y}_{-j}, \overline{x}) = 0, \ \forall\, \nu \in \mathbb{N}$$

Thus, the sequence $(y^\nu)_{\nu \in \mathbb{N}}$ is included in the set $Y(\overline{x}, \overline{y})$ given by (1). Summing $\Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Phi^M(\xi^\nu, \tau^\nu) = 0$ we have $\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu = \sum_{h \in \mathcal{H}} e_h(\tau^\nu)$ for every $\nu \in \mathbb{N}$. Using the definition of $\widetilde{e}_h$ given by (7) and Proposition 11, one gets $\sum_{h \in \mathcal{H}} e_h(\tau) = r$ for every $\tau \in [0,1]$, and so $\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu = r$ for every $\nu \in \mathbb{N}$. Thus, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}} \subseteq A(\overline{x}, \overline{y}; r) \subseteq K(r)$. Consequently, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ is included in $\operatorname{cl} K(r)$ which is a compact set. Then, up to a subsequence, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} K(r) \subseteq \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$, and so $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.

Step 1.2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* By $\Phi^{h.1}(\xi^\nu, \tau^\nu) = \Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$ and KKT sufficient conditions, x_h^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ u_h(x_h, \overline{x}_{-h}, \overline{y}) \\ &\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot [\tau^\nu e_h + (1 - \tau^\nu)\widetilde{x}_h] + p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh}(y_j^\nu - (1 - \tau^\nu)\widetilde{y}_j) \end{aligned} \tag{18}$$

We first claim that for every $\nu \in \mathbb{N}$, the following bundle

$$\widehat{e}_h(\tau^\nu) := \tau^\nu e_h + (1 - \tau^\nu)\widetilde{x}_h \tag{19}$$

belongs to the budget constraint of the problem above. By $\Phi^{j.1}(\xi^\nu, \tau^\nu) = \Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$ and KKT sufficient conditions, y_j^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ &\text{subject to } t_j(y_j, \overline{y}_{-j}, \overline{x}) \leq 0 \end{aligned} \tag{20}$$

$t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0$ since $G^{j.2}(\widetilde{\xi}) = 0$, see (8). By Point 2 of Assumption 1, $t_j(0, \overline{y}_{-j}, \overline{x}) = 0$. So, we get $t_j((1 - \tau^\nu)\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) < 0$ since $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-convex, that is, the production plan $(1-\tau^\nu)\widetilde{y}_j$ belongs to the constraint set of problem (20). Thus, $p^\nu \cdot (y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$ for every j, and so $p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh}(y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$ which completes the proof of the claim.

Therefore, for every $\nu \in \mathbb{N}$, $u_h(x_h^\nu, \overline{x}_{-h}, \overline{y}) \geq u_h(\widehat{e}_h(\tau^\nu), \overline{x}_{-h}, \overline{y})$. By Point 2 of Assumption 5, for every $\varepsilon > 0$ we get $u_h(x_h^\nu + \varepsilon \mathbf{1}, \overline{x}_{-h}, \overline{y}) > u_h(\widehat{e}_h(\tau^\nu), \overline{x}_{-h}, \overline{y})$ where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. Taking the limit over ν and using Point 1 of Assumption 5, $x_h^* \gg 0$ since it belongs to the closure of the upper counter set associated with $u_h(\widehat{e}_h(\tau^*), \overline{x}_{-h}, \overline{y})$ which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 5. Thus, $x^* \gg 0$.

Step 1.3. *Up to a subsequence,* $(\lambda^\nu, p^{\nu\setminus})_{\nu \in \mathbb{N}}$ *converges to some* $\lambda^* \in \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++}$. The proof is similar to the proof of Step 2.3.

Step 1.4. *Up to a subsequence,* $(\alpha^\nu)_{\nu \in \mathbb{N}}$ *converges to some* $\alpha^* \in \mathbb{R}^J_{++}$. The proof is similar to the proof of Step 2.4.

