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# *Strict Fairness of Equilibria in Mixed and Asymmetric Information Economies*

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**Chiara Donnini [*] and Marialaura Pesce[**]**

**Abstract**

We investigate the fairness property of equal-division competitive market equilibria (CME) in asymmetric information economies with a space of agents that may contain non-negligible (large) traders. We first propose an extension to our framework of the notion of strict fairness due to Zhou (1992). We prove that once agents are asymmetrically informed, any equal-division CME allocation is strictly fair, but a strictly fair allocation might not be supported by an equilibrium price. Then, we investigate the role of large traders and we provide two sufficient conditions under which, in the case of complete information economies, a redistribution of resources is strictly fair if and only if it results from a competitive mechanism.

**JEL Classification**: D43, D60, D82

**Keywords**: Asymmetric information, mixed markets, strict fairness, competitive equilibrium.

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# 1 Introduction

In this paper we study the fairness properties of equal-division competitive market equilibrium (CME) allocations in exchange economies with uncertainty and asymmetrically informed agents. We adopt a model presenting a two-fold generality: regarding the space of agents and the informational asymmetries among traders. Concerning the space of agents we consider economies having both atoms (large or non-negligible traders) and an atomless sector (small or negligible traders). This enables us to consider simultaneously finite economies, non-atomic economies as well as the so-called mixed markets. Concerning the agents' private information we assign to each individual two partitions of the set of states of nature representing respectively the information at the time of contracting and the information at the time of contracts delivery. This allows us to cover as particular cases most of the models of asymmetric information economy presented in the literature (see Basile et al. (2014) for details).

We propose an extension to our framework of the notion of strict fairness introduced by Zhou (1992) in complete information economies, for which each agent compares her bundle with any other coalition's average bundle. Zhou (1992) shows that in replica as well as in atomless economies, under a suitable set of assumptions, the only strictly fair allocations are the equal-division Walrasian allocations. In order to ensure that asymmetrically informed agents have equal initial opportunities, we require that each trader can look at only those individuals with the same initial endowment and the same private information signals in the states in which the envy is evaluated (not necessarily the same private information partitions). We prove that any equal-division CME allocation is strictly envy-free, guaranteeing in this way the existence of a strictly fair allocation. Furthermore, we show that in general the converse does not hold, that is the equivalence proved by Zhou (1992) fails in our context. With the goal to examine the responsability in this failure of the informational asymmetries and of the presence of non-negligigle traders, we first show that in asymmetric information economies, regardless the presence of the atoms, there might exist a strictly fair allocation not supported by an equilibrium price. Successively, to focus on the role of large agents we restrict our attention to complete information economies. We show that even with an arbitrarily large finite number of non-negligible traders with identical initial endowment, tastes and weight, there might exist a strictly fair allocation that is not a Walrasian allocation. This might be judged not surprising, but what is interesting and motivates our analysis is that the Core-Walras equivalence theorem holds under milder assumptions (see Shitovitz (1973) and Greenberg and Shitovitz (1986)). The problem lies in the fact that the presence of identical non-negligible agents, although many, is not enough to manipulate the measure of an envied coalition. This is because the average value of an allocation over larger coalition might be less desirable for an envious individual. Furthermore, contrary to the core, an enlargement of the envied coalition has no bearing on the set of envious agents, but it does only affect the preferred bundle.

This incites us to look for different assumptions in order to crumble the market power of the atoms. We provide two sufficient conditions under which a redistribution of resources is strictly fair if and only if it results from a competitive mechanism. The former needs the existence of countable infinitely many large traders with the same tastes, imposing implicitly a condition on the measure of the atoms. The latter requires for each atom the presence of a fringe of negligible traders with the same characteristics. This assumption is introduced in Gabszewicz and Mertens (1971) to prove the Core-Walras equivalence theorem in mixed markets. It also appears in Shitovitz (1992) to characterize competitive equilibria via a notion of coalitional fairness in a monopoly in which the measure of the atomless fringe is larger than the measure of the unique atom. Moreover, it is also used in a stronger version by Basile et al. (2016) for economies with public goods. Under one of these two hypotheses we prove that the mixed market can be considered equivalent to an associated atomless economy in terms of strictly fair allocations. We show that this one-to-one correspondence is not valid in general neither under conditions that are stronger than those used by Greenberg and Shitovitz (1986) for the core.

The paper is organized as follows. In Section 2 we present the model, the main definitions and we investigate the notion of strict fairness in asymmetric information economies. In Section 3 we focus on complete information economies and we provide two sufficient conditions under which the coincidence between the sets of equal-division Walrasian equilibria and of strictly fair allocations is restored. Section 4 contains some concluding remarks. The proofs are collected in the Appendix.

## 2 Strict fairness in asymmetric information economies

In this section we analyze the notion of strict fairness in economies under uncertainty and asymmetric information. The space of agents is a complete probability space $(T, \mathcal{T}, \mu)$ where $T$ is the set of agents and $\mathcal{T}$ is the $\sigma$-field of all eligible coalitions, whose economic weight on the market is given by the measure $\mu$. According to the atomless-atomic decomposition of measures, $T$ is partitioned into an atomless set $T_0$ and a set $T_1 = T \setminus T_0$ which is the union of at most countably many $\mu$-atoms. The set $T_0$ is representative of the *"small"* traders (price takers, negligible agents), while the family $\{A_1, A_2, \dots, A_k, \dots\}$ of $\mu$-atoms represents the non-negligible or *"large"* traders (oligopolies, cartels, syndicates). Any $\mu$-atom $A$ is treated as a single agent, then with an abuse of notation $T_1 = \{A_1, A_2, \dots, A_k, \dots\}$ and we use $A \in T_1$ instead of $A \subseteq T_1$. Such a model covers the case of discrete economies, non-atomic economies as well as mixed economies in which both sets $T_0$ and $T_1$ have positive $\mu$-measure. We identify physical commodities with elements of $\mathbb{R}^\ell_+$.

For what concerns the uncertainty, we consider a probability space $(\Omega, 2^\Omega, \pi)$ where $\Omega$ is the finite set of possible states of nature; $2^\Omega$ is the power set of $\Omega$ containing all the events and $\pi$ is a strictly positive common prior which describes the relative probability of the states. The private information of agents is represented by means of partitions of $\Omega$. With an abuse of notation, we use the same symbol for a partition and the algebra of subsets of $\Omega$ it generates.[1] There are several models in the literature of asymmetric information economies depending on what agents know when they write contracts and what they know when consumption takes place, with consequent several equilibrium notions not always related each other. In order to unify different settings with a unique model, we consider the so-called *generalized interim model*, introduced by Basile et al. (2014), according to which each agent $t$ is endowed with two exogenous partitions $\mathcal{F}_t$ and $\mathcal{G}_t$ of $\Omega$, with $\mathcal{G}_t$ finer than $\mathcal{F}_t$, representing respectively the information at the time of contracting and the information at the time of contracts delivery (see Basile et al. (2014) for details). The interpretation is as usual: if $\omega$ is the current state of nature, agent $t$ with information partition $\mathcal{G}_t$ observes the unique event $\mathcal{G}_t(\omega)$ of $\mathcal{G}_t$ containing $\omega$. Agents are assumed to consume the same bundles in states they do not distinguish at the time of consumption. Therefore, the consuption set of $t$, denoted by $M_t$, consists of random bundles constant over any event of $\mathcal{G}_t$, i.e., $M_t = \left\{x : \Omega \to I\!R^\ell_+ \ : \ x(\cdot) \text{ is } \mathcal{G}_t - \text{measurable}\right\}$. We assume that $M_t$ is non empty as it contains at least the initial endowment named $e_t$. The preferences of agent $t$ are represented by a state-dependent utility function $u_t : \Omega \times \mathbb{R}^\ell_+ \to \mathbb{R}$. At the time of contracting $t$ might be differently informed and receive a coarser private information signal $\mathcal{F}_t(\omega)$ based on which she evaluates contingent bundles, that is by means of the so-called *interim expected utility function*

$$V_t(x|\mathcal{F}_t(\omega)) = \sum_{\omega' \in \mathcal{F}_t(\omega)} u_t(\omega', x(\omega')) \frac{\pi(\omega')}{\pi\left(\mathcal{F}_t(\omega)\right)}.$$

