On the regularity of smooth production economies with externalities: Competitive equilibrium *à la Nash* [*]

Elena L. del Mercato and Vincenzo Platino [1]

Paris School of Economics – Université Paris 1 Panthéon-Sorbonne

Abstract

We consider a general equilibrium model of a private ownership economy with consumption and production externalities. The choices of all agents (households and firms) may affect utility functions and production technologies. The allocation of a competitive equilibrium is a Nash equilibrium. We provide an example showing that, under standard assumptions, competitive equilibria are indeterminate in an open set of the household's endowments. We introduce the model with firms' endowments, following Geanakoplos, Magill, Quinzii and Drèze (1990). In our model, firms' endowments impact the technologies and the marginal productivities of the other firms. We then prove that almost all economies are regular in the space of endowments of households and firms.

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[1] Elena del Mercato (corresponding author) and Vincenzo Platino, address: Centre d'Economie de la Sorbonne, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France, e-mails: Elena.delMercato@univ-paris1.fr; vincenzo.platino@gmail.com.

1 Introduction

The Arrow–Debreu model of general equilibrium has been extended to economies with consumption and production externalities. For such extensions, one has to choose un equilibrium notion. From a normative point of view, markets can be extended in order to obtain the two Fundamental Theorems of Welfare Economics also for such economies, i.e., a *perfect internalization* of the externalities. This is the idea sketched by Arrow (1969) and first analyzed by Laffont (1976). They enlarge in an appropriate manner the choice sets of the agents and introduce personalized prices that agents face on markets for rights on the consumption and production of any other agent in the economy. Recent contributions by Magill, Quinzii and Rochet (2015), and Crès and Tvede (2013), explore instead corporate governance policies that induce firms to internalize the externalities by maximizing some "social criterion" in order to increase the welfare of stakeholders or shareholders, depending on their respective contribution. Their analysis then focuses on legal systems that allow these goals to be achieved.

On the other hand, the positive theory of competitive equilibrium leads to a definition of equilibrium that combines Arrow–Debreu with Nash, that is, agents (households and firms) maximize their goals by taking as given both the commodity prices and the choices of every other agent in the economy. This is the notion given in Arrow and Hahn (1971), and Laffont (1988). This notion includes as a special case the classical equilibrium definition without externalities. However, with such an equilibrium notion, agents cannot choose the consumption and production of other agents, i.e., externalities cannot be internalized, and competitive markets may prevent equilibrium allocations from being Pareto optimal. This raises the question of which types of taxes and subsidy policies can Pareto improve the equilibrium allocations.

We consider a private ownership economy with a finite number of commodities, households and firms. Utility and transformation functions may be affected by the consumption and production activities of all other agents. We take the conventional non-cooperative view of market equilibrium. Our purpose is to provide the genericity of regular economies. We recall that an economy is regular if it has a finite (odd) number of equilibria and every equilibrium locally depends in a continuous or differentiable manner on the parameters describing the economy. Therefore, the equilibria of a regular economy are locally unique and persistent under small perturbations of the economy. Furthermore, at a regular economy, it is possible to perform classical comparative statics, see Smale (1981), Mas-Colell (1985), and Balasko (1988). There is an extensive literature on the results concerning the generic regularity of economies with externalities, see Villanacci and Zenginobuz (2005), Kung (2008), Bonnisseau and del Mercato (2010), and Balasko (2015). However, all these contributions deal with externalities on the consumer side only.

Apart from the intrinsic interest of a regularity result, regular economies are also important for the study of Pareto improving policies in terms of taxes and subsidies. In the presence of other sources of market failure, such as incomplete financial markets and public goods, there is a well established methodology for analyzing these policies, see Geanakoplos and Polemarchakis (1986), Geanakoplos, Magill, Quinzii and Drèze (1990), Citanna, Kajii and Villanacci (1998), Citanna, Polemarchakis and Tirelli (2006), Villanacci and Zenginobuz (2006, 2012). Such a methodology applies to the set of regular economies, since it requires equilibria to be differentiable maps of the fundamentals. Therefore, our contribution provides a solid foundation for the analysis of these kinds of policies in the spirit of Greenwald and Stiglitz (1986) and Geanakoplos and Polemarchakis (2008).[2]

We make basic assumptions on utility and transformation functions that are standard in "smooth" equilibrium models without externalities. These assumptions guarantee the non-emptiness and the compactness of the set of equilibria.[3] However, even in the simpler case of consumption externalities, they are not sufficient for establishing classical generic regularity. For economies with consumption externalities, establishing generic regularity requires the introduction of an additional assumption on the second order effects of externalities on individual utility functions, see Bonnisseau and del Mercato (2010).[4] However, the analogous assumption on the effects of production externalities on transformation functions is not going to work. We provide an example of a private ownership economy with one household and two firms where, despite well behaved second order external effects, equilibria are indeterminate in an open set of the household's endowments (and the indeterminacy is payoff relevant).

In order to overcome indeterminacy, we introduce firms' endowments into the model, following Geanakoplos, Magill, Quinzii and Drèze (1990).[5] Firms' en-

[2] The Pareto improving analysis of Greenwald and Stiglitz (1986) mainly focuses on economies with incomplete markets and imperfect information. However, in Section I, they consider a general equilibrium model of a private ownership economy with consumption and production externalities. They assume that the equilibrium exists and it is a differentiable map of the parameters describing the economy. The issue of the existence of such Pareto improving policies is not addressed for the general model provided in Section I. In Geanakoplos and Polemarchakis (2008), the Pareto improving analysis concerns economies with consumption externalities only.

[3] See, for instance del Mercato and Platino (2015.a).

[4] In the absence of this assumption, there is an example in which equilibria are indeterminate for all initial endowments.

[5] In Geanakoplos, Magill, Quinzii and Drèze (1990), there are no direct externalities in preferences and production sets. However, their model exhibits *pecuniary*

dowments consist of amounts of commodities initially held by the firms. In our model, firms' endowments have an impact on the production sets and the marginal productivities of the other firms. Consequently, perturbing these endowments affects the first and the second order effects of production externalities on transformation functions, thereby allowing us to establish generic regularity.[6] Our main theorem (Theorem 12) states that almost all economies are regular in the space of endowments of households and firms. Most of the classical properties of the equilibrium manifold then hold true.