Claim 2. $\Gamma^{-1}(0)$ *is compact.*

Let $(\xi^\nu, \tau^\nu)_{\nu \in \mathbb{N}}$ be a sequences in $\Gamma^{-1}(0)$. As in Claim 1, $(\tau^\nu)_{\nu \in \mathbb{N}}$ converges to $\tau^* \in [0,1]$. From Seps 2.1, 2.2, 2.3 and 2.4 below, up to a subsequence, $(\xi^\nu)_{\nu \in \mathbb{N}}$ converges to an element $\xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$. Since Γ is a continuous function, taking limit, we get $(\xi^*, \tau^*) \in \Gamma^{-1}(0)$.

We remind that for every $\tau \in [0,1]$, $x(\tau)$ and $y(\tau)$ are given by (9).

Step 2.1. *Up to a subsequence,* $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ *converges to some* $(x^*, y^*) \in \mathbb{R}^{CH}_{+} \times \mathbb{R}^{CJ}$. We show that for $r = \sum_{h \in \mathcal{H}} e_h$, the sequence $(x^\nu, y^\nu)_{\nu \in \mathbb{N}}$ is included in the bounded set $K(r)$ given by Lemma 4. Then, one completes the proof as in Step 1.1. By $\Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j we have that

$$t_j(y_j^\nu, y_{-j}^\nu(\tau^\nu), x^\nu(\tau^\nu)) = 0, \ \forall \nu \in \mathbb{N}$$

Thus, for every $\nu \in \mathbb{N}$, the production allocation y^ν belongs to the set $Y(x^\nu(\tau^\nu), y^\nu(\tau^\nu))$ given by (1). Now, summing $\Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Gamma^M(\xi^\nu, \tau^\nu) = 0$, we get $\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu = r$. So, for every $\nu \in \mathbb{N}$, the allocation (x^ν, y^ν) belongs to the set $A(x^\nu(\tau^\nu), y^\nu(\tau^\nu); r) \subseteq K(r)$, and consequently, $(x^\nu, y^\nu)_{\nu \in \mathbb{N}} \subseteq K(r)$.

Step 2.2. *The consumption allocation* x^* *is strictly positive, i.e.* $x^* \gg 0$. The argument is similar to the one used in Step 1.2 except for the last part which is quite different due to the presence of consumption externalities in the utility functions.

First, according to $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = \Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$, replace problem (18) with the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu)) \\ &\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot e_h + p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh} y^\nu_j \end{aligned} \tag{21}$$

and replace $\widehat{e}_h(\tau^\nu)$ given by (19) with $\widehat{e}_h(\tau^\nu) := e_h$. Second, according to $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = \Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$, replace problem (20) with the following problem

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ &\text{subject to } t_j(y_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) \leq 0 \end{aligned}$$

Finally, one follows the same strategy as for Step 1.2. But, differently from Step 1.2, in this case if $\tau^* = 1$, then $x^*_{-h}(\tau^*) = x^*_{-h}$ which *a priori* is not necessarily strictly positive. For this reason, in Points 1 and 4 of Assumption 5 we allow for consumption externalities on the boundary of $\mathbb{R}^{C(H-1)}_{++}$ so that $x^*_h \gg 0$ because x^*_h belongs to the closure of the upper contour set associated to $(e_h, x^*_{-h}(\tau^*), y^*(\tau^*))$.

Step 2.3. *Up to a subsequence, $(\lambda^\nu, p^{\nu\,\backslash})_{\nu \in \mathbb{N}}$ converges to some $(\lambda^*, p^{*\,\backslash}) \in \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++}$.* By $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = 0$, fixing commodity C, for every $\nu \in \mathbb{N}$ we have $\lambda^\nu_h = D_{x^C_h} u_h(x^\nu_h, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu))$. Taking the limit, by Points 1 and 2 of Assumption 5, we get $\lambda^*_h := D_{x^C_h} u_h(x^*_h, x^*_{-h}(\tau^*), y^*(\tau^*)) > 0$.