Throughout the paper we assume that for all $\omega \in \Omega$ the mapping $(t, x) \mapsto u_t(\omega, x)$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^\ell_+)-$measurable, where $\mathcal{B}(\mathbb{R}^\ell_+)$ is the $\sigma$-field of Borel subsets of $\mathbb{R}^\ell_+$; the function $u_t(\omega, \cdot)$ is continuous, strictly increasing and concave on $I\!R^\ell_+$. Since $\Omega$ is finite, there is a finite number of different information partitions of $\Omega$: $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ and $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$. We assume that the sets of

[1]See Hervés-Beloso and Monteiro (2013) for the relation among partitions, signals, information and $\sigma$-algebra.

information types $T_i^{\mathcal{F}} = \{t \in T : \mathcal{F}_t = \mathcal{F}_i\}$ and $T_j^{\mathcal{G}} = \{t \in T : \mathcal{G}_t = \mathcal{G}_j\}$ are measurable.

An *allocation* is a function $x : T \times \Omega \to I\!R_+^\ell$ such that for each $\omega \in \Omega$, $x(\cdot, \omega)$ is $\mu$-integrable and for each $t \in T$, $x(t, \cdot) \in M_t$.[2] An allocation $x$ is said to be *feasible* if $\int_T x_t(\omega) d\mu \leqslant \int_T e_t(\omega) d\mu$ for all $\omega \in \Omega$. A feasible allocation $x$ is *efficient* or *Pareto optimal* if there does not exist an alternative feasible allocation $y$ such that $V_t(y_t|\mathcal{F}_t(\omega)) \geqslant V_t(x_t|\mathcal{F}_t(\omega))$ for almost all $t \in T$ and for all $\omega \in \Omega$, with a strict inequality for members of a certain coalition $S$ in at least one state $\bar{\omega}$.[3] The notion of Walrasian equilibrium has been extended in this framework by Basile et al. (2014) as follows. A feasible allocation $x$ is a *competitive market equilibrium (CME) allocation* if there exists a price $p$, which is a non-zero function $p : \Omega \to I\!R_+^\ell$, such that for all $\omega \in \Omega$ and almost all agent $t \in T$, $x_t \in argmax_{y \in B_t(p,\omega)} V_t(y|\mathcal{F}_t(\omega))$, where

$$B_t(p, \omega) = \left\{ y \in M_t : \sum_{\omega' \in \mathcal{F}_t(\omega)} p(\omega') \cdot y(\omega') \leqslant \sum_{\omega' \in \mathcal{F}_t(\omega)} p(\omega') \cdot e_t(\omega') \right\}.$$

In the fairness literature, in order to avoid any initial advantage among individuals, it is common to assume that the total initial endowment is equally distributed among agents, i.e., $e_t(\cdot) = e(\cdot)$ for all $t \in T$. In this case, $x$ is called *equal-division CME allocation*.
By specifying the information algebra $\mathcal{F}_t$ and $\mathcal{G}_t$, we cover different models of asymmetric information economies described in the literature and the relative equilibrium concepts (see Remark 2.3 of Basile et al. (2014)). In particular, fixing appropriately the algebras $\mathcal{F}_t$ and $\mathcal{G}_t$, the CME allocations reduce to be the Walrasian expectations equilibria of Radner (1968), the constrained market equilibria of Wilson (1978) and the Walrasian equilibria in complete information economies. An exception is the concept of rational expectations equilibria, introduced by Radner (1979), according to which agents take into account also the information generated by the equilibrium prices. In our model $\mathcal{G}_t$ is indeed exogenously given and agents can not improve their information via the endogenous signal given by the prices.

In a complete information economy, an allocation $x$ is envy-free or equitable if each individual prefers to keep her own bundle rather than the bundle of any other, i.e., $u_t(x_t) \geq u_t(x_s)$ for any $t, s \in T$. A feasible allocation is fair if it is both envy-free and efficient. It is straighforward to prove that any equal-division Walrasian allocation is fair, but in general the converse does not hold. Zhou (1992) introduces a stronger notion of envy for which each

[2]We often denote by $x_t$ the random consumption bundle $x(t, \cdot)$ of agent $t$ and by $x_t(\omega)$ the bundle $x(t, \omega) \in I\!R_+^\ell$.

[3]Equivalently, since $\Omega$ is finite, different members of the coalition $S$ can be strictly better off with respect to different states, that is the state $\bar{\omega}$ can be agent-dependent.

individual $t$ compares her bundle not just with the bundle of other agents, but also with the average bundle of coalitions she does not belong to. Formally, $t$ strictly envies a coalition $S$, with $t \notin S$, if $u_t(\bar{x}(S)) > u_t(x_t)$, where $\bar{x}(S) = \frac{1}{\mu(S)} \int_S x_s \, d\mu$. In a sense, $t$ envies the possibility to join the coalition $S$ because she prefers what she would get in average being a member of $S$ rather than what she gets alone. Our first goal is to extend the notion of strict fairness to asymmetric information economies. To this, we must care about the information of the envied coalition. Indeed, each trader may be partially informed at the time of contracts delivery and any individual should compare her own bundle, which is measurable with respect to her own private information, with the average bundle of a certain coalition. But what is the information of this coalition? With respect to which algebra the average bundle must be measurable? Clearly there is not a unique extension of strict fairness to asymmetric information economies. A possibility could be for an agent $t$ to look at only those coalitions $S$ for which $\bar{x}(S)$ is $\mathcal{G}_t$-measurable, but this natural requirement leads a vacuous notion as shown below.

**Example 2.1.** Consider the economy described in de Clippel (2008) with two equiprobable states of nature $\Omega = \{a, b\}$, one good and three agents $T = \{1, 2, 3\}$ such that

$$\begin{array}{lll} \mathcal{F}_1 = \{\{a,b\}\} & \mathcal{G}_1 = \{\{a\},\{b\}\} & u_1(a,x_1) = u_1(b,x_1) = x_1 \\ \mathcal{F}_2 = \{\{a,b\}\} & \mathcal{G}_2 = \{\{a\},\{b\}\} & u_2(a,x_2) = u_2(b,x_2) = \sqrt{x_2} \\ \mathcal{F}_3 = \{\{a\},\{b\}\} & \mathcal{G}_3 = \{\{a\},\{b\}\} & u_3(a,x_3) = u_3(b,x_3) = x_3. \end{array}$$

The total initial endowment is 1200 in state $a$ and 1800 in state $b$. The unique feasible strictly envy-free allocation $x^*$ equally shares the total initial endowment among the agents, that is $x_t^*(a) = 400$ and $x_t^*(b) = 600$ for $t = 1, 2, 3$. This allocation is Pareto dominated by $y(a) = (301, 498, 401)$ and $y(b) = (701, 498, 601)$. Therefore, the set of strictly fair allocations, as defined above, is empty and it does not include the set of equal-division CME allocations which contains $x(a) = (300, 500, 400)$ and $x(b) = (700, 500, 600)$. △

In asymmetric information economies the incompatibility between envy-freeness and efficiency relies on the fact that agents' budget set depends not only on their initial endowments but also on their private information. Thus, since asymmetrically informed agents have different budget set, an equal-division CME allocation may not be envy-free. It is then clear that if we want to start from an equity status and see if the CME is a redistribution mechanism that preserves a form of fairness, we should avoid any kind of advantage and ensure to individuals the same initial opportunities, which means in asymmetric information economies the same initial endowments and the same private information signals. This idea can be formalized by the following definition. Given an allocation $x$, for all $t \in T$ and $\omega \in \Omega$ let

$$C_t(\omega) = \{s \in T \, : \, \mathcal{F}_t(\omega) = \mathcal{F}_s(\omega) \text{ and } \mathcal{G}_t(\omega') = \mathcal{G}_s(\omega') \text{ for all } \omega' \in \mathcal{F}_t(\omega)\}$$

be the set of agents receiving in state $\omega$ the same private information signals of $t$, not necessary the same private information partitions.

