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Inner Regularizations and Viscosity Solutions for Pessimistic Bilevel Optimization Problems

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Abstract
Pessimistic bilevel optimization problems are not guaranteed to have a solution even when restricted classes of data are involved. Thus, we propose a concept of viscosity solution which can satisfactory obviate the lack of optimal solutions since it allows to achieve in appropriate conditions the security value. Differently from the viscosity solution concept for optimization problems, introduced by H. Attouch in 1996 and defined through a viscosity function that aims to regularize the objective function, viscosity solutions for pessimistic bilevel optimization problems are defined through regularization families of the solutions map to the lower level optimization These families are termed "inner regularizations" since they approach the optimal solutions map from the inside. First, we investigate several classical regularizations of parametric minimum problems giving sufficient conditions for getting inner regularizations; then, we establish existence results for the corresponding viscosity solutions.

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1 Introduction

In this paper, we are concerned with pessimistic bilevel optimization problems, $(PBOP)$, in Banach spaces, also called generalized Stackelberg or weak Stackelberg problems, [1, 2, 3, 4, 5, 6, 7, 8, 9]. The lack of solutions to such a problem, highlighted for example in [4, 5, 6], is a consequence of the possible lack of lower semicontinuity of the solutions map $\mathcal{M}$ for the lower level problem. In spite of this inherent difficulty, due to relevant motivations in Game Theory and its applications [2, 5], the research on pessimistic bilevel optimization problems has been intensified and several papers have been devoted to different aspects of these problems: optimality conditions [9, 10], reformulations [11], numerical methods [12]....

Very recently, in the setting of pessimistic bilevel problems (MinSup problems) we investigated two topics:

1) a method to approach the security value ω [13, Section 1] in the presence or not of perturbations,

2) a concept of viscosity solution which can take the place of the exact solutions if they do not exist.

Point 1) has been developed in [13] for pessimistic bilevel problems with constraints described by the solutions to a quasi-variational inequality in $\mathbb{R}^k$.

Point 2) has been developed in [14] when the constraints are described by a Nash equilibrium problem in $\mathbb{R}^k$, and in [15] when the constraints are described by a quasi-equilibrium problem in Banach spaces.

In this paper, which aims to be a continuation of [6], we investigate several possibilities of obviating the lack of lower semicontinuity of the solutions map $\mathcal{M}$ to the lower level problem, and we define different concepts of viscosity solution for $(PBOP)$ by the aid of suitable regularizations. Then, we investigate existence results for each of them in Banach spaces.

The definition of viscosity solution given in [15] is based on the concept of *inner regularization* for a parametric quasi-equilibrium problem. Here, we consider pessimistic bilevel problems with constraints described by optimization problems. First, for a family of parametric minimum problems, we define inner regularizations requiring, among other conditions, the lower semicontinuity of the approximation maps of the lower level problem. Then, we suggest several possible regularizations of the lower level, two of which in line with [4, 6] and one inspired by [19], but it turns out that the classical strict ε-solutions map gives rise to an inner regularization family under the less restrictive assumptions, having also the advantage of satisfying a Slater constraint qualification condition that could be useful to produce necessary conditions for pessimistic bilevel optimization problems.

The rest of the paper is divided in five sections. In Section 2, first we present the problem statement, main definitions and notations, then we consider classical approximations of minimum problems in order to define inner regularizations of the lower level. In Section 3, other possible inner regularizations of the lower level problem are investigated; in Section 4, existence results of viscosity solutions are presented, whereas Section 5 is devoted to illustrative examples and counterexamples and Section 6 to conclusive remarks.

2 Preliminaries and inner regularizations

2.1 Statement of the problem, notations and definitions

We define a Pessimistic Bilevel Optimization Problem as follows:

$$(PBOP) \quad \text{find } \widetilde{t} \in T' \text{ such that } \sup_{y \in \mathcal{M}(\widetilde{t})} L(\widetilde{t}, y) \;=\; \inf_{t \in T'} \sup_{y \in \mathcal{M}(t)} L(t, y) \;=\; \omega,$$

where: (T, τ) is a Hausdorff topogical space, Y is a Banach space, $T' \subseteq T$ and $Y' \subseteq Y$ are nonempty sets, $F : T' \times Y' \to \mathbb{R}$ and $L : T' \times Y' \to \mathbb{R}$ are real-valued functions, $\mathcal{K} : t \in T' \to \mathcal{K}(t) \subseteq Y'$ is a set-valued map with nonempty values from T' to Y' and $\mathcal{M}(t)$ denotes, for any $t \in T'$, the optimal solutions set of the parametric optimization problem

$$P(t) \qquad \text{find } \widetilde{y} \in \mathcal{K}(t) \text{ such that } F(t, \widetilde{y}) \;=\; \inf_{y \in \mathcal{K}(t)} F(t, y).$$

The number ω is called the security value of $(PBOP)$. Everywhere in the paper, we indicate by s and w the strong and the weak topology on Y respectively, by $cl(H)$ and $int(H)$ the closure and the interior of a subset H of Y respectively, and by $(\varepsilon_n)_n$ a sequence of positive real numbers which decreases to zero. We use the notions of sequential lower semicontinuity, sequential closedness and sequential subcontinuity for set-valued maps (see, for example, Section 1 in [15]) omitting the term "sequential" and we do similarly for continuity properties of real-valued functions. Moreover, we assume that for every $t \in T'$ the set $\mathcal{M}(t)$ is nonempty, that is the optimization problem $P(t)$ has at least a solution, and that the objective function $F(t, \cdot)$ is bounded from below for every $t \in T'$. This last condition is made purely for the sake of simplicity in defining different concepts of approximate solutions to $P(t)$.

Definition 2.1 *A family of maps $\mathfrak{T} = \{\mathcal{T}^\varepsilon, \varepsilon > 0\}$, where $\mathcal{T}^\varepsilon : t \in T' \to \mathcal{T}^\varepsilon(t) \subseteq Y'$, is an inner regularization for the family of minimum problems $\{P(t), t \in T'\}$ if the conditions below are satisfied:*

- $R_1)$ $\mathcal{M}(t) \subseteq \mathcal{T}^\varepsilon(t)$ *for every $t \in T'$ and $\varepsilon > 0$;*
- $R_2)$ *for any $t \in T'$, any $(t_n)_n$ τ-converging to t in T' and any sequence $(\varepsilon_n)_n$, one has*

$$w - \limsup_n \mathcal{T}^{\varepsilon_n}(t_n) \subseteq \mathcal{M}(t);$$

- $R_3)$ $\mathcal{T}^\varepsilon$ *is a (τ, s)-lower semicontinuous set-valued map on T', for every $\varepsilon > 0$.*

In line with Mosco convergence for functions and sets [16, 3.3] and in order to avoid too much restrictive assumptions in infinite dimensional spaces, we consider the weak topology in condition R_2) and the strong topology in condition R_3).

Then, inspired by [17], we give the following definition:

Definition 2.2 *Let $\mathfrak{T}$ be an inner regularization for the family $\{P(t), t \in T'\}$. A point $\widetilde{t} \in T'$ is said to be a $\mathfrak{T}$-viscosity solution for the pessimistic bilevel optimization problem $(PBOP)$ if for every $(\varepsilon_n)_n$ there exists $(t_{\varepsilon_n})_n$, $t_{\varepsilon_n} \in T'$ for any $n \in \mathbb{N}$, such that:*

- $V_1)$ $\widetilde{t} \in cl^\tau_{seq}\{t_{\varepsilon_n}, n \in \mathbb{N}\}$, *i.e. there exists a subsequence $(t_{\varepsilon_{n'}})_{n'}$ τ-converging to $\widetilde{t}$;*
- $V_2)$ *for any $n \in \mathbb{N}$, t_{ε_n} is a minimum point for the marginal function*

$$i_{\varepsilon_n} : t \in T' \to i_{\varepsilon_n}(t) = \sup_{y \in \mathcal{T}^{\varepsilon_n}(t)} L(t, y); \tag{1}$$

i.e. $\sup_{y \in \mathcal{T}^{\varepsilon_n}(t_{\varepsilon_n})} L(t_{\varepsilon_n}, y) = \min_{t \in T'} \sup_{y \in \mathcal{T}^{\varepsilon_n}(t)} L(t, y)$;

V_3) *the sequence* $(i_{\varepsilon_n}(t_{\varepsilon_n}))_n$ *converges towards the security value* ω, *i.e.*

$$\lim_n \sup_{y \in \mathcal{T}^{\varepsilon_n}(t_{\varepsilon_n})} L(t_{\varepsilon_n}, y) \;=\; \inf_{t \in T'} \sup_{y \in \mathcal{M}(t)} L(t, y) \;=\; \omega.$$

We show in Proposition 4.2 that if the sequence of functions $(i_{\varepsilon_n})_n$ τ-epiconverges over the set T' [16] towards the function C defined by $C(t) = cl^{\tau}_{seq} \sup_{y \in \mathcal{M}(t)} L(t, y)$, the sequentially τ-lower semicontinuous regularization of $\sup_{y \in \mathcal{M}(\cdot)} L(\cdot, y)$, then any viscosity solution for the problem $(PBOP)$ turns out to be a minimum point for C. Therefore, a viscosity solution does not depend on the class used to find it and one can deal with the most suitable inner regularizations for the family $\{P(t), t \in T'\}$.