By $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = 0$, for all commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have $p^{\nu\,c} = \dfrac{D_{x^c_h} u_h(x^\nu_h, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu))}{\lambda^\nu_h}$. Taking the limit, by Points 1 and 2 of Assumption 5, we get $p^{*\,c} := \dfrac{D_{x^c_h} u_h(x^*_h, x^*_{-h}(\tau^*), y^*(\tau^*))}{\lambda^*_h} > 0$. Therefore, $p^{*\,\backslash} \gg 0$.

Step 2.4. *Up to a subsequence, $(\alpha^\nu)_{\nu \in \mathbb{N}}$ converges to some $\alpha^* \in \mathbb{R}^J_{++}$.* For every firm j, fix a commodity $c(j) \in \mathcal{C}$. By $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = 0$, for every $\nu \in \mathbb{N}$ we have that

$$\alpha^\nu_j = \frac{p^{\nu\,c(j)}}{D_{y^{c(j)}_j} t_j(y^\nu_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu))}$$

which is strictly positive by Point 4 of Assumption 1. Taking the limit, by Points 1 and 4 of Assumption 1, we get $\alpha^*_j := \dfrac{p^{*\,c(j)}}{D_{y^{c(j)}_j} t_j(y^*_j, y^*_{-j}(\tau^*), x^*(\tau^*))} > 0$.

■

6 Appendix

We introduce a definition of the *degree modulo 2* for continuous functions, see Appendix B in Geanakoplos and Shafer (1990), and Chapter 7 in Villanacci et al. (2002).

Let M and N be two C^2 manifolds of the same dimension contained in euclidean spaces. Let $\mathcal{A}$ be the set of triples (f, M, y) where

(1) $f : M \to N$ is a continuous function,
(2) $y \in N$ and $f^{-1}(y)$ is compact.

Theorem 16 *There exists a unique function, called degree modulo 2 and denoted by* $\deg_2 : \mathcal{A} \to \{0, 1\}$ *such that*

(1) (Normalisation) $\deg_2(id_M, M, y) = 1$ *where* $y \in M$ *and* id_M *denotes the identity of* M*.*

(2) (Non–triviality) If $(f, M, y) \in \mathcal{A}$ *and* $\deg_2(f, M, y) = 1$*, then* $f^{-1}(y) \neq \emptyset$*.*

(3) (Excision) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *such that* $f^{-1}(y) \subseteq U$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U, y)$$

(4) (Additivity) If $(f, M, y) \in \mathcal{A}$ *and* U_1 *and* U_2 *are open and disjoint subsets of* M *such that* $f^{-1}(y) \subseteq U_1 \cup U_2$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U_1, y) + deg_2(f, U_2, y)$$

(5) (Local constantness) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *with compact closure such that* $f^{-1}(y) \subseteq U$*, then there is an open neighborhood* V *of* y *in* N *such that for every* $y' \in V$*,*

$$\deg_2(f, U, y') = \deg_2(f, U, y)$$

(6) (Homotopy invariance) Let $L : (z, \tau) \in M \times [0, 1] \to L(z, \tau) \in N$ *be a continuous homotopy. If* $y \in N$ *and* $L^{-1}(y)$ *is compact, then*

$$\deg_2(L_0, U, y) = \deg_2(L_1, U, y)$$

where $L_0 := L(\cdot, 0) : M \to N$ *and* $L_1 := L(\cdot, 1) : M \to N$*.*

If there is no possible confusion on the manifold M, we simply denote $\deg_2(f, y)$ the degree modulo 2 of the triple (f, M, y).

As stated in the following proposition, in the case of C^1 functions and regular values, the degree modulo 2 is computed using the *residue class modulo 2*.

Proposition 17 *If $(g, M, y) \in \mathcal{A}$, g is a C^1 function and y is a regular value of g (i.e., for all $z^* \in g^{-1}(y)$, the differential mapping $Dg(z^*)$ is onto), then $g^{-1}(y)$ is finite (possibly empty) and the degree modulo 2 of g is given by*

$$\deg_2(g, M, y) = [\#g^{-1}(y)]_2 = \begin{cases} 0 & \text{if } \#g^{-1}(y) \text{ is even} \\ 1 & \text{if } \#g^{-1}(y) \text{ is odd} \end{cases}$$

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