**Definition 2.2.** *An agent $t$ strictly envies a coalition $S$ in state $\omega \in \Omega$ at $x$ if $\mu(S) > 0$, $t \notin S$, $S \subseteq C_t(\omega)$ and $V_t(\bar{x}(S)|\mathcal{F}_t(\omega)) > V_t(x_t|\mathcal{F}_t(\omega))$. Agent $t$ is said to be strictly envious at $x$ if she strictly envies some coalition $S$ in some state $\omega$. The allocation $x$ is* **strictly envy-free** *or* **strictly equitable** *if the set of strictly envious agents at $x$ has $\mu$-zero measure. Finally, $x$ is said to be* **strictly fair** *if it is both strictly equitable and efficient.*

We denote by $SF$ the set of strictly fair allocations. If the bundle-comparison is with one single agent, we get the notion of (individual) fairness introduced by Basile et al. (2014), and in complete information economies we end up with the definition of Zhou (1992). According to Definition 2.2, each trader does not care the tastes of the others, she just considers their private information structures, their consumption bundles and she needs to aggregate.[4] Note that we do not force agent $t$ to look at only those individuals with the same private information algebras $\mathcal{F}_t$ and $\mathcal{G}_t$. We only require that at least in the states in which equity is evaluated, agent $t$ can not envy individuals with different private information signals, that is with potential edge. This limitation also ensures that agent $t$ learns no additional information after the bundle-comparison with the others, and we show below that it is not comparable with the $\mathcal{G}_t$-measurability requirement of the average bundle.

**Example 2.3.** Consider the same asymmetric information economy illustrated in Example 2.1. We have already observed that by limiting any agent $t$ to look at only those coalitions with a $\mathcal{G}_t$-measurable average bundle, the set of strictly fair allocations is empty. On the other hand, Definition 2.2 returns a non vacuous notion because the set $SF$ contains the equal-division CME allocation $x(a) = (300, 500, 400)$ and $x(b) = (700, 500, 600)$. This inclusion, which always holds (see Proposition 2.4 below), is strict because the set $SF$ also contains, for instance, the allocation $x(a) = (500, 500, 200)$ and $x(b) = (500, 500, 800)$.

Conversely, consider an economy with four equiprobable states of nature $\Omega = \{a, b, c, d\}$, one good and four agents $T = \{1, 2, 3, 4\}$ such that $u_t(\cdot, x_t) =$

[4] This is consistent with the assumption that the economy is common knowledge, which is a standard requirement in the asymmetric information framework.

$x_t$ for any $t \in T$ and

$$\mathcal{F}_t = \mathcal{G}_t = \begin{cases} \{\{a,b\},\{c\},\{d\}\} & \text{if } t = 1, \\ \{\{a,b\},\{c,d\}\} & \text{if } t = 2, \\ \{\{a,b,c,d\}\} & \text{if } t = 3, \\ \{\{a\},\{b\},\{c\},\{d\}\} & \text{if } t = 4. \end{cases}$$

The total initial endowment is $(e(a), e(b), e(c), e(d)) = (7, 6, 9, 7)$. Consider the following efficient allocation

$$\begin{aligned} (x_1(a), x_1(b), x_1(c), x_1(d)) &= (2, 2, 3, 2) \\ (x_2(a), x_2(b), x_2(c), x_2(d)) &= (1, 1, 2, 2) \\ (x_3(a), x_3(b), x_3(c), x_3(d)) &= (1, 1, 1, 1) \\ (x_4(a), x_4(b), x_4(c), x_4(d)) &= (3, 2, 3, 2) \end{aligned}$$

and notice that for any $t \in T$ and any coalition $S$ with $t \notin S$, $\bar{x}(S)$ is either not $\mathcal{G}_t$-measurable or $V_t(\bar{x}(S)|\mathcal{G}_t(\omega)) \leq V_t(x_t(\omega)|\mathcal{G}_t(\omega))$ for all $\omega \in \Omega$. On the other hand, $x$ is not strictly fair according to Definition 2.2, because agent 2 strictly envies agent 1 in states $a$ and $b$. Observe that $x_1$ is not $\mathcal{G}_2$-measurable. $\triangle$

We now show that Definition 2.2 is not groundless because any equal-division CME allocation is strictly fair and under certain assumptions a CME allocation exists (Theorem 4.2 in Basile et al. (2014)). Moreover, it is a stronger concept of envy because any strictly fair allocation is (individual) fair.

**Proposition 2.4.** *Any equal-division CME allocation is strictly fair and any strictly fair allocation is (individual) fair.*

*Proof.* See Appendix.

In the light of Proposition 2.4, if initially agents have the same opportunities, the CME is an arrangement mechanism of resources that leads to strictly fair allocations, and so it keeps, after the redistribution, some principle of equity. We now show that once agents are asymmetrically informed, nevertheless the absence of large traders, there might exist a strictly fair allocation not supported by an equilibrium price. In the next example we consider a continuum of negligible traders to underline the role of the informational asymmetries among individuals in the failure of the equivalence between equal-division CME allocations and strictly fair allocations. The effect of large traders is investigated in the next section. However, Example 2.5 below can be rewritten to allow the presence of atoms.

**Example 2.5.** Consider an atomless asymmetric information economy with two equiprobable states of nature $\Omega = \{a, b\}$, with $\mathbb{R}^2_{++}$ as commodity set and $T = (0, 1)$ as the set of agents. The total initial endowment is

$e = (1,1) \gg 0$. Agents' utility functions are given by $u_t(x,y) = \log(x+y)$ for all $t \in T$ and all $\omega \in \Omega$; while their private information by

$$\mathcal{F}_t = \mathcal{G}_t = \begin{cases} \{\{a,b\}\} & \text{if } t \in \left(0, \frac{1}{2}\right) =: I_1 \\ \{\{a\},\{b\}\} & \text{if } t \in \left[\frac{1}{2}, 1\right) =: I_2, \end{cases}$$

hence agents in $I_1$ are uninformed and members of $I_2$ are perfectly informed. Consider the following feasible allocation

$$(x(t,\cdot), y(t,\cdot)) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } t \in I_1 \\ \left(\frac{3}{2}, \frac{3}{2}\right) & \text{if } t \in I_2, \end{cases}$$

which is clearly efficient and notice that it is also strictly equitable. Indeed, for all $\omega$ and for all $i \in \{1,2\}$, if $t \in I_i$ then $C_t(\omega) = I_i$, moreover $(x,y)$ is constant on each set $I_i$. On the other hand, $(x,y)$ is not an equal-division CME allocation, because any uniformed individual prefers the initial endowment. $\triangle$

**Remark 2.6.** The above example points out that the presence of asymmetrically informed agents breaks the coincidence, obtained by Zhou (1992) for complete information economies, between the sets of equal-division Walrasian allocations and of strictly fair allocations. This originates from the restriction imposed on the coalitions that an agent can look at, but at the same time, we have shown that without this constraint a strictly fair allocation might not exist (see Example 2.1) and the equivalence would fail as well.