2.2 Inner regularizations

For any given positive real number ε, we consider the maps:

$$\mathcal{M}^{\varepsilon} : t \in T' \to \mathcal{M}^{\varepsilon}(t) = \left\{ y \in \mathcal{K}(t) : F(t, y) \leq \inf_{y' \in \mathcal{K}(t)} F(t, y') + \varepsilon \right\},$$

$$\widetilde{\mathcal{M}}^{\varepsilon} : t \in T' \to \widetilde{\mathcal{M}}^{\varepsilon}(t) = \left\{ y \in \mathcal{K}(t) : F(t, y) < \inf_{y' \in \mathcal{K}(t)} F(t, y') + \varepsilon \right\},$$

$$\widetilde{\mathcal{D}}^{\varepsilon} : t \in T' \to \widetilde{\mathcal{D}}^{\varepsilon}(t) = \{ y \in \mathcal{K}(t) : d(y, \mathcal{M}(t)) < \varepsilon \}$$

and the families they describe

$$\mathfrak{M} = \{\mathcal{M}^{\varepsilon}, \varepsilon > 0\} \qquad \widetilde{\mathfrak{M}} = \left\{\widetilde{\mathcal{M}}^{\varepsilon}, \varepsilon > 0\right\} \qquad \widetilde{\mathfrak{D}} = \left\{\widetilde{\mathcal{D}}^{\varepsilon}, \varepsilon > 0\right\}.$$

We point out that the maps $\widetilde{\mathcal{M}}^{\varepsilon}$ and $\mathcal{M}^{\varepsilon}$ have been fruitfully used in a bilevel setting [4, 6, 18] whereas the use of the maps $\widetilde{\mathcal{D}}^{\varepsilon}$ is new and it has been inspired by results in [19].

Although it is clear that

$$\mathcal{M}(t) \subseteq \widetilde{\mathcal{D}}^{\varepsilon}(t) \;\text{ and }\; \mathcal{M}(t) \subseteq \widetilde{\mathcal{M}}^{\varepsilon}(t) \subseteq \mathcal{M}^{\varepsilon}(t),$$

inclusion relations between $\mathcal{M}^{\varepsilon}$ and $\widetilde{\mathcal{D}}^{\varepsilon}$, as well between $\widetilde{\mathcal{M}}^{\varepsilon}$ and $\widetilde{\mathcal{D}}^{\varepsilon}$, may change depending on the function $F(t, \cdot)$.

Example 2.1 Assume that $T = Y = \mathbb{R}$, $\mathcal{K}(t) = \mathbb{R}$ and that $F(t, 0) = -1$, $F(t, y) = |y|$ if $y \neq 0$ and $t \in T$. Then, for $\varepsilon < 1$, $\mathcal{M}(t) = \mathcal{M}^\varepsilon(t) = \{0\}$ and $\widetilde{\mathcal{D}}^\varepsilon(t) =]-\varepsilon, \varepsilon[$, so, in this case, $\mathcal{M}^\varepsilon(t) \subset \widetilde{\mathcal{D}}^\varepsilon(t)$. In the same assumptions, if $F(t, y) = |y|$ for every $y \in \mathbb{R}$ and $t \in T$, then

$$\widetilde{\mathcal{M}}^\varepsilon(t) \;=\; \widetilde{\mathcal{D}}^\varepsilon(t) \;=\,]-\varepsilon, \varepsilon[\;\subset\; [-\varepsilon, \varepsilon] \;=\; \mathcal{M}^\varepsilon(t),$$

whereas if $F(t, y) = (y-1)^2$ for every $y \in \mathbb{R}$ and $t \in T$, then $\widetilde{\mathcal{M}}^\varepsilon(t) =]1-\sqrt{\varepsilon}, 1+\sqrt{\varepsilon}[$ and $\widetilde{\mathcal{D}}^\varepsilon(t) =]1-\varepsilon, 1+\varepsilon[$, so that $\widetilde{\mathcal{D}}^\varepsilon(t) \subset \widetilde{\mathcal{M}}^\varepsilon(t) \subset \mathcal{M}^\varepsilon(t) = [1-\sqrt{\varepsilon}, 1+\sqrt{\varepsilon}]$.

Nevertheless, we can prove the condition below.

Proposition 2.1 *Let $t \in T'$. If $F(t, \cdot)$ is w-lower semicontinuous over Y' and $\mathcal{K}(t)$ is sequentially weakly compact, then*

$$\forall\, \varepsilon > 0\; \exists\, \eta > 0 \text{ such that } \mathcal{M}^\eta(t) \;\subseteq\; \widetilde{\mathcal{D}}^\varepsilon(t). \tag{2}$$

The same holds if $F(t, \cdot)$ is w-lower semicontinuous and sequentially w-coercive over Y' [15] and $\mathcal{K}(t)$ is w-closed.

Proof

If we suppose that (2) is not true, there exists a positive number ε and a sequence $(y_n)_n$, $y_n \in \mathcal{K}(t)$, such that $F(t, y_n) \leq \inf\limits_{y' \in \mathcal{K}(t)} F(t, y') + 1/n$ and $d(y_n, \mathcal{M}(t)) \geq \varepsilon$ for every $n \in \mathbb{N}$. A subsequence of $(y_n)_n$ has to w-converge to $\widetilde{y} \in \mathcal{K}(t)$ and $F(t, \widetilde{y}) \leq \inf\limits_{y' \in \mathcal{K}(t)} F(t, y')$. Then $\widetilde{y} \in \mathcal{M}(t)$ and we get a contradiction. The proof in the second case is analogous to the previous one so it is omitted.

First, we focus our attention on property R_2) of Definition 2.1.

Proposition 2.2 *Assume that:*

i) the set-valued map $\mathcal{K}$ is (τ, s)-lower semicontinuous and (τ, w)-closed over T';

ii) the function F is $(\tau \times w)$-lower semicontinuous and $(\tau \times s)$-upper semicontinuous over $T' \times Y'$.

Then, the families $\widetilde{\mathfrak{M}}$ and $\mathfrak{M}$ satisfy condition R_2).

Proof Let $(t_n)_n$, $t_n \in T'$, τ-converging to $\widetilde{t} \in T'$ and let $(y_n)_n$ w-converging to $\widetilde{y}$ in Y' such that $y_n \in \mathcal{M}^{\varepsilon_n}(t_n)$ for any $n \in \mathbb{N}$. Since $F(t_n, y_n) \leq \inf_{y \in \mathcal{K}(t_n)} F(t_n, y) + \varepsilon_n$ and $y_n \in \mathcal{K}(t_n)$, (τ, w)-closedness of $\mathcal{K}$ and $(\tau \times w)$-lower semicontinuity of F imply that $\widetilde{y} \in \mathcal{K}(\widetilde{t})$ and

$$F(\widetilde{t}, \widetilde{y}) \leq \liminf_n F(t_n, y_n) \leq \limsup_n \inf_{y \in \mathcal{K}(t_n)} F(t_n, y) = \limsup_n \mathfrak{m}(t_n),$$

where the function $\mathfrak{m}$ is the marginal function defined by

$$\mathfrak{m} : t \in T' \to \inf_{y' \in \mathcal{K}(t)} F(t, y') \in \mathbb{R}. \tag{3}$$

Then, the τ-upper semicontinuity of $\mathfrak{m}$, which is ensured by assumption $ii)$ (see, for example, [20]), implies that $F(\widetilde{t}, \widetilde{y}) \leq \mathfrak{m}(\widetilde{t})$, so that $\widetilde{y} \in \mathcal{M}(\widetilde{t})$.

The result holds true also for the family $\widetilde{\mathfrak{M}}$ since $\widetilde{\mathcal{M}}^{\varepsilon}(t) \subseteq \mathcal{M}^{\varepsilon}(t)$ for any t and any $\varepsilon > 0$. $\square$

Proposition 2.3 *Assume that the assumptions of Proposition 2.2 and the following hold:*

$i)$ the set-valued map $\mathcal{K}$ is (τ, w)-subcontinuous over T'.

Then, the family $\widetilde{\mathfrak{D}}$ satisfies condition R_2).

Proof Let $y_n \in \widetilde{\mathcal{D}}^{\varepsilon_n}(t_n)$ and let $(y_n)_n$ be w-converging to $\widetilde{y}$. Since the map $\mathcal{K}$ is (τ, w)-subcontinuous, then also the map $\mathcal{M}$ is (τ, w)-subcontinuous over T' and we can apply the second part of Lemma 2.2 in [21] with $S_n = \mathcal{M}$ for all $n \in \mathbb{N}$. Then, we get $d(\widetilde{y}, \mathcal{M}(\widetilde{t})) \leq \liminf_n d(y_n, \mathcal{M}(t_n)) \leq 0$ and $\widetilde{y} \in \mathcal{M}(\widetilde{t})$ since $\mathcal{M}(t)$ is w-closed by the assumptions of Proposition 2.2. $\square$

Now, we investigate condition R_3), that is the (τ, s)-lower semicontinuity for these maps for any given $\varepsilon > 0$.