In order to prove Theorem 12, we adapt *Smale's approach* to economies with consumption and production externalities. Smale's approach has been used for instance in Cass, Siconolfi and Villanacci (2001), Villanacci and Zenginobuz (2005), and Bonnisseau and del Mercato (2010) for other economic environments. This approach is an alternative to the aggregate excess demand-excess supply approach. Notice that in our economic environment, the aggregate excess demand-excess supply approach is problematic. This is because the individual demands and supplies are interdependent, making it difficult to define the aggregate excess demand-excess supply out of equilibrium.

The paper is organized as follows. In Section 2, we introduce the classical model, basic assumptions and definitions. In Section 3, we first discuss the difficulties of establishing classical generic regularity. We then provide our example of indeterminacy and we analyze the second order external effects in the example. In Section 4, we first introduce the model with firms' endowments. We then provide our regularity result and we sketch the outlines of its proof. All the lemmas are proved in Section 5.

2 The model

We consider a private ownership economy. There is a finite number of commodities labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$ and a finite number of households labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$.

The production plan of firm j is $y_j := (y_j^1, .., y_j^c, .., y_j^C)$. As usual, if $y_j^c > 0$ then commodity c is produced as an output, if $y_j^\ell < 0$ then commodity ℓ is used as an input. The production plan of firms other than j is $y_{-j} := (y_f)_{f \neq j}$, and

externalities arising from the incompleteness of financial markets.

[6] We do not rely on logarithmic or quadratic perturbations of the payoff functions that are commonly used to establish generic regularity in non-cooperative games, see for instance Harsanyi (1973), Ritzberger (1994), and van Damme (2002).

let $y := (y_j)_{j\in\mathcal{J}}$. The consumption of household h is $x_h := (x_h^1, .., x_h^c, .., x_h^C)$. The consumption of households other than h is $x_{-h} := (x_k)_{k\neq h}$, and let $x := (x_h)_{h\in\mathcal{H}}$.

The production set of firm j is described by a transformation function t_j. The main innovation of this paper is that the transformation function t_j may depend on the production and consumption activities of all other agents. That is, t_j describes both the technology of firm j and the way in which its technology is affected by the activities of the other agents. More precisely, for a given externality (y_{-j}, x), the production set of the firm j is

$$Y_j(y_{-j}, x) := \left\{y_j \in \mathbb{R}^C \colon t_j(y_j, y_{-j}, x) \leq 0\right\}$$

where t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. Let $t := (t_j)_{j\in\mathcal{J}}$.

The preferences of household h are described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

$u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). That is, u_h also describes the way in which the preferences of household h are affected by the activities of the other agents. Let $u := (u_h)_{h\in\mathcal{H}}$.

The endowment of household h is $e_h := (e_h^1, .., e_h^c, .., e_h^C) \in \mathbb{R}^C_{++}$, and let $e := (e_h)_{h\in\mathcal{H}}$. The total resources are $r := \sum_{h\in\mathcal{H}} e_h \in \mathbb{R}^C_{++}$. The share of firm j owned by household h is $s_{jh} \in [0,1]$, and let $s := (s_{jh})_{j\in\mathcal{J}, h\in\mathcal{H}}$. As usual, $\sum_{h\in\mathcal{H}} s_{jh} = 1$ for every firm $j \in \mathcal{J}$. A private ownership economy is $E := ((u, e, s), t)$.

The price of one unit of commodity c is $p^c \in \mathbb{R}_{++}$, and let $p := (p^1, .., p^c, .., p^C)$. Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, denote $w^\setminus := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

2.1 Basic assumptions

In this subsection, we make the following set of assumptions to establish the non-emptiness and the compactness of the equilibrium set.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^2 function.
(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) = 0$.
(3) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly quasi-convex, i.e., for all $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ is positive definite on $\operatorname{Ker} D_{y_j} t_j(y'_j, y_{-j}, x)$.

(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$*,* $D_{y_j} t_j(y'_j, y_{-j}, x) \gg 0$ *for all* $y'_j \in \mathbb{R}^C$*.*

For a given externality, the assumptions on t_j are standard in smooth general equilibrium models. Assumptions 1.1 and 1.4 imply that the production set is a C^2 manifold with boundary of dimension C and its boundary is a C^2 manifold of dimension $C-1$. Assumption 1.2 states that inaction is possible. Consequently, using standard arguments from profit maximization, the wealth of household h associated with his endowment $e_h \in \mathbb{R}^C_{++}$ and his profit shares is strictly positive for every price $p \in \mathbb{R}^C_{++}$. Thus, one deduces the non-emptiness of the interior of the individual budget constraint. Assumption 1.3 implies that the production set is convex. Furthermore, if the profit maximization problem has a solution, then it is unique since the function $t_j(\cdot, y_{-j}, x)$ is continuous and strictly quasi-convex. We remark that t_j is not required to be quasi-convex with respect to all the variables, i.e., we do not require the production set to be convex with respect to the externalities. Assumption 1.4 implies that the function $t_j(\cdot, y_{-j}, x)$ is strictly increasing and then the production set satisfies the classical "free disposal" condition.

Remark 2 *Our analysis holds true if some commodities are not involved in the technological process of some firms. In this case, the previous assumptions must apply only to the sets of commodities that are involved in the productions.*

Given $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, the set of the production plans that are consistent with the externality (x, y) is

$$Y(x, y) := \{\tilde{y} \in \mathbb{R}^{CJ} : t_j(\tilde{y}_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J}\} \tag{1}$$

The next assumption provides a boundedness condition on all the sets of feasible production plans that are consistent with the externalities.[7]

Assumption 3 (Uniform Boundedness) *Given* $r \in \mathbb{R}^C_{++}$*, there exists a bounded set* $C(r) \subseteq \mathbb{R}^{CJ}$ *such that for every* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$*,*

$$Y(x, y) \cap \{\tilde{y} \in \mathbb{R}^{CJ} : \sum_{j \in \mathcal{J}} \tilde{y}_j + r \gg 0\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 3.