## 3 Strict fairness in mixed economies

In the light of the above considerations, in order to characterize Walrasian equilibria by means of strictly fair allocations, we need to restrict our attention to the particular case of symmetrically informed agents, or equivalently to complete information economies, in which the interim expected utility reduces to be the ex-post utility function $u$ and the measurability constraints play no role. For simplicity we skip the dependence on the state in the notations. In this setting the feasibility can be equivalently written with an equality and the notion of Pareto dominance with a strict inequality for each agent $t \in T$. Moreover the concavity assumption can be weakened with the quasi-concavity and Definition 2.2 coincides with the notion of strict fairness due to Zhou (1992).

For some results we need $u$ to be strictly quasi-concave, differentiable on $\mathbb{R}^\ell_{++}$ and satisfying the boundary condition: for each $x \in \partial \mathbb{R}^\ell_+$ and $y \in \mathbb{R}^\ell_{++}$, $u(x) < u(y)$. The strict quasi-concavity is necessary for the equal treatment property (Lemma 3.7) and it is also used by Zhou (1992) together with the differentiability to obtain his equivalence results. The boundary condition is technical and replaced in some papers by focusing on the interior of the consumption set (see Zhou (1992) and also Remark 3.3 below). A stronger assumption used in the literature imposes that boundary bundles are indifferent, that is $u(x) = u(0)$ for all $x \in \partial I\!R^{\ell}_{+}$. We avoid it because it is incompatible with the strict monotonicity and the strict quasi-concavity.

For the rest of the paper, we denote by $e$ the total initial endowment of the economy which is assumed to be strictly positive and equally distributed among traders, i.e., $e = \int_T e(t)d\mu \in I\!R^{\ell}_{++}$ and $e(t) = e$ for all $t \in T$; and by $\mathcal{E}$ the mixed complete information economy $\{(T, \mathcal{T}, \mu), I\!R^{\ell}_{+}, e, (u_t)_{t \in T}\}$. Remember that two agents are said to be identical or of the same type if they have the same characteristics, that is the same utility function and same initial endowment. Two identical traders are said to be of the same kind if they also have the same measure (see Shitovitz (1973)). Some interesting considerations on the notion of strict envy-freeness are remarked below.

**Remark 3.1.** It can be proved that given an allocation $y$, a vector $k \in I\!R^{\ell}_{+}$ and a coalition $S$ such that $u(y(t)) > u(k)$ for almost all $t \in S$, $u(\bar{y}(S)) > u(k)$. The implication holds also with "$\geq$" instead of "$>$" (see Lemma in García-Cutrín and Hervés-Beloso (1993) and Lemma 7.1 in Basile et al. (2017)). Thus, Definition 2.2 can be rewritten by requiring that an agent $t$ strictly envies a coalition $S$ at an allocation $x$, if for some alternative allocation $y$ with the same average value over $S$ of $x$, $t$ prefers the bundle of each member of $S$ to her own, that is

$$
\begin{array}{ll}
(i) & u_t(y(s)) > u_t(x(t)) \text{ for almost all } s \in S, \text{ and} \\
(ii) & \bar{y}(S) = \bar{x}(S).
\end{array}
$$

**Remark 3.2.** Notice that in atomless economies (i.e., $T_1 = \emptyset$) any feasible strictly envy-free allocation $x$ is individually rational. Indeed, if there is some coalition $I$ with positive measure such that $u_t(e) > u_t(x(t))$ for almost all $t \in I$, then almost any member $t$ of $I$ strictly envies the coalition $T \setminus \{t\}$ because $u_t(x(t)) < u_t(e) = u_t(\bar{x}(T)) = u_t(\bar{x}(T \setminus \{t\}))$, which is a contradiction.

**Remark 3.3.** We can remark that under the boundary condition, any strictly fair allocation $x$ is strictly positive. If there exists a coalition $S \subseteq T_0$ such that for almost every $t$ in $S$, $x(t) \in \partial I\!R^{\ell}_{+}$; since $e \in I\!R^{\ell}_{++}$ it follows that $u_t(\bar{x}(T \setminus \{t\})) = u_t(\bar{x}(T)) = u_t(e) > u_t(x(t))$, which is a contradiction. Hence, $x(t) \gg 0$ for almost every $t$ in $T_0$ and a fortiori $\bar{x}(T_0) \in I\!R^{\ell}_{++}$. Now, if $\mu(T_0) > 0$ and there exists an atom $A$ for which $x(A) \in \partial I\!R^{\ell}_{+}$, then $u_A(\bar{x}(T_0)) > u_A(x(A))$, which is again a contradiction. On the other hand, if $\mu(T_0) = 0$ and there exist $A_1$ and $A_2$ in $T_1$ such that $x(A_1) \gg 0$ and $x(A_2) \in \partial I\!R^{\ell}_{+}$, $A_2$ strictly envies $A_1$. Finally, if for all agent $A$ in $T_1$, $x(A) \in \partial I\!R^{\ell}_{+}$ then $e$ blocks $x$, which denies the efficiency of $x$.

**Remark 3.4.** With similar arguments used for the core (Claim 1 in Greenberg and Shitovitz (1994)), it can be proved that given a non equitable allocation $x$, any envious agent at $x$ strictly envies a coalition containing at most a finite number of atoms.

In Figure 3 Zhou (1992) illustrates a mixed economy with two non-negligible traders with the same utility function, in which a strictly fair allocation is not an equal-division Walrasian allocation. This means that the hypotheses guaranteeing the Core-Walras equivalence theorem (see Shitovitz (1973)) are not enough to characterize the competitive equilibria through the strictly fair allocations. Actually, we next prove that even allowing the presence of an arbitrary large finite number of atoms of the same kind, there might exist a strictly fair allocation not supported by an equilibrium price.

**Proposition 3.5.** *In a mixed economy with $n$ atoms of the same kind, there might exist a strictly fair allocation which is not an equal-division Walrasian allocation.*

*Proof.* See Appendix.

With $n = 2$, Proposition 3.5 formalizes the idea presented in figure 3 by Zhou (1992). It also implies that in a mixed economy with two or more identical atoms, thanks to the Core-Walras equivalence theorem, the core and the set of coalitional fair allocations as defined in Varian (1974) (see also Gabszewicz (1975) and Yannelis (1985) for a notion of coalitional equitable net trade), are properly contained in the set of strictly fair allocations.

Following the idea of Greenberg and Shitovitz (1986), it is possible to construct an atomless economy $\mathcal{E}^*$ associated to the mixed market $\mathcal{E}$ by splitting each atom $A$ into a coalition $A^*$ of negligible traders all identical to $A$ (see Appendix for details). Each Walrasian allocation $x$ of $\mathcal{E}$ corresponds to a Walrasian allocation $x^*$ of $\mathcal{E}^*$ and vice versa. The same holds for efficient allocations, whereas for the core only if $T_1$ contains identical atoms and they are at least two. In what follows we show that, although the presence of an arbitrary large finite number of non-negligible identical traders, a strictly fair allocation $x$ of a mixed market $\mathcal{E}$ might correspond to an allocation $x^*$ which is not strictly equitable in the associated atomless economy $\mathcal{E}^*$. This is mainly due to the fact that, contrary to the core, in the associated atomless economy it is not possible to enlarge the measure of an envied coalition as much as we want, because an envious agent might not prefer the average bundle of a larger coalition. Moreover, a widening of the envied coalition impacts only on the preferred bundle, not on the set of envious individuals.

**Proposition 3.6.** *Let $\mathcal{E}$ be a mixed economy with $n$ large traders of the same kind and let $x$ be a strictly envy-free allocation. The corresponding allocation $x^*$ of the associated atomless economy $\mathcal{E}^*$ might not be strictly equitable.*

*Proof.* See Appendix.