Proposition 2.4 [6] *If assumptions of Proposition 2.2 and the following hold:*

$i)$ the set-valued map $\mathcal{K}$ is (τ, w)-subcontinuous over T';

then the map $\widetilde{\mathcal{M}}^{\varepsilon}$ is (τ, s)-lower semicontinuous over T', i.e. for any $t \in T'$ and any $(t_n)_n$ τ-converging to t in T' we have

$$\widetilde{\mathcal{M}}^{\varepsilon}(t) \;\subseteq\; s - \liminf_n \widetilde{\mathcal{M}}^{\varepsilon}(t_n). \tag{4}$$

Proof

Let $t \in T'$ and let $(t_n)_n$ be a sequence τ-converging to t in T'. Consider $y \in \widetilde{\mathcal{M}}^\varepsilon(t)$, that is $y \in \mathcal{K}(t)$ and $F(t,y) < \inf_{y' \in \mathcal{K}(t)} F(t,y') + \varepsilon$. There exists a sequence $(y_n)_n$ s-converging to y such that $y_n \in \mathcal{K}(t_n)$ for n sufficiently large and, due to the (τ,s)-upper semicontinuity of F, we get $\limsup_n F(t_n,y_n) \leq F(t,y) < \inf_{y' \in \mathcal{K}(t)} F(t,y') + \varepsilon$. On the other hand, the (τ,w)-lower semicontinuity of F and the properties of the map $\mathcal{K}$ imply that the marginal function $\mathfrak{m}$, defined in (3), is τ-lower semicontinuous over T' (see, for example, [20]), so that

$$\limsup_n F(t_n,y_n) < \liminf_n \mathfrak{m}(t_n) + \varepsilon.$$

Then, $F(t_n,y_n) < \inf_{y' \in \mathcal{K}(t_n)} F(t_n,y') + \varepsilon$, i.e. $y_n \in \widetilde{\mathcal{M}}^\varepsilon(t_n)$, for n sufficiently large. □

Proposition 2.5 [6] *If the assumptions of Proposition 2.4 and the following hold:*

i) the map $\mathcal{K}$ is convex-valued over T';

ii) $F(t,\cdot)$ is strictly quasiconvex on $\mathcal{K}(t)$ [22], for every $t \in T'$;

then the map $\mathcal{M}^\varepsilon$ is (τ,s)-lower semicontinuous over T'.

Proof

Assumptions *i)* and *ii)* imply that

$$\mathcal{M}^\varepsilon(t) \;\subseteq\; cl(\widetilde{\mathcal{M}}^\varepsilon(t)),$$

for any $t \in T'$ (see, for example, [6]). So, from Proposition 2.4 we infer that

$$\mathcal{M}^\varepsilon(t) \;\subseteq cl(s\text{-}\liminf_n \widetilde{\mathcal{M}}^\varepsilon(t_n)) = s\text{-}\liminf_n \widetilde{\mathcal{M}}^\varepsilon(t_n) \subseteq s\text{-}\liminf_n \mathcal{M}^\varepsilon(t_n)$$

for every $(t_n)_n$ τ-converging to t in T' and this completes the proof. □

For what concerning the lower semicontinuity property of the map $\widetilde{\mathcal{D}}^\varepsilon$, we can show that it may not be satisfied even in favourable conditions.

Example 2.2 Let $T = Y = \mathbb{R}$, $T' = Y' = [0,1]$, $\mathcal{K}(t) = [0,1]$ for any $t \in T'$ and let $F(t,y) = ty$ for every $(t,y) \in [0,1]^2$. It is easy to check that $\mathcal{M}(0) = [0,1]$ and $\mathcal{M}(t) = \{0\}$ if $t \in]0,1]$, so that $\widetilde{\mathcal{D}}^\varepsilon(0) = [0,1]$ and $\widetilde{\mathcal{D}}^\varepsilon(t) = [0,\varepsilon[$ if $t \in]0,1]$ and the map $\widetilde{\mathcal{D}}^\varepsilon$ is not lower semicontinuous over T'.

However, the proposition below proves that if, moreover, property (2) in Proposition 2.1 is satisfied "uniformly" with respect to the variable t, then the map $\widetilde{\mathcal{D}}^{\varepsilon}$ is lower semicontinuous.

Proposition 2.6 *If the assumptions of Proposition 2.4 and the following hold:*

i) the set-valued map $\mathcal{K}$ is convex-valued over T';

ii) for any $\widetilde{t} \in T'$ and every $\theta > 0$ there exist $\eta > 0$ and a τ-neighbourhood Q of $\widetilde{t}$ such that

$$\mathcal{M}^{\eta}(t) \;\subseteq\; \widetilde{\mathcal{D}}^{\theta}(t) \quad \forall\, t \in Q\,;$$

then the map $\widetilde{\mathcal{D}}^{\varepsilon}$ is (τ, s)-lower semicontinuous over T'.

Proof

Let $\widetilde{t} \in T'$ and $(t_n)_n$ be a sequence τ-converging to $\widetilde{t}$ in T'. If $\widetilde{y} \in \widetilde{\mathcal{D}}^{\varepsilon}(\widetilde{t})$, then $\widetilde{y} \in \mathcal{K}(\widetilde{t})$ and $d(\widetilde{y}, \mathcal{M}(\widetilde{t})) < \varepsilon$. Since the map $\mathcal{K}$ is (τ, s)-lower semicontinuous, there exists a sequence $(y_n)_n$ strongly converging to $\widetilde{y}$ and such that $y_n \in \mathcal{K}(t_n)$ for n sufficiently large. Let $z \in \mathcal{M}(\widetilde{t})$ such that $||\widetilde{y} - z|| < \varepsilon$ and let θ be a positive real number such that $||\widetilde{y} - z|| < \theta/2 < \theta < \varepsilon$. If η is the positive number existing by assumption $i)$ in correspondence to $\theta/2$, since $\mathcal{M}(\widetilde{t}) \subseteq \widetilde{\mathcal{M}}^{\eta}(\widetilde{t})$, we have that $z \in \widetilde{\mathcal{M}}^{\eta}(\widetilde{t})$ and from Proposition 2.4 we infer that there exists a sequence $(z_n)_n$ s-converging to z such that $z_n \in \widetilde{\mathcal{M}}^{\eta}(t_n)$ for n sufficiently large and, by condition $i)$, $z_n \in \widetilde{\mathcal{D}}^{\theta/2}(t_n)$, that is $z_n \in \mathcal{K}(t_n)$ and $d(z_n, \mathcal{M}(t_n)) < \theta/2$, for such indexes n. Then, consider $(\widetilde{y}_n)_n$ defined by $\widetilde{y}_n = \alpha_n z_n + (1 - \alpha_n) y_n$, where $(\alpha_n)_n$, $\alpha_n \in [0, 1]$, is a sequence of real numbers converging to zero: we have that $\widetilde{y}_n \in \mathcal{K}(t_n)$, since $\mathcal{K}$ is convex-valued, and that $(\widetilde{y}_n)_n$ s-converges to $\widetilde{y}$, so it remains only to prove that $d(\widetilde{y}_n, \mathcal{M}(t_n)) < \varepsilon$ for n sufficiently large. If $z'_n \in \mathcal{M}(t_n)$ is such that $||z'_n - z_n|| < \theta/2$, we get that $d(\widetilde{y}_n, \mathcal{M}(t_n)) \leq ||\widetilde{y}_n - z'_n|| = ||\alpha_n z_n + (1 - \alpha_n) y_n - z'_n|| = ||\alpha_n z_n - z_n + (1 - \alpha_n) y_n + z_n - z'_n|| \leq (1 - \alpha_n) ||y_n - z_n|| + ||z_n - z'_n|| < \theta < \varepsilon$ and $d(\widetilde{y}_n, \mathcal{M}(t_n)) < \varepsilon$ for n sufficiently large. $\square$

Corollary 2.1 *The family $\widetilde{\mathfrak{M}} = \left\{ \widetilde{\mathcal{M}}^{\varepsilon}, \varepsilon > 0 \right\}$ is an inner regularization for the problem $(PBOP)$ whenever the assumptions of Proposition 2.4 are satisfied.*

The family $\mathfrak{M} = \{\mathcal{M}^{\varepsilon}, \varepsilon > 0\}$ is an inner regularization for the problem $(PBOP)$ whenever the assumptions of

Proposition 2.5 are satisfied.

The family $\widetilde{\mathfrak{D}} = \left\{\widetilde{\mathcal{D}}^{\varepsilon}, \varepsilon > 0\right\}$ *is an inner regularization for the problem* $(PBOP)$ *whenever the assumptions of Proposition 2.6 are satisfied.*

In the rest of this section, we show that the security value ω can be approximated by the security values of suitable regularizations that do not need to be necessarily inner regularizations.

Consider a family of maps $\mathfrak{T} = \{\mathcal{T}^{\varepsilon},\, \varepsilon > 0\}$, where $\mathcal{T}^{\varepsilon} : t \in T' \to \mathcal{T}^{\varepsilon}(t) \subseteq Y'$ and, given any $t \in T'$, define

$$i_{\varepsilon}(t) \;=\; \sup_{y \in \mathcal{T}^{\varepsilon}(t)} L(t, y) \;\text{ and }\; \varphi_{\varepsilon} \;=\; \inf_{t \in T'} i_{\varepsilon}(t).$$

First, we establish a general result that will be next applied to the specific approximations families $\widetilde{\mathfrak{M}}$, $\mathfrak{M}$ and $\widetilde{\mathfrak{D}}$.