Lemma 4 *Given* $r \in \mathbb{R}^C_{++}$*, there exists a bounded set* $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$

[7] Assumption 3 is analogous to several conditions used to establish the existence of an equilibrium with externalities in production sets, that is, the boundedness condition given in Arrow and Hahn (1971), page 134 of Section 2 in Chapter 6; Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001); Assumption P(3) in Mandel (2008); Assumption 3 in del Mercato and Platino (2015.a).

such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$*, the following set is included in* $K(r)$*.*

$$A(x, y; r) := \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \colon \tilde{y} \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} \tilde{x}_h - \sum_{j \in \mathcal{J}} \tilde{y}_j \leq r\}$$

It is well known that the boundedness of the set of feasible allocations is a crucial condition for proving the existence of an equilibrium. However, Assumption 3 is a stronger version of the standard boundedness condition used for economies without externalities, because it guarantees that the set of feasible allocations $A(x, y; r)$ is uniformly bounded with respect to the externalities. It means that the bounded set $K(r)$ that includes the set of feasible allocations is independent of the externality effects. In particular, it implies the boundedness of the set of feasible allocations that are *mutually consistent*, i.e., the set $\mathcal{F}(r) = \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \colon t_j(y_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j \leq r\}$. Notice that in order to prove the existence of an equilibrium it would not be sufficient to assume only the boundedness of the set $\mathcal{F}(r)$.[8]

Assumption 5 *For all* $h \in \mathcal{H}$*,*

(1) *The function* u_h *is continuous in its domain and* C^2 *in the interior of its domain.*
(2) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$*, the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly increasing, i.e.,* $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ *for all* $x'_h \in \mathbb{R}^C_{++}$*.*
(3) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$*, the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly quasi-concave, i.e., for all* $x'_h \in \mathbb{R}^C_{++}$*,* $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ *is negative definite on* $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$*.*
(4) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ *and for every* $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$*,* $\operatorname{cl}_{\mathbb{R}^C} \{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}^C_{++}$*.*

For a given externality, the assumptions on u_h are standard in smooth general equilibrium models. Assumption 5.3 implies that the upper contour sets are convex. Furthermore, if the utility maximization problem has a solution, then it is unique. Note that u_h is not required to be quasi-concave with respect to all the variables, i.e., we do not require the preferences to be convex with respect to the externalities. Assumption 5.4 implies the classical Boundary Condition (BC), that is, the closure of the upper counter sets is included in

[8] In Chapter 6 of Arrow and Hahn (1971), the authors have recognized the need to assume the boundedness of a wider set of feasible production allocations than the ones that are mutually consistent, in order to extend their existence proof to the case of externalities. In Bonnisseau and Médecin (2001), Assumption UB is needed to find the cube to compactify the economy in order to use fixed point arguments. Assumption P(3) in Mandel (2008) or Assumption 3 in del Mercato and Platino (2015.a) are used to show that the set of feasible allocations is bounded once externalities move along a homotopy arc.

$\mathbb{R}^C_{++}$. Notice that in Assumptions 5.1 and 5.4, we allow for consumption externalities x_{-h} on the boundary of the set $\mathbb{R}^{C(H-1)}_{++}$ in order to handle the behavior of u_h as consumption externalities approach the boundary. Assumptions 5.1 and 5.4 imply that BC is still valid whenever consumption externalities converge to zero for some commodities. This property is used to prove properness properties of the equilibrium set, i.e., Lemma 13.

2.2 Competitive equilibrium and equilibrium function

In this subsection, we provide the definition of competitive equilibrium and the notion of equilibrium function.

We use commodity C as the "numeraire good". Given $p^\setminus \in \mathbb{R}^{C-1}_{++}$, let $p := (p^\setminus, 1) \in \mathbb{R}^C_{++}$.

Definition 6 $(x^*, y^*, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for the economy E if for all $j \in \mathcal{J}$, y^*_j solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y^*_{-j}, x^*) \leq 0 \end{aligned} \tag{2}$$

for all $h \in \mathcal{H}$, x^*_h solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^*_{-h}, y^*) \\ &\text{subject to } \; p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j) \end{aligned} \tag{3}$$

and (x^*, y^*) satisfies market clearing conditions, that is

$$\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y^*_j \tag{4}$$

The proof of the following proposition is standard, because in problems (2) and (3) each agent takes as given both the price and the choices of every other agent in the economy.

Proposition 7

(1) From Assumption 1, if y^*_j is a solution to problem (2), then it is unique and it is completely characterized by KKT conditions.[9]

[9] From now on, "KKT conditions" means Karush–Kuhn–Tucker conditions.

(2) *From Assumption 1.2 and Assumption 5, there exists a unique solution* x_h^* *to problem (3) and it is completely characterized by KKT conditions.*
(3) *From Assumption 5.2, household h's budget constraint holds with an equality. Thus, at equilibrium, due to the Walras law, the market clearing condition for commodity C is "redundant". Therefore, one replaces condition (4) with* $\sum_{h\in\mathcal{H}} x_h^{*\backslash} = \sum_{h\in\mathcal{H}} e_h^{\backslash} + \sum_{j\in\mathcal{J}} y_j^{*\backslash}$.

Let $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}^{C-1}_{++}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^{\backslash}) := ((x_h, \lambda_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}}, p^{\backslash})$ where λ_h denotes the Lagrange multiplier associated with household h's budget constraint, and α_j denotes the Lagrange multiplier associated with firm j's technological constraint. We describe the competitive equilibria associated with the economy E using the *equilibrium function* $F_E : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F_E(\xi) := ((F_E^{h.1}(\xi), F_E^{h.2}(\xi))_{h\in\mathcal{H}}, (F_E^{j.1}(\xi), F_E^{j.2}(\xi))_{j\in\mathcal{J}}, F_E^M(\xi)) \tag{5}$$

where $F_E^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F_E^{h.2}(\xi) := -p\cdot(x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh} y_j)$, $F_E^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F_E^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F_E^M(\xi) := \sum_{h\in\mathcal{H}} x_h^{\backslash} - \sum_{j\in\mathcal{J}} y_j^{\backslash} - \sum_{h\in\mathcal{H}} e_h^{\backslash}$.