We now provide two sufficient conditions for the coincidence between the sets of equal-division Walrasian equilibria and of strictly fair allocations. Our first equivalence holds under the presence of countable infinitely many atoms with same tastes, by implicitly imposing, contrary to Shitovitz (1973), a condition on their measure. The second result requires the existence for each atom of a coalition of negligible identical agents (atomless fringe), but does not impose any restriction on their number and any relation among them. This second assumption is in the same spirit of Gabszewicz and Mertens (1971) and it has been used by Shitovitz (1992) to characterize Walrasian allocations by means of a notion of coalitional fairness in a monopoly. Actually, Shitovitz (1992) imposes an additional restriction on the measure of the atomless fringe. Similar but stronger assumptions are made in Basile et al. (2016) for a different goal in a different context; precisely all atoms must be of the same type even once there are small traders identical to them. With these two sufficient conditions we prove the one-to-one correspondence between the economies $\mathcal{E}$ and $\mathcal{E}^*$ in terms of strict envy-freeness. From this and from the equivalence obtained by Zhou (1992) in atomless economies, the coincidence in mixed markets between the sets of equal-division Walrasian allocations and of strictly fair allocations directly follows.

In proving our main result, two interesting preliminary properties of strict fairness are needed. First, in atomless economies, thanks to the Lyapunov convexity theorem one can arbitrarily reduce the measure of an envied coalition (see also footnote 3 in Zhou (1992)). Second, a strictly equitable allocation assigns the same bundle to identical agents. Thus, strict fairness satisfies a natural fundamental principle of any equity concept that is "*equals are treated equally*".

**Lemma 3.7.** *Let $\mathcal{E}$ be a mixed economy. Let $x$ be a strictly equitable allocation and $C$ be a coalition containing agents with the same strictly quasi-concave utility function $u$. If $\mu(C \setminus T_1) > 0$ or if $C$ contains at least three atoms, then $x(t) = \bar{x}(C)$ for almost all $t \in C$.*

*Proof.* See Appendix.

**Proposition 3.8.** *If one of the following statements holds*

1. *$T_1$ is countably infinite and all atoms have the same strictly quasi-concave utility function,*

2. *for each atom $A$ there exists a coalition with positive $\mu$-measure of negligible agents having the same strictly quasi-concave utility function of $A$,*

*then any strictly envy-free allocation of the mixed economy $\mathcal{E}$ corresponds to a strictly equitable allocation of the associated atomless economy $\mathcal{E}^*$.*

*Proof.* See Appendix.

The converse implication also holds without any extra assumption (see Proposition 5.1 in the Appendix). From the above result, our main theorem derives.

**Theorem 3.9.** *Assume that for almost every $t$ in $T$, $u_t$ is strictly quasi-concave, differentiable on $\mathbb{R}^{\ell}_{++}$ and satisfying the boundary condition. If one of the following statements holds*

1. *$T_1$ consists in countable infinitely many identical atoms,*
2. *for each atom $A$ there exists a coalition with positive $\mu$-measure of negligible agents identical to $A$,*

*then the equal-division Walrasian allocations are the only strictly fair allocations.*

*Proof.* See Appendix.

**Remark 3.10.** Notice that in the proof of Propositions 3.5 and 3.6 none of the two sufficient conditions is satisfied. Theorem 3.9 gives a very nice interpretation of the Walrasian equilibria in terms of economic equity, as it states that a redistribution of resources, that initially are equally divided among agents, is strictly fair if and only if it emerges from a competitive mechanism. Furthermore, it allows us to compare the set of strictly fair allocations with the core of a mixed economy, because they both coincide with the set of equal-division Walrasian allocations.

# 4 Conclusions

Throughout the paper we assume that the initial endowment is equally shared among individuals, because we want to characterize those redistribution processes that inherit a form of fairness by dividing fairly a fixed amount of resources among agents. A related study involves the notion of strictly fair net trade and the problem to conduct fair trades among agents with given initial endowment. This line of investigation is followed for example by Basile et al. (2017) who consider a coalitional notion of envy-free net trade based on the Aubin approach and compare it with the core and the set of equilibria of a mixed economy with infinitely many commodities. By combining their results with the equivalence obtained in our paper, further interpretations of strictly fair allocations arise at least in finite dimensional economies. It would be interesting to extend our results to economies with infinitely many commodities in which the Lyapunov convexity theorem does not hold. The concept of fair net trade is particularly useful when agents are asymmetrically informed because the equal-division restriction prevent us from employing the type-agent representation (see de Clippel (2007)).

Each agent $t$ is endowed with two exogenous and arbitrary partitions of $\Omega$, $\mathcal{F}_t$ and $\mathcal{G}_t$, with $\mathcal{G}_t$ finer than $\mathcal{F}_t$. Although the information $\mathcal{G}_t$ is arbitrary, given a coalition $S$ containing $t$, $\mathcal{G}_t$ can not be viewed as the information got by $t$ after any information sharing process among members of $S$ that guarantees to forget nothing.[5] Indeed, contrary to the idea of information sharing rules, the information $\mathcal{G}_t$ does not vary according to the coalition that $t$ joins. Furthermore, since $\mathcal{G}_t$ is exogenous, it does not consider the information inferred by some endogenous signal as for the case of rational expectations equilibria (REE). Under a suitable set of assumptions, Basile et al. (2017) characterize REE with the core and the set of coalitional fair allocations with personalized participation rates. Similar arguments can be used to extend their results in terms of strictly fair allocations.

A requirement behind the concept of strict fairness is that every agent knows the bundle of all the others and is able to compare the averages of these bundles with her own. One could judge this a strong requirement especially in large economies or in situations in which agents' aggregation ability is limited for some reasons. In atomless economies, the Lyapunov convexity theorem ensures that the measure of an envied coalition can be arbitrarily reduced. On the other hand, in a companion paper, we prove that it might be not possible to enlarge the measure of an envied coalition as much as we want, unless the agents are allowed to use only a fraction of their endowment, that is with a weaker notion of envy-freeness based on the Aubin approach. We formalize this notion that we call Aubin strict fairness, and we show that competitive market equilibria are the only Aubin strictly fair allocations regardless the number and the characterists of the atoms.

Zhou (1992) suggests in his concluding remarks that *"a more reasonable assumption is that an agent is aware of and sensitive to consumption bundles of only those agents to whom he can relate himself"*. Motivated by this observation, Cato (2010) introduces a notion of local fairness according to which every agent looks at only the bundle of her neighbors. Cato (2010) shows that in atomless economies local and global fairness coincide because they both fully characterize the equal-division Walrasian allocations. It is worthwhile to investigate on the possibility to extend our equivalence results in terms of local fairness.

[5]This is consistent with assumption ($P4$) of Hervés-Beloso et al. (2014).

# 5 Appendix

**The associated atomless economy $\mathcal{E}^*$**

Given a mixed economy $\mathcal{E} = \{(T, \mathcal{T}, \mu), I\!R^\ell_+, e, (u_t)_{t\in T}\}$ we construct the associated atomless economy $\mathcal{E}^* = \{(T^*, \mathcal{T}^*, \mu^*), I\!R^\ell_+, e, (u_t)_{t\in T^*}\}$, by splitting each atom $A$ of $T_1$ into a continuum of negligible traders identical to $A$, that forms an atomless coalition $A^*$ with the same measure of $A$. Denoting by $T^*_1$ the disjoint union of the atomless coalitions $A^*$, the measure space of agents $(T^*, \mathcal{T}^*, \mu^*)$ is the direct sum of $(T_0, \mathcal{T}_0, \mu)$ and $T^*_1$ endowed with the Lebesgue measure. The consumption set $I\!R^\ell_+$ and the initial endowment $e$ are unchanged; while agents' utility function are such that $u_t = u_A$ for each member $t$ of $A^*$.
For an allocation $x$ of the economy $\mathcal{E}$, we define over $T^*$ the allocation $x^*$ of the economy $\mathcal{E}^*$ by setting $x^* = x\chi_{T_0} + \sum_{A\in T_1} x(A)\chi_{A^*}$, where $\chi_C$ is the characteristic function of a set $C$. Reciprocally, given an allocation $x^*$ of $\mathcal{E}^*$, we define the allocation $x$ of $\mathcal{E}$ by setting $x = x^*\chi_{T_0} + \sum_{A\in T_1} \bar{x}^*(A^*)\chi_A$.