Proposition 2.7 *Assume that:*

i) the family $\mathfrak{T} = \{\mathcal{T}^{\varepsilon},\, \varepsilon > 0\}$ *satisfies conditions* R_1) *and* R_2)*;*

ii) the function $-L$ *is* $(\tau \times w)$*-coercive on* $T' \times Y'$*;*

iii) for every $t \in T'$ *there exists a sequence* $(t_n)_n$ τ*-converging to* t *in* T' *such that for every* $y \in Y'$ *and any* $(y_n)_n$ w*-converging to* y *in* Y' *one has*

$$\limsup_{n} L(t_n, y_n) \;\leq\; L(t, y).$$

Then we have:

$$\omega = \lim_{\varepsilon \to 0} \varphi_{\varepsilon}.$$

Proof

Observe that $\omega \leq \varphi_{\varepsilon}$ for every $\varepsilon > 0$, since $\mathcal{M}(t) \subseteq \mathcal{T}^{\varepsilon}(t)$ for any $t \in T'$, so when $\omega = +\infty$ then also $\varphi_{\varepsilon} = +\infty$ and there is nothing to prove. When $\omega \in \mathbb{R} \cup \{-\infty\}$, it is sufficient to prove that

$$\lim_{\varepsilon \to 0} \varphi_{\varepsilon} \;=\; \inf_{\varepsilon > 0} \varphi_{\varepsilon} \;\leq\; \omega,$$

since φ_ε increases with respect to ε.

If we assume that there exists $\eta \in \mathbb{R}$ such that

$$\omega \;<\; \eta \;<\; \inf_{\varepsilon>0} \varphi_\varepsilon \tag{5}$$

we can determine a point $\widetilde{t} \in T'$ such that $\sup\limits_{y\in\mathcal{M}(\widetilde{t})} L(\widetilde{t}, y) < \eta$, so that

$$L(\widetilde{t}, y) \;<\; \eta \qquad \forall\, y \in \mathcal{M}(\widetilde{t}). \tag{6}$$

We infer from condition *iii*) that a sequence $(\widetilde{t}_n)_n$, which τ-converges to $\widetilde{t}$ in T', is such that

$$\limsup_n L(\widetilde{t}_n, y_n) \;\leq\; L(\widetilde{t}, y) \tag{7}$$

for any $y \in Y'$ and any $(y_n)_n$ w-converging to y in T'.

Then, for any given $(\varepsilon_n)_n$, we have from (5) that $\eta \;<\; \varphi_{\varepsilon_n} \;\leq\; \sup\limits_{y\in\mathcal{T}^{\varepsilon_n}(\widetilde{t}_n)} L(\widetilde{t}_n, y)$ for any $n \in \mathbb{N}$ and we can determine $(\widetilde{y}_n)_n$ such that

$$\widetilde{y}_n \in \mathcal{T}^{\varepsilon_n}(\widetilde{t}_n) \;\text{ and }\;\; \eta \;<\; L(\widetilde{t}_n, \widetilde{y}_n) \quad \forall\, n \in \mathbb{N}.$$

A subsequence $(\widetilde{y}_{n_k})_k$ of $(\widetilde{y}_n)_n$ w-converges to $\widetilde{y} \in Y'$, due to condition *ii*), and $\widetilde{y} \in \mathcal{M}(\widetilde{t})$ because the family $\mathfrak{T}$ satisfies condition R_2).

Then, by (7) we get that $\eta \leq L(\widetilde{t}, \widetilde{y})$ and this gives a contradiction with (6). $\square$

In order to give specific results for the security values related to the families $\mathfrak{M}$, $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{D}}$, it is useful to introduce the notations below:

$$\omega_\varepsilon \;=\; \inf_{t\in T'} \sup_{y\in\mathcal{M}^\varepsilon(t)} L(t, y), \qquad \widetilde{\omega}_\varepsilon \;=\; \inf_{t\in T'} \sup_{y\in\widetilde{\mathcal{M}}^\varepsilon(t)} L(t, y), \qquad \widetilde{\mathfrak{d}}_\varepsilon \;=\; \inf_{t\in T'} \sup_{y\in\widetilde{\mathcal{D}}^\varepsilon(t)} L(t, y).$$

Therefore, from Propositions 2.2 and 2.7 we infer an approximation result for the security value of the problem $(PBOP)$ which does not require any convexity assumption.

Corollary 2.2 *Assume that all conditions in Propositions 2.2 and 2.7 hold. Then, we have:*

$$\lim_{\varepsilon\to 0} \omega_\varepsilon \;=\; \lim_{\varepsilon\to 0} \widetilde{\omega}_\varepsilon \;=\; \omega.$$

If conditions i) in Propositions 2.3 and 2.7 hold, then we have:

$$\lim_{\varepsilon \to 0} \widetilde{\mathfrak{d}}_\varepsilon \;=\; \omega.$$

3 Further types of inner regularizations

In this section, we present an overview of further possible inner regularizations for the lower level problem that could be useful in applications.

For any given positive real number ε, we consider the maps:

$$\mathcal{P}^\varepsilon : t \in T' \to \mathcal{P}^\varepsilon(t) = \left\{ y \in \mathcal{K}(t) : F(t,y) \le \inf_{y' \in \mathcal{K}(t)} \left(F(t,y') + ||y - y'||^2 \right) + \varepsilon \right\},$$

$$\mathcal{E}^\varepsilon : t \in T' \to \mathcal{E}^\varepsilon(t) = \left\{ y \in \mathcal{K}(t) : F(t,y) \le \inf_{y' \in \mathcal{K}(t)} \left(F(t,y') + \varepsilon ||y - y'|| \right) \right\},$$

and their "strict" version:

$$\widetilde{\mathcal{P}}^\varepsilon : t \in T' \to \widetilde{\mathcal{P}}^\varepsilon(t) = \left\{ y \in \mathcal{K}(t) : F(t,y) < \inf_{y' \in \mathcal{K}(t)} \left(F(t,y') + ||y - y'||^2 \right) + \varepsilon \right\},$$

$$\widetilde{\mathcal{E}}^\varepsilon : t \in T' \to \widetilde{\mathcal{E}}^\varepsilon(t) = \left\{ y \in \mathcal{K}(t) : F(t,y) < F(t,y') + \varepsilon ||y - y'|| \ \forall \ y' \in \mathcal{K}(t) - \{y\} \right\},$$

which are all nonempty-valued, since $\mathcal{M}(t)$ is contained both in $\widetilde{\mathcal{P}}^\varepsilon(t)$ and in $\widetilde{\mathcal{E}}^\varepsilon(t)$.

The letters choosen to denote these approximations recall that the maps $\widetilde{\mathcal{P}}^\varepsilon$ and $\mathcal{P}^\varepsilon$ have been inspired by the proximal approximation method [23, 24, 25] and the maps $\widetilde{\mathcal{E}}^\varepsilon$ and $\mathcal{E}^\varepsilon$ have been inspired by the Ekeland variational principle [26].

The families they describe are

$$\mathfrak{P} = \{\mathcal{P}^\varepsilon,\, \varepsilon > 0\}, \quad \widetilde{\mathfrak{P}} = \left\{\widetilde{\mathcal{P}}^\varepsilon,\, \varepsilon > 0\right\}, \qquad \mathfrak{E} = \{\mathcal{E}^\varepsilon,\, \varepsilon > 0\}, \qquad \widetilde{\mathfrak{E}} = \left\{\widetilde{\mathcal{E}}^\varepsilon,\, \varepsilon > 0\right\}.$$

Moreover, in the spirit of [15], we can deal with strict approximate solutions for $P(t)$ that are allowed to violate the constraint $\mathcal{K}(t)$ though lying close to it, so, we also introduce the following maps:

$$\widetilde{\mathcal{M}}_d^\varepsilon : t \in T' \to \widetilde{\mathcal{M}}_d^\varepsilon(t) = \left\{ y \in K : d(y, \mathcal{K}(t)) \le \varepsilon \text{ and } F(t,y) < \inf_{y' \in \mathcal{K}(t)} F(t,y') + \varepsilon \right\},$$

$$\widetilde{\mathcal{P}}_d^\varepsilon : t \in T' \to \widetilde{\mathcal{P}}_d^\varepsilon(t) = \left\{ y \in K : d(y, \mathcal{K}(t)) \leq \varepsilon \text{ and } F(t,y) < \inf_{y' \in \mathcal{K}(t)} \left(F(t,y') + ||y - y'||^2 \right) + \varepsilon \right\},$$

$$\widetilde{\mathcal{E}}_d^\varepsilon : t \in T' \to \widetilde{\mathcal{E}}_d^\varepsilon(t) = \left\{ y \in K : d(y, \mathcal{K}(t)) \leq \varepsilon \text{ and } F(t,y) < F(t,y') + \varepsilon ||y - y'|| \ \forall \ y' \in \mathcal{K}(t) - \{y\} \right\},$$

and we denote by $\widetilde{\mathfrak{M}}_d, \widetilde{\mathfrak{P}}_d, \widetilde{\mathfrak{E}}_d$ the families they describe.

We proceed, as in Section 2, by investigating firstly property R_2) and, then, property R_3) for all the above families.

Proposition 3.1 *Under the assumptions of Proposition 2.2, the following assertions hold:*

1. *The maps $\mathcal{E}^\varepsilon$, $\widetilde{\mathcal{E}}^\varepsilon$, $\widetilde{\mathcal{E}}_d^\varepsilon$ and $\widetilde{\mathcal{M}}_d^\varepsilon$ satisfy condition R_2).*

2. *If the hypotheses below also hold:*

   i) the map $\mathcal{K}$ is convex-valued over T';

   ii) $F(t, \cdot)$ is convex on $\mathcal{K}(t)$, for every $t \in T'$;

   then the maps $\mathcal{P}^\varepsilon$, $\widetilde{\mathcal{P}}^\varepsilon$ and $\widetilde{\mathcal{P}}_d^\varepsilon$ satisfy condition R_2).