The vector $\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$ is an *extended equilibrium* for the economy E if and only if $F_E(\xi^*) = 0$. We simply call ξ^* an equilibrium.

Theorem 8 (Existence and compactness) *The equilibrium set* $F_E^{-1}(0)$ *is non-empty and compact.*

del Mercato and Platino (2015.a) provides a proof of Theorem 8.

3 Regular economies and indeterminacy of equilibria

We recall below the formal notion of a regular economy.

Definition 9 *E is a regular economy if F_E is a C^1 mapping and 0 is a regular value of F_E, i.e., for every $\xi^* \in F_E^{-1}(0)$, the differential mapping $D_\xi F_E(\xi^*)$ is onto.*

The fact that 0 is a regular value of the equilibrium function F_E implies that the equilibria of the economy E are determinate. [10]

[10] First, since the equilibrium set $F_E^{-1}(0)$ is non-empty and compact, as a consequence of the Regular Value Theorem, the economy E has a finite number of equilibria. Second, the Implicit Function Theorem implies that, locally, every equi-

In the presence of externalities, the possibility of equilibrium indeterminacy cannot be excluded by making standard assumptions. Indeed, the equilibrium notion given in Definition 6 has the characteristics described in what follows. All the agents take as given both the price and the choice of every other agent in the economy. Given the price and the choices of the other agents, the individual optimal solutions are completely determined. But, for a given price, the equilibrium allocation $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ is a Nash equilibrium, and the problem is that, under standard assumptions, one may get indeterminacy in Nash equilibrium. We illustrate the reason below.

Consider the equilibrium function F_E defined in (5). The economy E remains fixed, thus we omit the subscript E. The price $p^\backslash$ is fixed. Consider all the equilibrium equations except the $L-1$ market clearing conditions, by defining the following function G.

$$G(q) := ((F^{h.1}(q,p^\backslash), F^{h.2}(q,p^\backslash))_{h\in\mathcal{H}}, (F^{j.1}(q,p^\backslash), F^{j.2}(q,p^\backslash))_{j\in\mathcal{J}}) \tag{6}$$

where $q := ((x_h, \lambda_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}})$, so that we write ξ as $(q, p^\backslash)$.

Every $q^* = ((x_h^*, \lambda_h^*)_{h\in\mathcal{H}}, (y_j^*, \alpha_j^*)_{j\in\mathcal{J}})$ such that $G(q^*) = 0$ provides the Nash demands and supplies $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ at the price $p^\backslash$.

Consider the Jacobian matrix of that system,

$$D_qG(q^*) = D_q((F^{h.1}(q^*,p^\backslash), F^{h.2}(q^*,p^\backslash))_{h\in\mathcal{H}}, (F^{j.1}(q^*,p^\backslash), F^{j.2}(q^*,p^\backslash))_{j\in\mathcal{J}}) \tag{7}$$

In the absence of externalities, the Jacobian matrix $D_qG(q^*)$ is nonsingular. This is because the transformation and utility functions are respectively differentiably strictly quasi-convex and strictly quasi-concave in the individual choice. Then, the Implicit Function Theorem implies that, locally, the individual demands and supplies are a C^1 mapping of the price.

It turns out that in the presence of externalities, under standard assumptions, the Jacobian matrix $D_qG(q^*)$ is not necessarily nonsingular. Consequently, the Nash demands and supplies $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ may not even be a well defined mapping of the price. The matrix $D_qG(q^*)$ may fail to be nonsingular due to the presence of some of the following effects:

(1) the second order effects of externalities on utility functions arising from the derivatives of $F^{h.1}(q,p^\backslash)$ with respect to (x_{-h}, y),
(2) the second order effects of externalities on transformation functions arising from the derivatives of $F^{j.1}(q,p^\backslash)$ with respect to (y_{-j}, x),
(3) the first order effects of externalities on transformation functions arising from the derivatives of $F^{j.2}(q,p^\backslash)$ with respect to (y_{-j}, x).

librium is a continuous or differentiable mapping of the parameters describing the economy.

In economies without externalities, all these effects are equal to zero.

One might believe that if the utility and transformation functions are respectively differentiably strictly quasi-convex and strictly quasi-concave with respect to all the variables (i.e., individual choice and externalities), then the matrix $D_q G(q^*)$ is nonsingular. This belief is wrong, since the matrix $D_q G(q^*)$ does not actually involve the whole Hessian matrix of the utility and transformation functions. The matrix $D_q G(q^*)$ involves only a partial block of rows of those Hessian matrices. This is because the first order effects of externalities on utility and transformation functions do not appear in the first order conditions associated with the individual maximization problems, i.e., in the system $G(q) = 0$.

In the case of pure exchange economies with externalities, Bonnisseau and del Mercato (2010) have introduced a specific assumption on the second order external effects on utility and *possibility functions.*[11] We adapt their assumption on utility functions to our framework.[12]

Assumption 10 (Bonnisseau and del Mercato, 2010) *Let* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ *such that the gradients* $(D_{x_h} u_h(x_h, x_{-h}, y))_{h \in \mathcal{H}}$ *are positively collinear. Let* $v \in \mathbb{R}^{CH}$ *such that* $\sum_{h \in \mathcal{H}} v_h = 0$ *and* $v_h \in \operatorname{Ker} D_{x_h} u_h(x_h, x_{-h}, y)$ *for every* $h \in \mathcal{H}$*. Then,* $v_h \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k) < 0$ *whenever* $v_h \neq 0$*.*

Under Assumption 10, the classical result of generic regularity holds true if there are no externalities in the production sets.

In the presence of external effects on the production side, the analogous assumption on transformation functions is not sufficient for establishing generic regularity in the space of households' endowments.[13] This is shown next by means of an example.

Example. There are two commodities. There are no externalities on the consumption side. There is one household, his consumption is $x = (x^1, x^2)$, his initial endowment is $e = (e^1, e^2)$ and his utility function is given by

[11] The *possibility functions* represent general consumption sets with externalities.