**Proof of Proposition 2.4** Let $x$ be an equal-division CME allocation and assume on the contrary that it is not strictly fair. Since it is efficient (Proposition 4.3 in Basile et al. (2014)), it follows that in some state $\bar{\omega}$ the set of envious individuals has positive $\mu$-measure. Any envious agent $t$ strictly envies a certain coalition $S$ at $x$, so that $t \notin S$, $S \subseteq C_t(\bar{\omega})$ and $V_t(\bar{x}(S)|\mathcal{F}_t(\bar{\omega})) > V_t(x_t|\mathcal{F}_t(\bar{\omega}))$. From $S \subseteq C_t(\bar{\omega})$, it follows that $\mathcal{F}_t(\bar{\omega}) = \mathcal{F}_s(\bar{\omega}) = \mathcal{F}(\bar{\omega})$ and $\mathcal{G}_t(\omega') = \mathcal{G}_s(\omega') = \mathcal{G}(\omega')$ for all $\omega' \in \mathcal{F}(\bar{\omega})$ and for all $s \in S$. Let $y : T \times \Omega \to I\!R^\ell_+$ be such that $y_s(\omega) = \bar{x}(S, \omega)$ if $s \in S$ and $\omega \in \mathcal{F}(\bar{\omega})$ and $y_s(\omega)$ equals to a state-independent positive vector $\bar{y}$ otherwise. Thus, $y_s(\cdot)$ is $\mathcal{G}_s$-measurable for all $s \in S \cup \{t\}$. Moreover $V_t(\bar{x}(S)|\mathcal{F}_t(\bar{\omega})) > V_t(x_t|\mathcal{F}_t(\bar{\omega}))$ implies that $V_t(y_s|\mathcal{F}_t(\bar{\omega})) > V_t(x_t|\mathcal{F}_t(\bar{\omega}))$ for all $s \in S$, and hence

$$
\begin{aligned}
0 &< \sum_{\omega'\in\mathcal{F}_t(\bar{\omega})} \big[p(\omega')\cdot y_s(\omega') - p(\omega')\cdot e(\omega')\big] = \\
&= \sum_{\omega'\in\mathcal{F}(\bar{\omega})} \left[p(\omega')\cdot \frac{1}{\mu(S)}\int_S x_s(\omega')\,d\mu - p(\omega')\cdot e(\omega')\right] = \\
&= \frac{1}{\mu(S)}\int_S \left[\sum_{\omega'\in\mathcal{F}(\bar{\omega})} \big[p(\omega')\cdot x_s(\omega') - p(\omega')\cdot e(\omega')\big]\right] d\mu \leqslant 0,
\end{aligned}
$$

which is a contradiction.

Now, let $x$ be a strictly envy-free allocation and assume to the contrary that the set of (individual) envious agents at $x$ has positive $\mu$-measure. According to Definition 3.6 of Basile et al. (2014), this means that for some state $\bar{\omega}$ and for any envious agent $t$ in $\bar{\omega}$, the set $S := \{s \in C_t(\bar{\omega}) \ : \ V_t(x_s|\mathcal{F}_t(\bar{\omega})) >$

$V_t(x_t|\mathcal{F}_t(\bar{\omega}))\}$ has positive $\mu$-measure. Then, $t \notin S$ and $S \subseteq C_t(\bar{\omega})$, moreover from concavity, $V_t(\bar{x}(S)|\mathcal{F}_t(\omega)) > V_t(x_t|\mathcal{F}_t(\omega))$, which contradicts that $x$ is strictly envy-free. $\square$

**Proof of Proposition 3.5** Consider a mixed economy whose consumption set is $I\!R^2_{++}$, $T = T_0 \cup T_1$ where $T_0 = \left[0, \frac{1}{2}\right]$ and $T_1 = \{A_1, \ldots, A_n\}$ with $\mu(A_i) = \frac{1}{2n}$ for every $i = 1, ..., n$. Note that $\mu(T_1) = \frac{1}{2}$. The total initial endowment is $e = (1,1) \gg 0$. Agents' utility function is given by

$$u_t(x,y) = \begin{cases} xy & \text{if } t \in T_0 \\ x^2 y & \text{if } t \in T_1. \end{cases}$$

Consider the following family of feasible allocations

$$(x(t), y(t)) = (a,b)\chi_{T_0} + (2-a, 2-b)\chi_{T_1} \tag{1}$$

with $b = \dfrac{(n-1)a+2}{n(2a-1)+1}$ and $a \in \left[\dfrac{2(3n+1)}{7n+3}, 1\right)$.

*FACT 1.* Any $(x,y)$ is strictly envy-free in $\mathcal{E}$.

For every coalition $S$ let us denote by $S_0$ and $S_1$ respectively the sets $S \cap T_0$ and $S \cap T_1$. Let $\alpha = \frac{\mu(S_0)}{\mu(S)}$, and therefore $1 - \alpha = \frac{\mu(S_1)}{\mu(S)}$. Then, the average bundle of $(x,y)$ over $S$ is

$$(\bar{x}(S), \bar{y}(S)) = \left(2\alpha(a-1) + 2 - a, 2\alpha(n+1)\frac{1-a}{n(2a-1)+1} + \frac{(3n+1)a - 2n}{n(2a-1)+1}\right).$$

With some algebraic operations, we can notice that $ab > (2-a)(2-b)$ and $a^2 b < (2-a)^2(2-b)$. Hence, if $\mu(S_1) = 0$, $u_t(x(t), y(t)) = u_t(\bar{x}(S), \bar{y}(S))$ for all $t \in T_0$ and $u_t(x(t), y(t)) > u_t(\bar{x}(S), \bar{y}(S))$ for all $t$ in $T_1$, meaning that none strictly envies a coalition of negligible traders. Similarly if $\mu(S_0) = 0$, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$ for all $t$ in $T$, that is none strictly envies a coalition containing only atoms. Assume now that $\mu(S_0) > 0$ and $\mu(S_1) > 0$, then $0 < \alpha \leqslant \frac{n}{n+1}$.

For every $t$ in $T_0$, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$ that is

$$a\frac{(n-1)a+2}{n(2a-1)+1} \geqslant [2\alpha(a-1)+2-a]\frac{2\alpha(n+1)(1-a) + (3n+1)a - 2n}{n(2a-1)+1}, \tag{2}$$

because by easy computation (2) is equivalent to $(n+1)\alpha^2 - (2n+1)\alpha + n \geqslant 0$, which holds, in particular, for any $\alpha$ in $(0, \frac{n}{n+1}]$.

For every $t$ in $T_1$, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$ that is

$$(2-a)^2\frac{(3n+1)a-2n}{n(2a-1)+1} \geqslant [2\alpha(a-1)+2-a]^2\frac{2\alpha(n+1)(1-a) + (3n+1)a - 2n}{n(2a-1)+1}, \tag{3}$$

because by algebraic computation (3) is equivalent to $4\alpha^2(n+1)(1-a)^2 + 2\alpha(1-a)(a(5n+3)-6n-4)+(2-a)(-a(7n+3)+6n+2) \leqslant 0$, which holds, in particular, for any $\alpha \in (0,1)$. Therefore, $(x,y)$ is strictly envy-free in $\mathcal{E}$.