Proof

Point 1.

Let $(t_n)_n$, $t_n \in T'$, τ-converging to $\widetilde{t} \in T'$ and let $(y_n)_n$ w-converging to $\widetilde{y}$ in Y' such that $y_n \in \mathcal{E}^{\varepsilon_n}(t_n)$ for any $n \in \mathbb{N}$. In order to prove that $\widetilde{y} \in \mathcal{M}(\widetilde{t})$, consider $v \in \mathcal{K}(\widetilde{t})$. There exists $(v_n)_n$ s-converging to v such that $v_n \in \mathcal{K}(t_n)$ for n sufficiently large. So, for such indexes n, $F(t_n, y_n) \leq F(t_n, v_n) + \varepsilon_n ||y_n - v_n||$ and we infer that $F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, v)$ because the sequence $(||y_n - v_n||)_n$ is bounded and $\lim_n \varepsilon_n ||y_n - v_n|| = 0$. Since $\widetilde{y} \in \mathcal{K}(\widetilde{t})$ due to the (τ, w)-closedness of $\mathcal{K}$, we get that $\widetilde{y} \in \mathcal{M}(\widetilde{t})$. The result holds true also for the maps $\widetilde{\mathcal{E}}^\varepsilon$ since $\widetilde{\mathcal{E}}^\varepsilon(t) \subseteq \mathcal{E}^\varepsilon(t)$ for any t.

Now, assume that $(y_n)_n$ w-converges to $\widetilde{y}$, $y_n \in \widetilde{\mathcal{E}}_d^{\varepsilon_n}(t_n)$ for any n, that is

$$d(y_n, \mathcal{K}(t_n)) \leq \varepsilon_n \text{ and } F(t_n, y_n) < F(t_n, y') + \varepsilon_n ||y_n - y'|| \ \forall y' \in \mathcal{K}(t_n) - \{y_n\}\,.$$

For any $z \in \mathcal{K}(\widetilde{t}) - \{\widetilde{y}\}$ there exists $(z_n)_n$ s-converging to z such that $z_n \in \mathcal{K}(t_n) - \{y_n\}$ for n sufficiently large. Therefore, we get $F(t_n, y_n) < F(t_n, z_n) + \varepsilon_n ||z_n - y_n||$ for such indexes n and $F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, z)$. Moreover,

$\widetilde{y} \in \mathcal{K}(\widetilde{t})$ since $d(\widetilde{y}, \mathcal{K}(\widetilde{t})) \leq \liminf_n d(\widetilde{y}_n, \mathcal{K}(t_n)) = 0$ (see, for example, [21]) and $\mathcal{K}(\widetilde{t})$ is closed, so we obtain that $\widetilde{y} \in \mathcal{M}(\widetilde{t})$.

Now, assume that $(y_n)_n$ w-converges to $\widetilde{y}$, $y_n \in \widetilde{\mathcal{M}}_d^{\varepsilon_n}(t_n)$ for any n, that is

$$d(y_n, \mathcal{K}(t_n)) \leq \varepsilon_n \text{ and } F(t_n, y_n) < \inf_{y \in \mathcal{K}(t_n)} F(t_n, y) + \varepsilon_n.$$

For any $z \in \mathcal{K}(\widetilde{t})$ there exists $(z_n)_n$ s-converging to z such that $z_n \in \mathcal{K}(t_n)$ for n sufficiently large. Therefore, we get $F(t_n, y_n) < F(t_n, z_n) + \varepsilon_n$ for such indexes n and $F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, z)$, so $\widetilde{y} \in \mathcal{M}(\widetilde{t})$ since we get $d(\widetilde{y}, \mathcal{K}(\widetilde{t})) = 0$ arguing as before.

Point 2.

Now, assume that $y_n \in \mathcal{P}^{\varepsilon_n}(t_n)$ and that $(y_n)_n$ w-converges to $\widetilde{y}$.

Since $F(t_n, y_n) \leq \inf_{y \in \mathcal{K}(t_n)} \left(F(t_n, y) + ||y - y_n||^2\right) + \varepsilon_n$ and $y_n \in \mathcal{K}(t_n)$, the (τ, w)-closedness of $\mathcal{K}$ and the $(\tau \times w)$-lower semicontinuity of F imply that $\widetilde{y} \in \mathcal{K}(\widetilde{t})$ and $F(\widetilde{t}, \widetilde{y}) \leq \liminf_n F(t_n, y_n) \leq \limsup_n \inf_{y \in \mathcal{K}(t_n)} \left(F(t_n, y) + ||y - y_n||^2\right)$.

Then, from the τ-upper semicontinuity of the marginal function $\inf_{y \in \mathcal{K}(\cdot)} \left(F(\cdot, y) + ||\widetilde{y} - y||^2\right)$, which is guaranteed under the assumptions of Proposition 2.2, we infer that

$$F(\widetilde{t}, \widetilde{y}) \leq \inf_{y \in \mathcal{K}(\widetilde{t})} \left(F(\widetilde{t}, y) + ||\widetilde{y} - y||^2\right) \tag{8}$$

and therefore $\widetilde{y} \in \mathcal{M}(\widetilde{t})$. Indeed, for any given $v \in \mathcal{K}(\widetilde{t})$ and $n \in \mathbb{N}$ consider the sequence defined by $v_n = (1 - \alpha_n)\widetilde{y} + \alpha_n v$ where $(\alpha_n)_n$ converges to 0 in]0,1]. Since $v_n \in \mathcal{K}(\widetilde{t})$ by assumption $i)$, we infer from $ii)$ and (8) that

$$F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, v_n) + ||\widetilde{y} - v_n||^2 \leq (1 - \alpha_n)F(\widetilde{t}, \widetilde{y}) + \alpha_n F(\widetilde{t}, v) + \alpha_n^2 ||\widetilde{y} - v||^2.$$

So, dividing by α_n, we get $F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, v) + \alpha_n ||\widetilde{y} - v||^2$ for any n and $F(\widetilde{t}, \widetilde{y}) \leq F(\widetilde{t}, v)$.

The result holds also true for the map $\widetilde{\mathcal{P}}^{\varepsilon}$ since $\widetilde{\mathcal{P}}^{\varepsilon}(t) \subseteq \mathcal{P}^{\varepsilon}(t)$ for any t.

Now, assume that $(y_n)_n$ w-converges to $\widetilde{y}$, $y_n \in \widetilde{\mathcal{P}}_d^{\varepsilon_n}(t_n)$ for any n, that is

$$d(y_n, \mathcal{K}(t_n)) \leq \varepsilon_n \text{ and } F(t_n, y_n) < \inf_{y \in \mathcal{K}(t_n)} (F(t_n, y) + ||y_n - y||^2) + \varepsilon_n.$$

For any $z \in \mathcal{K}(\widetilde{t})$ there exists $(z_n)_n$ s-converging to z such that $z_n \in \mathcal{K}(t_n)$ for n sufficiently large. Therefore, we get $F(t_n, y_n) < F(t_n, z_n) + ||y_n - z_n||^2 + \varepsilon_n$ for such indexes n and by the assumptions we infer that

$F(\widetilde{t},\widetilde{y}) \leq F(\widetilde{t},z)+||\widetilde{y}-z||^2$. Arguing as before, we have that $d(\widetilde{y},\mathcal{K}(\widetilde{t}))=0$, that is $\widetilde{y}\in K(\widetilde{t})$, and $F(\widetilde{t},\widetilde{y}) \leq F(\widetilde{t},v)$ for every $v\in K(\widetilde{t})$. So, we can conclude that $\widetilde{y}\in\mathcal{M}(\widetilde{t})$. □

Now, we investigate property R_3) starting with "strict" approximations.

Proposition 3.2 *Under the assumptions of Proposition 2.4 the following assertions are true:*

1. *The map $\widetilde{\mathcal{P}}^\varepsilon$ is (τ,s)-lower semicontinuous over T'.*

2. *If also the following hold:*

   i) the map $\mathcal{K}$ is convex-valued over T';

   ii) for any τ-converging sequence $(t_n)_n$, $t_n\in T'$, there exists $\nu\in\mathbb{N}$ such that

$$int \bigcap_{n\geq\nu}\mathcal{K}(t_n)\neq\emptyset;$$

   then the maps $\widetilde{\mathcal{M}}_d^\varepsilon$ and $\widetilde{\mathcal{P}}_d^\varepsilon$ are (τ,s)-lower semicontinuous over T'.

3. *If also the following holds:*

   iii) for every $t\in T'$ and every $(t_n)_n$ τ-converging to t in T' one has

$$\liminf_n \frac{F(t_n,u_n)-F(t_n,v_n)}{||u_n-v_n||}<\varepsilon$$

   whenever $(u_n)_n$ and $(v_n)_n$ weakly converge towards the same point $u\in\mathcal{K}(t)$,

   then the map $\widetilde{\mathcal{E}}^\varepsilon$ is (τ,s)-lower semicontinuous over T'.

4. *If also conditions i), ii) and iii) hold, then the map $\widetilde{\mathcal{E}}_d^\varepsilon$ is (τ,s)-lower semicontinuous over T'.*

Proof

POINT 1.