[12] Assumption 10 is in the same spirit as the assumption of *diagonally strict concavity* introduced in Rosen (1965) on a weighted sum of the payoff functions of the agents. However, compared with the condition of Theorem 6 of Rosen (1965), Assumption 10 is easier to read, it does not involve any vector of weights and it focuses only on directions $(v_h)_{h \in \mathcal{H}}$ that sum to zero where v_h is orthogonal to the gradient $D_{x_h} u_h(x_h, x_{-h}, y)$.

[13] Further research is open to find some kind of auxiliary assumption on the first order effects of externalities on transformation functions. In Section 4, we choose another approach.

$u(x^1, x^2) = x^1 x^2.$

There are two firms, the production plan of firm j is $y_j = (y_j^1, y_j^2)$. Without loss of generality, for simplicity of exposition, the subscript f denotes the subscript $-j$, so that the production plan of the firm other than j is $y_f = (y_f^1, y_f^2)$. Both firms use commodity 2 to produce commodity 1.

The production technology of firm j is affected by the production plan of the other firm in the following way. Given y_f, the production set of firm j is

$$Y_j(y_f) = \{y_j \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } y_j^1 \leq f_j(y_j^2, y_f)\}$$

where the production function f_j is defined by $f_j(y_j^2, y_f) := 2\phi(y_f)\sqrt{-y_j^2}$ with

$$\phi(y_f) := \begin{cases} \dfrac{y_f^1}{2\sqrt{-y_f^2}} & \text{if} \quad y_f^2 < -\varepsilon^2 \\ \dfrac{y_f^1}{2\varepsilon} & \text{if} \;\; y_f^2 \in [-\varepsilon^2, 0] \end{cases}$$

where $0 < \varepsilon < 1$. The production function f_j is not completely smooth with respect to the externality.[14] But, it goes in the essence of the problem.

For every firm $j = 1, 2$, Assumption 1 is satisfied for every externality $y_f = (y_f^1, y_f^2)$ with $y_f^1 > 0$ and $y_f^2 < -\varepsilon^2$. Therefore, in what follows,

(1) we focus on equilibria where the amounts of output are strictly positive and the amounts of input are strictly lower than $-\varepsilon^2$,
(2) $\phi(y_f)$ is then given by $\dfrac{y_f^1}{2\sqrt{-y_f^2}}$ according to the definition above.

The price of commodity 2 is normalized to 1. Let $(x^*, y_1^*, y_2^*, (p^*, 1))$ be a competitive equilibrium of the economy $e = (e^1, e^2) \in \mathbb{R}^2_{++}$.

For each firm $j = 1, 2$, let α_j be the Lagrange multiplier associated with firm j's technological constraint. The KKT conditions associated with firm j's maximization problem are given by

$$\begin{cases} p^* = \alpha_j, \; 1 = \dfrac{\alpha_j \phi(y_f^*)}{\sqrt{-y_j^2}} \\ y_j^1 = 2\phi(y_f^*)\sqrt{-y_j^2} \end{cases}$$

[14] In order to get a smooth approximation, one might approximate the function ϕ around $-\varepsilon^2$ by a polynomial function.

Consequently, one gets the following equilibrium equations for every $j = 1, 2$,

$$y_j^{*2} = -(p^*)^2[\phi(y_f^*)]^2 \text{ and } y_j^{*1} = 2p^*[\phi(y_f^*)]^2$$

and one easily deduces that

$$y_1^{*1} = y_2^{*1} = -\frac{2y_2^{*2}}{p^*} \text{ and } y_1^{*2} = y_2^{*2} \text{ for any } y_2^{*2} < -\varepsilon^2 \tag{8}$$

Thus, at equilibrium, the aggregate profit is given by $\pi^* := -2y_2^{*2}$ and the optimal solution of the household is

$$x^{*1} = \frac{1}{2p^*}(p^*e^1 + e^2 + \pi^*) \text{ and } x^{*2} = p^*x^{*1} \tag{9}$$

Using the market clearing condition for commodity 2, any bundle $(x^*, y_1^*, y_2^*, (p^*, 1))$ given by (8), (9) and

$$p^* = \frac{e^2 + 6y_2^{*2}}{e^1} \quad \text{with} \quad y_2^{*2} \in \left]-\frac{e^2}{6}, -\varepsilon^2\right[$$

is a competitive equilibrium. Thus, equilibria are indeterminate for all the initial endowments that belong to the open set $\{e = (e^1, e^2) \in \mathbb{R}^2_{++} \colon e^2 > 6\varepsilon^2\}$.

We show that the economy of the example exhibits well behaved second order external effects. For this purpose, we provide below the condition on the transformation functions analogous to Assumption 10.

Let $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $t_j(y_j, y_{-j}, x) = 0$ for every $j \in \mathcal{J}$ and the gradients $(D_{y_j} t_j(y_j, y_{-j}, x))_{j \in \mathcal{J}}$ are positively collinear. Let $z \in \mathbb{R}^{CJ}$ such that $\sum_{j \in \mathcal{J}} z_j = 0$ and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_{-j}, x)$ for every $j \in \mathcal{J}$. Then, $z_j \sum_{f \in \mathcal{J}} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f) > 0$ whenever $z_j \neq 0$.[15]

For every firm j, the transformation function is given by $t_j(y_j, y_f) = y_j^1 - 2\phi(y_f)\sqrt{-y_j^2}$. As above, we focus on production plans for which $\phi(y_f)$ is given by $\dfrac{y_f^1}{2\sqrt{-y_f^2}}$. Take $z = (z_1, z_2) \in \mathbb{R}^4$ such that

$$z_1 + z_2 = 0 \tag{10}$$

[15] If there are no external effects, this condition is satisfied because t_j is differentiably strictly quasi-convex in y_j. Therefore, the sign of the inequality is strictly positive, whereas in Assumption 10 the analogous sign is strictly negative since u_h is differentiably strictly quasi-concave in x_h.