*FACT 2.* The allocation

$$\begin{aligned}(\tilde{x}(t),\tilde{y}(t)) &= \left(\frac{3n-2+\sqrt{9n^2+16n+8}}{7n+1}, \frac{3n-4+\sqrt{9n^2+16n+8}}{5n-1}\right)\chi_{T_0}+ \\ &+ \left(\frac{11n+4-\sqrt{9n^2+16n+8}}{7n+1}, \frac{7n+2-\sqrt{9n^2+16n+8}}{5n-1}\right)\chi_{T_1},\end{aligned}$$

belonging to the family of strictly envy-free allocations (1), is efficient.

Assume to the contrary that for some feasible allocation $(c,d)$,

$$\begin{cases} c(s)d(s) > \tilde{x}(s)\tilde{y}(s) & \text{if } s \in T_0 \\ c^2(s)d(s) > \tilde{x}^2(s)\tilde{y}(s) & \text{if } s \in T_1 \\ \int_T (c(s),d(s))d\mu = (1,1). \end{cases}$$

From Remark 3.1, also the allocation $(\bar{c},\bar{d}) := (\bar{c}(T_0),\bar{d}(T_0))\chi_{T_0}+(\bar{c}(T_1),\bar{d}(T_1))\chi_{T_1}$ blocks $(\tilde{x},\tilde{y})$. From the feasibility of $(\bar{c},\bar{d})$, it follows that $(\bar{c}(T_1),\bar{d}(T_1)) = (2-\bar{c}(T_0), 2-\bar{d}(T_0))$, with $\bar{c}(T_0)$, $\bar{c}(T_1)$, $\bar{d}(T_0)$ and $\bar{d}(T_1)$ in $(0,2)$. Therefore,

$$\begin{cases} \bar{c}(T_0)\bar{d}(T_0) > \frac{3n-2+\sqrt{9n^2+16n+8}}{7n+1}\frac{3n-4+\sqrt{9n^2+16n+8}}{5n-1} \\ (2-\bar{c}(T_0))^2(2-\bar{d}(T_0)) > \left(\frac{11n+4-\sqrt{9n^2+16n+8}}{7n+1}\right)^2 \frac{7n+2-\sqrt{9n^2+16n+8}}{5n-1}. \end{cases}$$

By the first inequality,

$$\bar{d}(T_0) > \frac{2[9n^2-n+8+3(n-1)\sqrt{9n^2+16n+8}]}{(7n+1)(5n-1)\bar{c}(T_0)}$$

which implies in the second condition

$$\begin{aligned}(2-\bar{c}(T_0))^2 &\left[2-\frac{2(9n^2-n+8+3(n-1)\sqrt{9n^2+16n+8})}{(7n+1)(5n-1)\bar{c}(T_0)}\right] > \\ &> \left(\frac{11n+4-\sqrt{9n^2+16n+8}}{7n+1}\right)^2 \frac{7n+2-\sqrt{9n^2+16n+8}}{5n-1},\end{aligned}$$

that is, after algebraic computation,

$$(7n+1)^2\left(\bar{c}(T_0)-\frac{3n-2+\sqrt{9n^2+16n+8}}{7n+1}\right)^2\left(\bar{c}(T_0)-\frac{17n-\sqrt{9n^2+16n+8}}{5n-1}\right) > 0,$$

that has no solution in $(0,2)$. This means that $(\tilde{x},\tilde{y})$ is efficient.

*Therefore, $(\tilde{x},\tilde{y})$ is strictly fair but not a Walrasian allocation, because the unique equal-division Walrasian allocation is* $\left(\frac{6}{7},\frac{6}{5}\right)\chi_{T_0}+\left(\frac{8}{7},\frac{4}{5}\right)\chi_{T_1}$. This concludes the proof. □

**Proof of Proposition 3.6** Consider the same economy described in the proof of Proposition 3.5 and the same family of strictly envy-free allocations (1). We have already observed that for any coalition $S$ with $\mu(S \cap T_0) > 0$ and $\mu(S \cap T_1) > 0$, $\alpha = \frac{\mu(S \cap T_0)}{\mu(S)} \leqslant \frac{n}{n+1}$. This restriction does not hold for the coalitions $S^*$ of the associated atomless economy $\mathcal{E}^*$, because $\mu^*(S^* \cap T_1^*)$ can be any real number in $(0, \frac{1}{2}]$, while $\mu(S \cap T_1) \in [\frac{1}{2n}, \frac{1}{2}]$.[6]
Let $(x^*, y^*)$ be the associated family of allocations given by

$$(x^*(t), y^*(t)) = \begin{cases} (x(t), y(t)) & \text{if } t \in T_0 \\ (x(A), y(A)) & \text{if } t \in A^* \quad \text{and} \quad A \in T_1. \end{cases}$$

Notice that for all $t$ in $T_0$ and for all $S^*$ such that $\frac{\mu^*(S^* \cap T_0)}{\mu^*(S)} > \frac{n}{n+1}$,

$$u_t(x^*(t), y^*(t)) < u_t(\bar{x}(S^* \setminus \{t\}), \bar{y}(S^* \setminus \{t\})) = u_t(\bar{x}(S^*), \bar{y}(S^*));$$

meaning that any agent $t$ in $T_0$ is strictly envious and hence any allocation of the family $(x^*, y^*)$ is not strictly equitable in $\mathcal{E}^*$. This concludes the proof. □

**Proof of Lemma 3.7** Since $x$ is strictly envy-free, $u(x(A_1)) = u(x(A_2))$ for all $A_1, A_2 \in C \cap T_1$. If $C$ contains at least three atoms, by strict quasi-concavity of $u$,

$$x(A_1) = x(A_2) \quad \text{for all } A_1, A_2 \in C \cap T_1, \tag{4}$$

otherwise a third atom $A_3$ would strictly envy the coalition $\{A_1, A_2\}$. Notice that if $x$ is also efficient, (4) holds regardless the number of large traders of $C$, otherwise the feasible allocation $y = x\chi_{T \setminus (C \cap T_1)} + \bar{x}(C \cap T_1)\chi_{C \cap T_1}$ would block $x$. The presence of at least three atoms is needed because according to the notion of strict envy-freeness an individual can not be member of the coalition she envies.

Consider the following sets $B = \{t \in C : u(\bar{x}(C)) > u(x(t))\}$ and $D = \{t \in C : u(\bar{x}(C)) < u(x(t))\}$. We want to show that $\mu(B) = \mu(D) = 0$.

First, assume to the contrary that $\mu(B) > 0$ and notice that $\mu(B \cap T_0) = 0$, otherwise every agent $t$ in $B \cap T_0$ would strictly envy the coalition $C \setminus \{t\}$. Similarly, by Remark 3.1, $\mu(C \setminus B) = 0$. Hence $\mu(C) = \mu(B) = \mu(B \cap T_1)$ and from (4) it follows that $u(x(t)) = u(\bar{x}(C))$ for almost all $t \in B$, which is impossible, as the set $B$ is defined. Hence $\mu(B) = 0$.

Now, assume to the contrary that $\mu(D) > 0$. Define $\alpha = \frac{\mu(D)}{\mu(C)}$ and notice that $\alpha < 1$, that is $\mu(C \setminus D) > 0$, otherwise Remark 3.1 would induce a

[6] Recall that $\mu(S \cap T_1) > 0$ means that $S$ contains at least one atom with measure $\frac{1}{2n}$, which by definition has no proper subcoalition with positive measure. Hence $\mu(S \cap T_1) \geqslant \frac{1}{2n}$.

contradiction. By the continuity of the utility function, there exist $\varepsilon \in (0,1)$ and a subset $E$ of $D$ with positive measure such that $u(\varepsilon x(t)) > u(\bar{x}(C))$ for almost every $t$ in $E$. Hence, $u(\varepsilon\bar{x}(E)) = u\left(\frac{1}{\mu(E)}\int_E \varepsilon x(s)d\mu\right) > u(\bar{x}(C))$, and $u(\bar{x}(C\setminus E)) = u\left(\frac{1}{\mu(C\setminus E)}\int_{C\setminus E} x(s)d\mu\right) \geqslant u(\bar{x}(C))$.