Let $t\in T'$ and let $(t_n)_n$ be a sequence τ-converging to t in T'. Consider $y\in\widetilde{\mathcal{P}}^\varepsilon(t)$, that is $y\in\mathcal{K}(t)$ and $F(t,y)<\inf_{y'\in\mathcal{K}(t)}(F(t,y')+||y-y'||^2)+\varepsilon$. There exists a sequence $(y_n)_n$ s-converging to y such that $y_n\in\mathcal{K}(t_n)$

for n sufficiently large, since the map $\mathcal{K}$ is lower semicontinuous, and we can prove that

$$\inf_{y'\in\mathcal{K}(t)}(F(t,y')+||y-y'||^2)\leq\liminf_n\inf_{y'\in\mathcal{K}(t_n)}(F(t_n,y')+||y_n-y'||^2) \tag{9}$$

which gives $y_n\in\widetilde{\mathcal{P}}^{\varepsilon}(t_n)$ for n sufficiently large since $\limsup_n F(t_n,y_n)\leq F(t,y)<\inf_{y'\in\mathcal{K}(t)}(F(t,y')+||y-y'||^2)+\varepsilon$. Indeed, if inequality (9) is not true, there exist $a\in\mathbb{R}$, a subsequence $(n_k)_k$ and a sequence $(v_k)_k$ such that $v_k\in\mathcal{K}(t_{n_k})$ and

$$F\left(t_{n_k},y_{n_k}\right)+||y_{n_k}-v_k||^2<a<F(t,y')+||y-y'||^2\quad\forall\ y'\in\mathcal{K}(t).$$

A subsequence of $(v_k)_k$ has to w-converge towards $v\in\mathcal{K}(t)$ and by the assumptions we infer that

$$F(t,v)+||y-v||^2\leq\liminf_k\left(F\left(t_{n_k},y_{n_k}\right)+||y_{n_k}-v_k||^2\right)<F(t,y')+||y-y'||^2\quad\forall\ \ y'\in\mathcal{K}(t)$$

getting a contradiction.

Point 2.

Let $t\in T'$ and let $(t_n)_n$ be a sequence τ-converging to t in T'. Consider $y\in\widetilde{\mathcal{M}}^{\varepsilon}_d(t)$, that is

$$d(y,\mathcal{K}(t))\leq\varepsilon\text{ and }F(t,y)<\inf_{y'\in\mathcal{K}(t)}F(t,y')+\varepsilon.$$

From Proposition 5.1 in [27] we infer that there exists a sequence $(y_n)_n$ strongly converging to y in Y' such that $d(y_n,\mathcal{K}(t_n))\leq\varepsilon$ for n sufficiently large and arguing as in Proposition 2.4 it can be shown that $F(t_n,y_n)<\inf_{y'\in\mathcal{K}(t_n)}F(t_n,y')+\varepsilon$, i.e. $y_n\in\widetilde{\mathcal{M}}^{\varepsilon}_d(t_n)$, that gives the lower semicontinuity of $\widetilde{\mathcal{M}}^{\varepsilon}_d$ over T'. The proof for $\widetilde{\mathcal{P}}^{\varepsilon}_d$ is similar and is omitted.

Point 3.

Let $t\in T'$ and let $(t_n)_n$ be a sequence τ-converging to t in T'. Consider $y\in\widetilde{\mathcal{E}}^{\varepsilon}(t)$, that is $y\in\mathcal{K}(t)$ and $F(t,y)<F(t,y')+\varepsilon||y-y'||$ for all $y'\in\mathcal{K}(t)-\{y\}$, and a sequence $(y_n)_n$ s-converging to y such that $y_n\in\mathcal{K}(t_n)$ for n sufficiently large. If we assume that $y_{n_k}\notin\widetilde{\mathcal{E}}^{\varepsilon}(t_{n_k})$ for an increasing sequence of integers $(n_k)_k$, we can determine a sequence $(v_{n_k})_k$ such that $v_{n_k}\in\mathcal{K}\left(t_{n_k}\right)-\{y_{n_k}\}$ and $F\left(t_{n_k},y_{n_k}\right)-F\left(t_{n_k},v_{n_k}\right)\geq\varepsilon||y_{n_k}-v_{n_k}||$ for any $k\in\mathbb{N}$. The map $\mathcal{K}$ being (τ,w)-closed and (τ,w)-subcontinuous, we get that a subsequence of $(v_{n_k})_k$ weakly converges towards $v\in\mathcal{K}(t)$ and we have a contradiction in any case. Indeed, if $v=y$ we would contradict

assumption iii), if $v \neq y$ we would infer that $y \notin \widetilde{\mathcal{E}}^{\varepsilon}(t)$.

Point 4.

Now, given $t \in T'$ and a sequence $(t_n)_n$ τ-converging to t in T', consider $y \in \widetilde{\mathcal{E}}^{\varepsilon}_d(t)$, that is

$$d(y, \mathcal{K}(t)) \leq \varepsilon \text{ and } F(t, y) < F(t, y') + \varepsilon ||y - y'|| \ \forall \ y' \in \mathcal{K}(t) - \{y\}\,.$$

As in Point 2, we can prove that there exists $(y_n)_n$ strongly converging to y such that $d(y_n, \mathcal{K}(t_n)) \leq \varepsilon$ for any $n \in \mathbb{N}$. If we assume that $y_{n_k} \notin \widetilde{\mathcal{E}}^{\varepsilon}(t_{n_k})$ for an increasing sequence of integers $(n_k)_k$, we reach a contradiction arguing as in Point 3. $\square$

Remark 3.1 Condition iii) is satisfied if, for instance, the function F is locally Lipschitz in the variable y uniformly with respect to t with the Lipschitz constant smaller than ε.

The next result is in line with Proposition 2.5, since it extends Point 1 in Proposition 3.2 to "large" approximate solutions maps $\mathcal{P}^{\varepsilon}$.

Proposition 3.3 *Under the assumptions of Proposition 2.5 the map $\mathcal{P}^{\varepsilon}$ is (τ, s)-lower semicontinuous over T'.*

Proof

Similarly as in Proposition 2.5, it is sufficient to prove that $\mathcal{P}^{\varepsilon}(t) \subseteq cl\left(\widetilde{\mathcal{P}}^{\varepsilon}(t)\right)$ for any given $t \in T'$ since the map $\widetilde{\mathcal{P}}^{\varepsilon}$ is (τ, s)-lower semicontinuous.

If $y \in \mathcal{P}^{\varepsilon}(t)$ and we assume that $F(t, y) < \inf_{y' \in \mathcal{K}(t)} \left(F(t, y') + ||y - y'||^2\right) + \varepsilon$, we have $y \in \widetilde{\mathcal{P}}^{\varepsilon}(t)$ and there is nothing to prove. Then, assume now that

$$F(t, y) = \inf_{y' \in \mathcal{K}(t)} \left(F(t, y') + ||y - y'||^2\right) + \varepsilon = F(t, \widetilde{y}) + ||y - \widetilde{y}||^2 + \varepsilon,$$

where the existence of $\widetilde{y} \in \mathcal{K}(t)$ such that $\inf_{y' \in \mathcal{K}(t)} \left(F(t, y') + ||y - y'||^2\right) + \varepsilon = F(t, \widetilde{y}) + ||y - \widetilde{y}||^2 + \varepsilon$ is guaranteed by the made assumptions. Consider a sequence $(\alpha_n)_n$ converging to 0 in [0,1] and the sequence defined by $y_n = \alpha_n \widetilde{y} + (1 - \alpha_n) y$ which s-converges to y in $\mathcal{K}(t)$ by condition i). From ii) we infer that $F(t, y_n) < max\left\{F(t, y), F(t, \widetilde{y})\right\} = F(t, y) = \inf_{y' \in \mathcal{K}(t)} \left(F(t, y') + ||y - y'||^2\right) + \varepsilon$, so $y_n \in \widetilde{\mathcal{P}}^{\varepsilon}(t)$ for any $n \in \mathbb{N}$ and $y \in cl\left(\widetilde{\mathcal{P}}^{\varepsilon}(t)\right)$. $\square$

Remark 3.2 We note that even if the map $\widetilde{\mathcal{E}}^\varepsilon$ is lower semicontinuous and all assumptions in Proposition 2.5 are satisfied, the map $\mathcal{E}^\varepsilon$ may be not lower semicontinuous. Indeed, consider $T = E = \mathbb{R}$, $T' = Y' = [0,1]$, $\mathcal{K}(t) = [0,1]$ for any $t \in T'$ and let $F(t,y) = ty$ for every $(t,y) \in [0,1]^2$. Then, it is easy to check that $\mathcal{E}^\varepsilon(t) = [0,1]$ for $t \in [0,\varepsilon]$ and that $\mathcal{E}^\varepsilon(t) = \{0\}$ for $t \in]\varepsilon, 1]$, so that $\mathcal{E}^\varepsilon$ is not lower semicontinuous at $t = \varepsilon$.

We stress that, among the numerous regularizations presented here, the unique which turns out to be an inner regularization only under the assumptions of Proposition 2.4 is $\widetilde{\mathfrak{M}}$, all the other ones needing supplementary hypotheses. This is emphasized in the next corollary.

Corollary 3.1 *Assume that the assumptions of Proposition 2.4 hold. Then, the following assertions are true:*

1. The family $\widetilde{\mathcal{E}}^\varepsilon$ is an inner regularization for the problem $(PBOP)$ whenever also condition iii) in Proposition 3.2 holds.