and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_f)$. Then, one gets

$$z_j^1 = -\frac{\phi(y_f)}{\sqrt{-y_j^2}} z_j^2 \tag{11}$$

Using the formulas in (10) and (11), it is an easy matter to verify that the quantity $z_j D^2_{y_j} t_j(y_j, y_f)(z_j) + z_j D^2_{y_f y_j} t_j(y_j, y_f)(z_f)$ is given by

$$\left[\frac{1}{(-y_j^2)} + \frac{1}{\sqrt{-y_j^2}\sqrt{-y_f^2}} - \frac{1}{(-y_f^2)}\right] \frac{\phi(y_f)}{2\sqrt{(-y_j^2)}} (z_j^2)^2 \tag{12}$$

Since $z_j \neq 0$, from (11) we have that $z_j^2 \neq 0$. Since the gradients $(D_{y_j} t_j(y_j, y_f))_{j=1,2}$ are positively collinear, one gets $y_j^1 = y_f^1$. Then $y_j^2 = y_f^2$, because $t_j(y_j, y_f) = 0$ for every $j = 1, 2$. Therefore, the quantity in (12) becomes

$$\frac{y_j^1}{4(-y_j^2)^2} (z_j^2)^2$$

which is strictly positive. ∎

4 The model with firms' endowments

In order to establish generic regularity, we introduce firms' endowments into the model of Section 2. Following Geanakoplos, Magill, Quinzii and Drèze (1990), every firm j is endowed with an *exogenously* given vector of commodities $\eta_j := (\eta_j^1, .., \eta_j^c, .., \eta_j^C) \in \mathbb{R}^C_+$. The description of a private ownership economy now also includes the vector of firms' endowments $\eta := (\eta_j)_{j \in \mathcal{J}}$, so that $E := ((u, e, s), (t, \eta))$.

The **total production plan** y_j **introduced in the market by firm** j is given by

$$y_j := y'_j + \eta_j \tag{13}$$

where y'_j is the **production decision** of firm j according to its technology and the externality (y_{-j}, x), that is, $y'_j \in Y_j(y_{-j}, x)$. Notice that in $Y_j(y_{-j}, x)$, the production externality is the total production plan $y_{-j} = y'_{-j} + \eta_{-j}$ introduced in the market by firms other than j, and not the production decision y'_{-j} of firms other than j. This seems reasonable in many economic applications.[16]

[16] One could consider another model where the production externality is the production decision y'_{-j} of firms other than j, i.e., $y'_j \in Y_j(y'_{-j}, x)$. In this case, despite the presence of firms' endowments, one obtains the same Jacobian matrix $D_q G(q^*)$ as in (7) of Section 3. In other words, firms' endowments do not impact the technologies

Given the price $p \in \mathbb{R}^C_{++}$ and the externality $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the optimal production decision of firm j maximizes the profit $p \cdot y'_j$ over the set $\{y'_j \in \mathbb{R}^C \colon t_j(y'_j, y_{-j}, x) \leq 0\}$. We write this problem in terms of the total production plan y_j introduced in the market by firm j, that is,

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p \cdot y_j \\ &\text{subject to } \; t_j(y_j - \eta_j, y_{-j}, x) \leq 0 \end{aligned} \tag{14}$$

Problem (14) is equivalent to the previous maximization problem, because $y'_j = y_j - \eta_j$ and firm j takes as given its endowment η_j.

The return from firm j generated by the price p, the production decision y'_j and the endowment η_j is given by $p \cdot (y'_j + \eta_j)$. We write this return in terms of the total production plan introduced in the market by firm j, that is, $p \cdot y_j$.

Competitive equilibrium and equilibrium function. The notions given in Subsection 2.2 are adapted according to the notation introduced in (13) and the firm behavior given in (14). More precisely, in Definition 6, we replace $t_j(y_j, y^*_{-j}, x^*)$ with $t_j(y_j - \eta_j, y^*_{-j}, x^*)$. Consequently, in this section, all the components of the equilibrium function F_E are defined as in (5), except for the first order conditions associated with problem (14) which are replaced by

$$F_E^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j - \eta_j, y_{-j}, x) \; \text{ and } \; F_E^{j.2}(\xi) := -t_j(y_j - \eta_j, y_{-j}, x)$$

From now on,

(1) $t = (t_j)_{j \in \mathcal{J}}$ is fixed and satisfies Assumption 1 and Assumption 3 for every $r \in \mathbb{R}^C_{++}$,

(2) $u = (u_h)_{h \in \mathcal{H}}$ is fixed and satisfies Assumptions 5 and 10,

(3) a private ownership economy is completely parametrized by the endowments of households and firms (e, η) in the open set $\Lambda := \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}_{++}$,

(4) we simply denote $F_{e,\eta}$ the equilibrium function, that is, $F_{e,\eta}(\xi) := F_E(\xi)$.

Note that the non-emptiness and the compactness of the equilibrium set $F^{-1}_{e,\eta}(0)$ does not immediately follow from Theorem 8. However, under the assumptions above, the result analogous to Theorem 8 holds true, see Section 5 for more details.

Lemma 11 *For every economy $(e, \eta) \in \Lambda$, the equilibrium set $F^{-1}_{e,\eta}(0)$ is non-empty and compact.*

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and the marginal productivities of the other firms. Consequently, introducing firms' endowments does not lead to the result of generic regularity.

4.1 The regularity result

In this subsection, we provide and we prove our result of generic regularity.

Theorem 12 *The set Λ^* of regular economies (e, η) is an open and full measure subset of Λ.*

In order to prove this theorem, we introduce the following notations and we provide two auxiliary lemmas, namely Lemmas 13 and 14. These lemmas are proved in Section 5.