Now, let $\beta = \frac{\mu(E)}{\mu(C)} \in (0,1)$ and notice that $\bar{x}(C) = \beta\bar{x}(E) + (1-\beta)\bar{x}(C\setminus E)$. Then,

$$\begin{aligned} u(\bar{x}(C)) &= u\left(\beta(1-\varepsilon)\bar{x}(E) + \beta\varepsilon\bar{x}(E) + (1-\beta)\bar{x}(C\setminus E)\right) > \\ &> u\left(\beta\varepsilon\bar{x}(E) + (1-\beta)\bar{x}(C\setminus E)\right) \geqslant u(\bar{x}(C)), \end{aligned}$$

which is a contradiction. Since $\mu(B) = \mu(D) = 0$, then $u(x(t)) = u(\bar{x}(C))$ for almost all $t \in C$. Using the strict quasi concavity we complete the proof. $\square$

**Proof of Proposition 3.8** Let $x$ be a strictly envy-free allocation for the economy $\mathcal{E}$. Assume to the contrary that $x^*$ is not strictly envy-free in the associated atomless economy $\mathcal{E}^*$. By Remark 3.1, for any envious agent $t$ there exist a coalition[7] $S^*$ and an allocation $y^*$ such that $u_t(y^*(s)) > u_t(x^*(t))$ $\mu^*$-almost everywhere on $S^*$ and

$$\int_{S^*} x^*(s)d\mu^* = \int_{S^*} y^*(s)d\mu^*. \tag{5}$$

Let us consider the statement 1.

By Remark 3.1, $y^*$ can be taken constant on $S^* \cap T_1^*$, i.e., $y^*(s) = \bar{y}$ for all $s \in S^* \cap T_1^*$, while Lemma 3.7 ensures that $x^*(s) = \bar{x}$ for all $s \in S^* \cap T_1^*$. Therefore from (5) it follows

$$\int_{S^*} [y^*(s) - x^*(s)]d\mu^* = \int_{S^* \cap T_0} [y^*(s) - x^*(s)]d\mu^* + (\bar{y} - \bar{x})\mu^*(S^* \cap T_1^*) = 0.$$

Notice that $\mu^*(S^* \cap T_1^*) > 0$ otherwise, by taking in $\mathcal{E}$, $S = S^*$ and $y = y^*$, we would contradict the strict equitability of $x$ in $\mathcal{E}$. Since $T_1$ is countably infinite and $\mu(T_1) = \sum_{n=1}^{+\infty} \mu(A_n) < +\infty$, then $\lim_{n\to+\infty} \mu(A_n) = 0$ and hence there exists $B$ belonging to $T_1$ such that $\mu(B) < \mu^*(S^* \cap T_1^*)$. Then for any $\alpha$, $\alpha\int_{S^*\cap T_0}[y^*(s) - x^*(s)]d\mu^* + (\bar{y} - \bar{x})\alpha\mu^*(S^* \cap T_1^*) = 0$, in particular for $\alpha = \frac{\mu(B)}{\mu^*(S^*\cap T_1^*)}$. By the Lyapunov convexity theorem there exists $R$ subset of $S^* \cap T_0$ such that $\int_R [y^*(s) - x^*(s)]d\mu + (\bar{y} - \bar{x})\mu(B) = 0$. Without loss of generality $t \notin R$, hence $t$ strictly envies $R \cup \{B\}$ at $x$ via $z = y^*\chi_R + \bar{y}\chi_B$.

[7] The coalition $S^*$ depends on $t$ but for simplicity we avoid to stress it.

Let us consider the statement 2.

From Remark 3.4, without loss of generality the set $J := \{n \in I\!N : \mu^*(S^* \cap A_n^*) > 0, \text{ with } A_n^* \in T_1^*\}$ is finite. For any $n \in J$, let $C_n^* = \{t \in T^* : u_t = u_{A_n^*}\}$ be the set of agents indentical to the atom $A_n$ and $C^* = \bigcup_{n\in J} C_n^*$. For any $n \in J$, by assumption $\mu(C_n^* \cap T_0) > 0$; moreover Remark 3.1 and Lemma 3.7 imply respectively that $y^*$ and $x^*$ can be taken constant on each $C_n^*$. Hence, $y^*(s) = y_n$ and $x^*(s) = x_n$ for almost all $s \in C_n^* \cap S^*$. From (5) it follows

$$\int_{S^*\setminus C^*} [y^*(s) - x^*(s)]d\mu^* + \sum_{n\in J}(y_n - x_n)\mu^*(C_n^* \cap S^*) = 0. \tag{6}$$

Let $J_1 := \{n \in J : \mu^*(C_n^* \cap S^*) > \mu(C_n^* \cap T_0)\}$. For any $n \in J_1$, let $\beta_n$ be $\frac{\mu(C_n^*\cap T_0)}{\mu^*(C_n^*\cap S^*)}$ and $\beta := \min\{\beta_n, n \in J_1\}$. Then, from (6) it follows that

$$\beta \int_{S^*\setminus C^*} [y^*(s) - x^*(s)]d\mu^* + \beta \sum_{n\in J}(y_n - x_n)\mu^*(C_n^* \cap S^*) = 0.$$

For any $n \in J$, $\beta\mu^*(C_n^* \cap S^*) \leqslant \mu(C_n^* \cap T_0)$, then there exists $B_n \subseteq C_n^* \cap T_0$ such that $\mu(B_n) = \beta\mu^*(C_n^* \cap S^*)$. Furthermore, by the Lyapunov convexity theorem there exists $B \subseteq S^* \setminus C^*$ such that

$$\int_B (y^*(s) - x^*(s))d\mu + \sum_{n\in J}(y_n - x_n)\mu(B_n) = 0.$$

This means that $t$ strictly envies the atomless coalition $\left(\bigcup_{n\in J} B_n\right) \cup B$ at $x$ via $y = y^*\chi_B + \sum_{n\in J} y_n\chi_{B_n}$, which is a contradiction. □

For sake of completeness we now prove the converse.

**Proposition 5.1.** *If $x^*$ is strictly envy-free in $\mathcal{E}^*$, the corresponding allocation $x$ is strictly envy-free in $\mathcal{E}$.*

*Proof.* Assume by contradiction that $x$ is not strictly equitable in $\mathcal{E}$. Thus, for every envious agent $t$ in $\mathcal{E}$ there exists a coalition $S$ such that $t \notin S$ and $u_t(\bar{x}(S)) > u_t(x(t))$. Consider the coalition $S^*$ of $\mathcal{E}^*$ obtained by splitting each member of $S \cap T_1$, then, by definition of $x^*$, $u_t\left(\frac{1}{\mu^*(S^*)}\int_{S^*} x^*(t)\, d\mu^*\right) = u_t(\bar{x}(S)) > u_t(x(t)) = u_t(x^*(t))$. This means that the set of envious agents in $\mathcal{E}^*$ has $\mu^*$-positive measure, which is a contradiction. □

**Proof of Theorem 3.9** We have already shown that any equal-division Walrasian allocation is strictly fair. To prove the converse, let $x$ be a strictly fair allocation and consider the corresponding allocation $x^*$. From Greenberg and Shitovitz (1986), $x^*$ is efficient in $\mathcal{E}^*$, while Proposition 3.8 ensures that it is also strictly envy-free. Therefore, $x^*$ is a strictly fair allocation for the associated atomless economy $\mathcal{E}^*$, and by Zhou (1992) it is an equal-division Walrasian allocation. Coming back to the original mixed economy $\mathcal{E}$, the associated allocation $x$ is an equal-division Walrasian allocation. This concludes the proof. $\square$

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