2. The families $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$ are inner regularizations for the problem $(PBOP)$ whenever also conditions i) and ii) in Proposition 3.1 hold.

3. The family $\widetilde{\mathfrak{M}}_d$ is an inner regularization for the problem $(PBOP)$ whenever also conditions i) and ii) in Proposition 3.2 hold.

4. The family $\widetilde{\mathfrak{P}}_d$ is an inner regularization for the problem $(PBOP)$ whenever also conditions i) and ii) in Proposition 3.2 and ii) in Proposition 3.1 hold.

5. The family $\widetilde{\mathcal{E}}^\varepsilon_d$ is an inner regularization for the problem $(PBOP)$ whenever also conditions i), ii) and iii) in Proposition 3.2 hold.

4 Existence of viscosity solutions

We first establish a general result about the existence of viscosity solutions.

Proposition 4.1 *If $\mathfrak{T} = \{\mathcal{T}^\varepsilon, \varepsilon > 0\}$ is an inner regularization for the family $\{P(t), t \in T\}$, the set T' is τ-sequentially compact and the following hold:*

i) the function $-L$ *is* $(\tau \times w)$*-coercive on* $T' \times Y'$*;*

ii) for every $t \in T'$ *there exists a sequence* $(t_n)_n$ τ*-converging to* t *in* T' *such that for every* $y \in Y'$ *and any* $(y_n)_n$ w*-converging to* y *in* Y' *one has*

$$\limsup_n L(t_n, y_n) \leq L(t, y);$$

iii) the function L *is* $(\tau \times w)$*-lower semicontinuous on* $T' \times Y'$*;*

then, there exists a $\mathfrak{T}$*-viscosity solution for the pessimistic problem* $(PBOP)$*.*

Proof

From the made assumptions we infer that for every $\varepsilon > 0$ the function $i_\varepsilon(t) = \sup_{y \in \mathcal{T}^\varepsilon(t)} L(t, y)$ is τ-lower semicontinuous over T', so there exists $t_\varepsilon \in T'$ such that $i_\varepsilon(t_\varepsilon) = \inf_{t \in T'} i_\varepsilon(t) = \varphi_\varepsilon$. Then, if $(\varepsilon_n)_n$ decreases to zero and $(t_{\varepsilon_n})_n$ is such that $i_{\varepsilon_n}(t_{\varepsilon_n}) = \varphi_{\varepsilon_n}$ for all $n \in \mathbb{N}$, there exists a subsequence of $(t_{\varepsilon_n})_n$ which τ-converges towards a point $\widetilde{t} \in T'$, so conditions $V_1)$ and $V_2)$ of Definition 1.2 are satisfied. The point $\widetilde{t}$ turns out to be a $\mathfrak{T}$-viscosity solution for the problem $(PBOP)$ because $\lim_n \varphi_{\varepsilon_n} = \omega$ thanks to Proposition 2.7. $\square$

Now, we explicitely give an existence result for the viscosity solutions generated by the inner regularization families introduced before.

Theorem 4.1 *Assume that the set* T' *is* τ*-sequentially compact and that conditions in Propositions 2.4 and 4.1 hold. Then, we have:*

1. *There exists a* $\widetilde{\mathfrak{M}}$*-viscosity solution for the pessimistic problem* $(PBOP)$*.*
2. *There exists a* $\widetilde{\mathfrak{D}}$*-viscosity solution for the pessimistic problem* $(PBOP)$ *whenever also conditions i) and ii) in Proposition 2.6 hold.*
3. *There exists a* $\widetilde{\mathfrak{E}}$*-viscosity solution for the pessimistic problem* $(PBOP)$ *whenever also condition iii) in Proposition 3.2 holds.*

4. *There exists a $\mathfrak{M}$-viscosity solution, $\mathfrak{P}$-viscosity solution and a $\widetilde{\mathfrak{P}}$-viscosity solution whenever also conditions i) and ii) in Proposition 3.1 hold.*

5. *There exists a $\widetilde{\mathfrak{M}}_d$-viscosity solution for the pessimistic problem $(PBOP)$ whenever also conditions i) and ii) in Proposition 3.2 hold.*

6. *There exists a $\widetilde{\mathfrak{E}}_d$-viscosity solution for the pessimistic problem $(PBOP)$ whenever also conditions i), ii) and iii) in Proposition 3.2 hold.*

7. *There exists a $\widetilde{\mathfrak{P}}_d$-viscosity solution for the pessimistic problem $(PBOP)$ whenever also conditions i) and ii) in Proposition 3.2 and ii) in Proposition 3.1 hold.*

Proof

The proof follows from Corollary 2.1, Corollary 3.1 and Proposition 4.1. □

Now, we prove that any viscosity solution solves a suitable minimization problem as announced in Section 2. We recall that a sequence of real-valued functions $(f_h)_h$ τ-epiconverges to a function f over T' if the following conditions hold:

- for every $t \in T'$ and every sequence $(t_h)_h$ τ-converging to t in T'

$$f(t) \leq \liminf_h f_h(t_h);$$

- for every $t \in T'$ there exists a sequence $(t'_h)_h$ τ-converging to t in T'

$$\limsup_h f_h(t_h) \leq f(t).$$

Then, we establish the following result.

Proposition 4.2 *If $\mathfrak{T} = \{\mathcal{T}^\varepsilon, \varepsilon > 0\}$ is an inner regularization for the family $\{P(t), t \in T\}$ and the following assumptions hold:*

i) for every $t \in T'$, the function $-L(t, \cdot)$ is w-coercive in y on Y';

ii) for every $t \in T'$, the function $L(t,\cdot)$ is s-upper semicontinuous on Y';

then the sequence $(i_{\varepsilon_n})_n$ τ-epiconverges towards the function C defined in T' by

$$C(t) = cl^{\tau}_{seq} \sup_{y \in \mathcal{M}(t)} L(t,y).$$

Consequently, a $\mathfrak{T}$-viscosity solution for $(PBOP)$, if there are any, is also a minimum point for C.

Proof

The sequence $(i_{\varepsilon_n})_n$ is monotone decreasing, so it is sufficient to prove that it pointwise converges over T' towards the function ψ defined by $\psi(t) = \sup_{y \in \mathcal{M}(t)} L(t,y)$ (see, for example, [16]).

Given any $t \in T'$ and $n \in \mathbb{N}$ one has $\psi(t) \le i_{\varepsilon_n}(t)$ and $\psi(t) \le \lim_n i_{\varepsilon_n}(t)$. If we assume that $\psi(t) < \lim_n i_{\varepsilon_n}(t)$, there exists $\alpha \in \mathbb{R}$ such that $\psi(t) < \alpha < \lim_n i_{\varepsilon_n}(t) = \inf_n i_{\varepsilon_n}(t)$. Therefore, for any $n \in \mathbb{N}$ there is $y_n \in \mathcal{T}^{\varepsilon_n}(t)$ such that $\alpha < L(t, y_n)$ and $\alpha \le \limsup_n L(t, y_n)$. Due to assumption $i)$, a subsequence $(y_{n_h})_h$ w-converges to $y \in Y'$ and by condition $ii)$ we get $\limsup_h L(t, y_{n_h}) \le L(t,y)$. Therefore, since $y \in \mathcal{M}(t)$ by condition R_2), we get the contradiction $\psi(t) < \alpha \le \sup_{y \in \mathcal{M}(t)} L(t,y) = \psi(t)$.

Therefore, due to a known property of the epiconvergence [16], every τ-sequence of minimum points for the functions i_{ε_n} has to converge towards a minimum point of the function C, which implies the second assertion.

□

5 Illustrative examples

The first example illustrates, in a simple case, the behavior of all regularization maps previously considered.

Example 5.1 Let $T = Y = \mathbb{R}$, $T' = Y' = [0,1]$, $\mathcal{K}(t) = [0,1]$ for any $t \in T'$ and let $F(t,y) = -y^2 + (1+t)y - t$ for every $(t,y) \in [0,1]^2$. It is easy to check that:

- $\mathcal{M}(0) = \{0,1\}$ and $\mathcal{M}(t) = \{0\}$ if $t \in]0,1]$;
- $\widetilde{\mathcal{D}}^{\varepsilon}(0) = [0,\varepsilon[\cup]1-\varepsilon, 1]$ and $\widetilde{\mathcal{D}}^{\varepsilon}(t) = [0,\varepsilon[$ if $t \in]0,1]$;

- $\widetilde{\mathcal{E}}^{\varepsilon}(t) = \{0, 1\}$ if $t \in [0, \varepsilon[$

  $\widetilde{\mathcal{E}}^{\varepsilon}(t) = \{0\}$ if $t = \varepsilon$

  $\widetilde{\mathcal{E}}^{\varepsilon}(t) = \{0\}$ if $t \in]\varepsilon, 1]$;

- $\mathcal{E}^{\varepsilon}(t) = \{0, 1\}$ if $t \in [0, \varepsilon[$

  $\mathcal{E}^{\varepsilon}(t) = \{0, 1\}$ if $t = \varepsilon$

  $\mathcal{E}^{\varepsilon}(t) = \{0\}$ if $t \in]\varepsilon, 1]$.