Assumptions 1.1 and 5.1 imply that the equilibrium function $F_{e,\eta}$ is C^1 everywhere. By Definition 9, the economy (e, η) is regular if

$$\forall\, \xi^* \in F_{e,\eta}^{-1}(0),\ \operatorname{rank} D_\xi F_{e,\eta}(\xi^*) = \dim \Xi$$

Define the set $G := \{(\xi, e, \eta) \in F^{-1}(0) :\ \operatorname{rank} D_\xi F(\xi, e, \eta) < \dim \Xi\}$, where the function $F : \Xi \times \Lambda \to \mathbb{R}^{\dim \Xi}$ is defined by

$$F(\xi, e, \eta) := F_{e,\eta}(\xi)$$

and denote Π the restriction to $F^{-1}(0)$ of the projection of $\Xi \times \Lambda$ onto Λ, i.e.

$$\Pi : (\xi, e, \eta) \in F^{-1}(0) \to \Pi(\xi, e, \eta) := (e, \eta) \in \Lambda$$

Write the set Λ^* given in Theorem 12 as $\Lambda^* = \Lambda \setminus \Pi(G)$. In order to prove Theorem 12, it is enough to show that $\Pi(G)$ is a closed set in Λ and $\Pi(G)$ has measure zero.

We first claim that $\Pi(G)$ is a closed set in Λ. From Assumptions 1.1 and 5.1, F and $D_\xi F$ are continuous on $\Xi \times \Lambda$. The set G is characterized by the fact that the determinant of all the square submatrices of $D_\xi F(\xi, e, \eta)$ of dimension $\dim \Xi$ is equal to zero. Since the determinant is a continuous function and $D_\xi F$ is continuous on $F^{-1}(0)$, the set G is closed in $F^{-1}(0)$. Thus, $\Pi(G)$ is closed since the projection Π is proper. The properness of the projection Π is provided in the following lemma.

Lemma 13 *The projection $\Pi : F^{-1}(0) \to \Lambda$ is a proper function.*

We then show that $\Pi(G)$ has measure zero in Λ. For this purpose, we need the following lemma.

Lemma 14 *0 is a regular value of F.*

Lemma 14 and the Transversality Theorem imply that there is a full measure subset Ω of Λ such that for each $(e, \eta) \in \Omega$ and for each ξ^* such that

$F(\xi^*, e, \eta) = 0$, rank $D_\xi F(\xi^*, e, \eta) = \dim \Xi$. Now, let $(e, \eta) \in \Pi(G)$, then there exists $\xi \in \Xi$ such that $F(\xi, e, \eta) = 0$ and rank $D_\xi F(\xi, e, \eta) < \dim \Xi$. So, $(e, \eta) \notin \Omega$. This prove that $\Pi(G)$ is included in the complementary of Ω, that is, in $\Omega^C := \Lambda \setminus \Omega$. Since Ω^C has zero measure, so too does $\Pi(G)$. Thus, the set of regular economies Λ^* has full measure since $\Omega \subseteq \Lambda^*$, which completes the proof of Theorem 12.

Finally, using the Regular Value Theorem and the Implicit Function Theorem, one deduces the following lemma from Lemma 11 and Theorem 12.

Proposition 15 *For each $(e, \eta) \in \Lambda^*$,*

(1) the equilibrium set associated with the economy (e, η) is a non-empty finite set, i.e.,

$$\exists\, r \in \mathbb{N} \setminus \{0\} : \; F_{e,\eta}^{-1}(0) = \{\xi^1, ..., \xi^r\}$$

(2) there exists an open neighborhood I of (e, η) in Λ^*, and for each $i = 1, \ldots, r$ there exist an open neighborhood U_i of ξ^i in Ξ and a C^1 function $g_i : I \to U_i$ such that

(a) $U_i \cap U_k = \emptyset$ if $i \neq k$,

(b) $g_i(e, \eta) = \xi^i$ and $\xi' \in F_{e',\eta'}^{-1}(0)$ holds for $(\xi', e', \eta') \in U_i \times I$ if and only if $\xi' = g_i(e', \eta')$.

5 Appendix

In this section, we prove all the lemmas stated in Section 4. The following notation helps in writing the proofs,

$$t_j(y_j - \eta_j, y_{-j}, x) = (t_j \circ g_j)(y_j, y_{-j}, x, \eta_j) \tag{15}$$

where the mapping g_j is defined by $g_j(y_j, y_{-j}, x, \eta_j) := (y_j - \eta_j, y_{-j}, x)$.

Proof of Lemma 11. In order to prove this lemma, one must pay attention to Assumption 1.2 and Lemma 4.

First, notice that for any given endowment η_j, the function given in (15) satisfies all the assumptions given in Assumption 1, except Assumption 1.2. However, the total production plan $y_j = \eta_j$ acts for $(t_j \circ g_j)(y_j, y_{-j}, x, \eta_j)$ as $y_j = 0$ acts for $t_j(y_j, y_{-j}, x)$. Indeed, Assumption 1.2 implies that $(t_j \circ g_j)(\eta_j, y_{-j}, x, \eta_j) = 0$, and then using standard maximization arguments, one gets $p \cdot y_j^* \geq p \cdot \eta_j$ if y_j^* solves problem (14). Consequently, the individual wealth $p \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y_j^*)$ of household h is greater than $p \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} \eta_j)$ which is strictly positive since $(e, \eta) \gg 0$.

Second, one needs to establish the result analogous to Lemma 4 for the functions $(t_j \circ g_j)_{j\in\mathcal{J}}$ and the vector $r = \sum_{h\in\mathcal{H}} e_h$. For this purpose, the set $K(r)$ must be replaced with the set $\left(K(r + \sum_{j\in\mathcal{J}} \eta_j)\right) + \widetilde{\eta}$, where $\widetilde{\eta} := (0, \eta)$ and the set $K(r + \sum_{j\in\mathcal{J}} \eta_j)$ is provided by Lemma 4 for the functions $(t_j)_{j\in\mathcal{J}}$ and the vector $r + \sum_{j\in\mathcal{J}} \eta_j$. The proof of this lemma is then established by adapting the proof of Theorem 8. ■

Proof of Lemma 13. The main difficulty of adapting classical arguments arises from the presence of consumption externalities in the preferences. More precisely, one must pay attention to equilibrium consumption allocations that may converge on the boundary of the positive orthant. However, Assumptions 5.1 and 5.4 imply that the limit point of the household h's equilibrium consumptions is strictly positive, even when the limit point of the equilibrium consumption of everybody else is on the boundary. Since the same argument applies for every household h, one concludes that the limit point of the equilibrium consumption allocations has to be strictly positive. A detailed proof of the lemma is given in del Mercato and Platino (2015.b). ■

Proof of Lemma 14. We show that for each $(\xi^*, e^*, \eta^*) \in F^{-1}(0)$, the Jacobian matrix $D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*)$ has full row rank. It is enough to prove that $\Delta D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*) = 0$ implies $\Delta = 0$, where

$$\Delta := ((\Delta x_h, \Delta\lambda_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^{\setminus}) \in \mathbb{R}^{H(C+1)} \times \mathbb{R}^{J(C+1)} \times \mathbb{R}^{C-1}$$

The system $\Delta D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*) = 0$ is written in detail below.[17]

[17] I_{C-1} is the identity matrix of size $(C-1)$, and $[I_{C-1}|0]$ is the $(C-1)\times C$ matrix where the last column is the vector 0.