Now, for $\varepsilon < 1/4$, we set $\alpha_\varepsilon(t) = \frac{1}{2}\left(t + 1 - \sqrt{(t+1)^2 - 4\varepsilon}\right)$ and $\beta_\varepsilon(t) = \frac{1}{2}\left(t + 1 + \sqrt{(t+1)^2 - 4\varepsilon}\right)$, the solutions to the equation

$$y^2 + (t+1)y + \varepsilon = 0.$$

So, we get that

- $\mathcal{M}^{\varepsilon}(t) = [0, \alpha_\varepsilon(t)] \cup [\beta_\varepsilon(t), 1]$ if $t \in [0, \varepsilon[$

  $\mathcal{M}^{\varepsilon}(t) = [0, \varepsilon] \cup \{1\}$ if $t = \varepsilon$

  $\mathcal{M}^{\varepsilon}(t) = [0, \alpha_\varepsilon(t)]$ if $t \in]\varepsilon, 1]$;

- $\widetilde{\mathcal{M}}^{\varepsilon}(t) = [0, \alpha_\varepsilon(t)[\cup]\beta_\varepsilon(t), 1]$ if $t \in [0, \varepsilon[$

  $\widetilde{\mathcal{M}}^{\varepsilon}(t) = [0, \varepsilon[$ if $t = \varepsilon$

  $\widetilde{\mathcal{M}}^{\varepsilon}(t) = [0, \alpha_\varepsilon(t)[$ if $t \in]\varepsilon, 1]$.

Now, we set $\gamma_\varepsilon(t) = 1/4\left(1 + t - \sqrt{(1+t)^2 - 8\varepsilon}\right)$ and $\delta_\varepsilon(t) = 1/4\left(1 + t + \sqrt{(1+t)^2 - 8\varepsilon}\right)$, the solutions to the equation

$$2y^2 - (1+t)y + \varepsilon = 0$$

and $\mu_\varepsilon(t) = 1/4\left(3 + t - \sqrt{(1-t)^2 - 8\varepsilon}\right)$ and $\nu_\varepsilon(t) = 1/4\left(3 + t + \sqrt{(1-t)^2 - 8\varepsilon}\right)$, the solutions to the equation

$$2y^2 - (3+t)y + 1 + t + \varepsilon = 0.$$

Then, we get that

- $\widetilde{\mathcal{P}}^\varepsilon(t) = [0, \gamma_\varepsilon(t)[\ \cup\ ]\delta_\varepsilon(t), \mu_\varepsilon(t)[\ \cup\ ]\nu_\varepsilon(t), 1]$ if $t \in \left[0, 1 - \sqrt{8\varepsilon}\right[$

  $\widetilde{\mathcal{P}}^\varepsilon(t) = [0, \gamma_\varepsilon(t)[\ \cup\ ]\delta_\varepsilon(t), 1]\ -\ \left\{1 - \sqrt{\varepsilon/2}\right\}$ if $t = 1 - \sqrt{8\varepsilon}$

  $\widetilde{\mathcal{P}}^\varepsilon(t) = [0, \gamma_\varepsilon(t)[\ \cup\ ]\delta_\varepsilon(t), 1]$ if $t \in \left]1 - \sqrt{8\varepsilon}, 1\right]$

- $\mathcal{P}^\varepsilon(t) = [0, \gamma_\varepsilon(t)]\ \cup\ [\delta_\varepsilon(t), \mu_\varepsilon(t)]\ \cup\ [\nu_\varepsilon(t), 1]$ if $t \in \left[0, 1 - \sqrt{8\varepsilon}\right[$

  $\mathcal{P}^\varepsilon(t) = [0, \gamma_\varepsilon(t)]\ \cup\ [\delta_\varepsilon(t), 1]$ if $t \in \left[1 - \sqrt{8\varepsilon}, 1\right]$.

Therefore, it is easy to see that:

- the maps $\mathcal{M}$, $\widetilde{\mathcal{D}}^\varepsilon$, $\mathcal{M}^\varepsilon$ and $\mathcal{E}^\varepsilon$ are not lower semicontinuous; more precisely, the maps $\mathcal{M}$ and $\widetilde{\mathcal{D}}^\varepsilon$ fail to be lower semicontinuous at $t = 0$, the maps $\mathcal{M}^\varepsilon$ and $\mathcal{E}^\varepsilon$ fail to be lower semicontinuous at $t = \varepsilon$;

- the maps $\widetilde{\mathcal{M}}^\varepsilon$, $\widetilde{\mathcal{E}}^\varepsilon$, $\widetilde{\mathcal{P}}^\varepsilon$ and $\mathcal{P}^\varepsilon$ are lower semicontinuous;

- the maps $\widetilde{\mathcal{M}}^\varepsilon$, $\widetilde{\mathcal{E}}^\varepsilon$, $\mathcal{M}^\varepsilon$, $\mathcal{E}^\varepsilon$ and $\widetilde{\mathcal{D}}^\varepsilon$ satisfy condition R_2);

- the maps $\widetilde{\mathcal{P}}^\varepsilon$ and $\mathcal{P}^\varepsilon$ do not satisfy condition R_2) at $t = 1$;

- the maps $\widetilde{\mathcal{M}}^\varepsilon$ and $\widetilde{\mathcal{E}}^\varepsilon$ generate an inner regularization class for the lower level problem.

We emphasize that the objective function of this example does not fulfill neither condition *ii*) in Proposition 2.5 nor condition *ii*) in Proposition 2.6, which explains the lack of lower semicontinuity by the maps $\mathcal{M}^\varepsilon$ and $\widetilde{\mathcal{D}}^\varepsilon$. It is easy to check that also assumption *ii*) in Proposition 3.1 is not satisfied and this explains the lack of property R_2) by $\mathcal{P}^\varepsilon$ and $\widetilde{\mathcal{P}}^\varepsilon$.

The following example illustrates the existence of $\widetilde{\mathfrak{M}}$-viscosity solutions and the lack of $\mathfrak{M}$-viscosity solutions where $\widetilde{\mathfrak{M}}$ and $\mathfrak{M}$ have been determined in Example 5.1.

Example 5.2 Consider the function $L(t, y) = t + y$ where $(t, y) \in [0, 1]^2$.

With the data of Example 5.1 is easy to check that:

$$\widetilde{\psi}_\varepsilon(t) = \sup_{y \in \widetilde{\mathcal{M}}^\varepsilon(t)} (t + y) = t + 1 \text{ if } t \in [0, \varepsilon[, \quad \widetilde{\psi}_\varepsilon(\varepsilon) = 2\varepsilon, \quad \widetilde{\psi}_\varepsilon(t) = t + \alpha_\varepsilon(t) \text{ if } t \in]\varepsilon, 1].$$

Then, we get that $\widetilde{\omega}_\varepsilon = \inf_{t\in[0,1]} \widetilde{\psi}_\varepsilon(t) = 2\varepsilon = \psi_\varepsilon(\varepsilon)$, so the unique $\widetilde{\mathfrak{M}}$-viscosity solution is $t_o = 0$.

On the contrary, if we compute the function $\psi_\varepsilon(t) = \sup_{y\in\mathcal{M}^\varepsilon(t)} (t+y)$ we get that:

$$\psi_\varepsilon(t) = t+1 \text{ if } t \in [0,\varepsilon], \qquad \psi_\varepsilon(t) = t + \alpha_\varepsilon(t) \text{ if } t \in]\varepsilon, 1],$$

and $\omega_\varepsilon = \inf_{t\in[0,1]} \psi_\varepsilon(t) = 2\varepsilon$. Then, $\lim_{\varepsilon\to 0} \omega_\varepsilon = 0 = \omega$, nevertheless it does not exist any minimum point for ψ_ε and, consequently, any $\mathfrak{M}$-viscosity solution for $(PBOP)$.

Finally, the last example shows that a viscosity solution for $(PBOP)$ does not necessarily solve the Optimistic Bilevel Optimization Problem [21]:

$$(OBOP) \quad \text{find } (\widetilde{t}, \widetilde{y}) \in T' \times Y' \text{ such that } \widetilde{y} \in \mathcal{M}(\widetilde{t}) \text{ and } L(\widetilde{t}, \widetilde{y}) \;=\; \inf_{t\in T'} \inf_{y\in\mathcal{M}(t)} L(t,y).$$

Example 5.3 Consider the map $K(t) = [0,1]$ for $t \in [0,1]$ and the functions $F(t,y) = t(t-1/2)y$, $L(t,y) = t+y$ for $(t,y) \in [0,1]^2$. It is easy to check that $(PBOP)$ does not have a solution whereas the unique viscosity solution, $t = 1/2$, is different from the unique solution $t = 0$ for $(OBOP)$.

6 Conclusions

The Pessimistic Bilevel Optimization Problem can modelize the behavior of a risk adverse Leader which would like to solve a *minsup* problem aiming to reach the *security value* ω. When this problem does not have a solution, which may arise even under strong assumptions, the leader can make do with alternative solution concepts and the $\mathfrak{T}$-viscosity solutions, as defined here, represent a possible and satisfactory surrogate to the lack of exact solutions since they allow to realize the value ω under suitable hypotheses. In this paper, we have presented several classical approximation maps for the lower level optimization problems $P(t)$ and we have investigated when they define inner regularizations finding that, among all of them, the family of strict ε-minimum maps needs fewer assumptions. In our definitions and results, we have appropriately balanced continuity and compactness-like properties using the strong and weak convergence property features, in line with the approach of U. Mosco [16], and we have got reasonable sufficient conditions for the existence of such solutions in infinite dimensional spaces. Future developments of this topic will concern stability properties as well necessary conditions for the existence of viscosity solutions, also possibly with respect to other inner regularization families.

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