We remind that $(t_j \circ g_j)$ is defined in (15).

$$
\begin{cases}
(1) \displaystyle\sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*) - \Delta\lambda_k p^* - \sum_{j\in\mathcal{J}} \alpha^*_j \Delta y_j D^2_{x_k y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) + \\
- \displaystyle\sum_{j\in\mathcal{J}} \Delta\alpha_j D_{x_k}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) + \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, k \in \mathcal{H} \\
(2) \ -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\
(3) \displaystyle\sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x^*_h, x^*_{-h}, y^*) + \sum_{h\in\mathcal{H}} \Delta\lambda_h s_{fh} p^* - \sum_{j\in\mathcal{J}} \alpha^*_j \Delta y_j D^2_{y_f y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) + \\
- \displaystyle\sum_{j\in\mathcal{J}} \Delta\alpha_j D_{y_f}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) - \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, f \in \mathcal{J} \\
(4) \ -\Delta y_j \cdot D_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = 0, \ \forall\, j \in \mathcal{J} \\
(5) \ \Delta\lambda_h p^* - \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, h \in \mathcal{H} \\
(6) \ -\displaystyle\sum_{h\in\mathcal{H}} \lambda^*_h \Delta x^{\setminus}_h - \sum_{h\in\mathcal{H}} \Delta\lambda_h (x^{*\setminus}_h - e^{*\setminus}_h - \sum_{j\in\mathcal{J}} s_{jh} y^{*\setminus}_j) + \sum_{j\in\mathcal{J}} \Delta y^{\setminus}_j = 0 \\
(7) \ -\alpha^*_j \Delta y_j D^2_{\eta_j y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) - \Delta\alpha_j D_{\eta_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = 0, \ \forall\, j \in \mathcal{J}
\end{cases}
$$

Using the definition of $(t_j \circ g_j)$, we have that

$$D_{\eta_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = -D_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j)$$

and

$$D^2_{\eta_j y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = -D^2_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j)$$

Therefore, for every $j \in \mathcal{J}$ equation (7) becomes

$$\alpha^*_j \Delta y_j D^2_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) + \Delta\alpha_j D_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = 0 \quad (16)$$

Multiplying the equation above by Δy_j and using equation (4), since $\alpha^*_j > 0$ one gets

$$\Delta y_j D^2_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j)(\Delta y_j) = 0$$

Then, equation (4) and Assumption 1.3 imply that $\Delta y_j = 0$.

Therefore, equation (16) becomes

$$\Delta\alpha_j D_{y_j}(t_j \circ g_j)(y^*_j, y^*_{-j}, x^*, \eta^*_j) = 0$$

and then $\Delta\alpha_j = 0$ by Assumption 1.4. Thus, we get $(\Delta y_j, \Delta\alpha_j) = 0$ for every $j \in \mathcal{J}$.

Since $p^{*C} = 1$, from equation (5) one gets $\Delta\lambda_h = 0$ for all $h \in \mathcal{H}$, and then

$\Delta p^{\setminus} = 0$. Thus, the above system becomes

$$\begin{cases} (1) \quad \sum_{h \in \mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*) = 0, \ \forall \, k \in \mathcal{H} \\ (2) \quad -\Delta x_h \cdot p^* = 0, \ \forall \, h \in \mathcal{H} \\ (3) \quad \sum_{h \in \mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x^*_h, x^*_{-h}, y^*) = 0, \ \forall \, f \in \mathcal{J} \\ (6) \quad -\sum_{h \in \mathcal{H}} \lambda^*_h \Delta x^{\setminus}_h = 0 \end{cases} \tag{17}$$

Since $F^{h.1}(\xi^*, e^*, \eta^*) = 0$, equation (2) in system (17) implies that for every $h \in \mathcal{H}$, $\Delta x_h \in \operatorname{Ker} D_{x_h} u_h(x^*_h, x^*_{-h}, y^*)$. Now, for every $h \in \mathcal{H}$ define $v_h := \lambda^*_h \Delta x_h$. The vector $(x^*_h, v_h)_{h \in \mathcal{H}}$ satisfies the following conditions.

$$(v_h)_{h \in \mathcal{H}} \in \prod_{h \in \mathcal{H}} \operatorname{Ker} D_{x_h} u_h(x^*_h, x^*_{-h}, y^*) \text{ and } \sum_{h \in \mathcal{H}} v_h = 0 \tag{18}$$

where the last equality comes from equation (6) in system (17). Multiplying both sides of equation (1) in system (17) by v_k, and using the definition of v_h, one gets $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda^*_h} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*)(v_k) = 0$ for every $k \in \mathcal{H}$. Summing up $k \in \mathcal{H}$, we obtain $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda^*_h} \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*)(v_k) = 0$.

Note that the gradients $(D_{x_h} u_h(x^*_h, x^*_{-h}, y^*))_{h \in \mathcal{H}}$ are positively proportional because $F^{h.1}(\xi^*, e^*, \eta^*) = 0$ for every $h \in \mathcal{H}$. Therefore, from (18) all the conditions of Assumption 10 are satisfied, and then $v_h = 0$ for each $h \in \mathcal{H}$ since $\lambda^*_h > 0$. Thus, we get $\Delta x_h = 0$ for all $h \in \mathcal{H}$. Consequently, one has $\Delta = 0$ which completes the proof. ∎

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