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Abstract

Both pessimistic and optimistic bilevel optimization problems may be not stable under perturbation when the lower-level problem has not a unique solution, meaning that the limit of sequences of solutions (resp. equilibria) to perturbed bilevel problems is not necessarily a solution (resp. an equilibrium) to the original problem. In this paper, we investigate the notion of lower Stackelberg equilibrium, an equilibrium concept arising as a limit point of pessimistic equilibria and of optimistic equilibria of perturbed bilevel problems. First, connections with pessimistic equilibria and optimistic equilibria are obtained in a general setting, together with existence and closure results. Secondly, the problem of finding a lower Stackelberg equilibrium is shown to be stable under general perturbation, differently from what happens for pessimistic and optimistic bilevel problems. Then, moving to the game theory viewpoint, the set of lower Stackelberg equilibria is proved to coincide with the set of subgame perfect Nash equilibrium outcomes of the associated Stackelberg game. These results allow to achieve a comprehensive look on various equilibrium concepts in bilevel optimization and in Stackelberg games as well as to add a new interpretation in terms of game theory to previous limit results on pessimistic equilibria and optimistic equilibria under perturbation.

Keywords: Pessimistic and optimistic bilevel optimization problems; Stackelberg game; lower Stackelberg equilibrium; Subgame perfect Nash equilibrium; Stability under perturbation.

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1 Introduction

The study of bilevel optimization problems (BOP for short) holds a relevant role in the landscape of both optimization theory and game theory. Starting from the work of von Stackelberg [56], where a BOP is associated to a sequential game modelling a duopolistic competition of firms, such topics have been the subject of deep investigation and increasing development in theory and applications. This is testified by the surveys that continually appear in literature (see, for example, Vincente and Calamai [55]; Colson, Marcotte and Savard [11]; Caruso, Lignola and Morgan [10]; Kleinert, Labbé, Ljubić and Schmidt [19]), by the recent volume edited by Dempe and Zemkoho [16] and by the fundamental and highly cited book by Dempe [14].

BOPs are hierarchical problems consisting of two optimization problems, one in the upper level and another one in the lower level, both involving two decision variables: x and y. In the lower level, y has to be chosen as an optimal solution to an optimization problem parameterized by x. In the upper level, taking into account the constraint given by the set of optimal solutions in the lower-level, another optimization problem has to be managed and a solution x should be found. This setting corresponds to the situation where a first agent acting in the upper level, the *leader*, wants to find an optimal solution for herself trying to forecast the behavior of a second agent acting in the lower level, the *follower*, who takes his optimal decision after observed the choice of the agent in the upper level.

Depending on the set of solutions to the lower-level problem and on the leader's approaches, various types of BOPs can be identified.

- *Uniqueness of solutions in the lower level.* In this case, the leader can perfectly forecast the lower-level solution and she faces the so-called classical Stackelberg problem. For first results on existence, approximation, well-posedness refer to Shimizu and Aiyoshi [51]; Loridan and Morgan [31]; Morgan [44], for algorithms to [14, Chapter 6] and for further discussion to [10, Sections 4.2.1 and 4.3.1].

- *Non-uniqueness of solutions in the lower level: pessimistic approach.* The leader believes that the follower could choose the worst action for her in the set of solutions to the lower-level problem. This situation leads to the so-called pessimistic BOP (or weak Stackelberg problem). For first investigations refer to Molodtsov and Fedorov [43]; Leitmann [21]; Loridan and Morgan [32]; Lignola and Morgan [29], for further discussion to Başar and Olsder [2, Section 3.6] and to [10, Sections 4.3.2 and 4.4.1].

- *Non-uniqueness of solutions in the lower level: optimistic approach.* The leader believes that the follower will choose the best action for her in the set of solutions to the lower-level problem. This situation leads to the so-called optimistic BOP (or strong Stackelberg problem). For first investigations refer to Bard [3]; Dempe [13]; Outrata [48]; Vicente, Savard and Júdice [54]; Ye and Zhu [57]; Lignola and Morgan [25], for further discussion to [14] and [10, Sections 4.3.3 and 4.4.2].

- *Non-uniqueness of solutions in the lower level: bayesian approach.* The leader has information that allows her to attribute some probabilities to the lower-level solutions. This circumstance leads to the so-called intermediate Stackelberg problem. For definitions and applications refer to Mallozzi and Morgan [39, 37, 38], for a survey on the bilevel optimization under uncertainty to Beck, Ljubić and Schmidt [4].

This paper concerns a new point of view, which is different from those described above and which arises from the analysis of stability issues in pessimistic and optimistic bilevel optimization. In fact, it is well-known that variational stability of pessimistic and optimistic BOPs (as well as of intermediate Stackelberg problems) may fail, meaning that the limit of sequences of solutions to perturbed bilevel problems may be not a solution to the original one (see, for example, [10, Sections 4.3.2 and 4.3.3] for a comprehensive discussion), unless the lower-level problem has a unique solution (see [31]). Nevertheless, relevant limit results have been achieved on the asymptotic behavior of pessimistic and optimistic BOPs under perturbation. They are summarized in Table 1, where only results on the "exact" problems are indicated but not those involving "approximated" problems. Focusing on the asymptotic behavior of pessimistic and optimistic equilibria of perturbed

	Bilevel problem	**Perturbation**	**Results on the exact bilevel problem under perturbation**
Loridan and Morgan [32, 33]	pessimistic	general perturbation on the objective functions	stability of the lower-level problem, limit theorems on the marginal functions and values of the upper-level problem
Loridan and Morgan [34]	pessimistic	Tikhonov and Least-Norm regularization on the lower-level objective function	limit theorems on the sequence of pessimistic equilibria of Tikhonov-perturbed (and least-norm-perturbed) problems
Loridan and Morgan [35]	pessimistic	Molodtsov regularization on the lower-level objective function	limit theorems on the sequence of optimistic equilibria of Molodtsov-perturbed problems
Dempe and Schmidt [15]	optimistic	inverse-Molodtsov (altruistic) regularization on the lower-level objective function	limit result on the sequence of lower-level solutions of inverse-Molodtsov-perturbed problems, definition of a descent algorithm and convergence to a stationary point
Lignola and Morgan [27]	optimistic	general perturbation on the constraints sets defined by inequalities and on the objective functions	stability of the lower-level problem, limit theorems on the marginal functions and values of the upper-level problem
Dempe [12]	optimistic	Tikhonov regularization on the lower-level objective function	definition of a bundle algorithm and convergence to a stationary point
Dempe [14, Section 7.2]	optimistic	general perturbation on the constraints sets defined by inequalities and on the objective functions	limit theorem on the sequence of optimistic equilibria of perturbed problems

Table 1: Limit results on exact pessimistic and optimistic BOPs under perturbation.

bilevel problems, in all the limit results [34, Theorem 3.6], [35, Propositions 12, 13 and 23] and [14, Theorem 7.12], the limit pair $(\bar{x}, \bar{y})$ satisfies the same conditions: $\bar{y}$ is a solution to the lower-level problem parameterized by $\bar{x}$ and the value of the upper-level objective function in $(\bar{x}, \bar{y})$ is not greater than the value of the pessimistic BOP. In literature, such a pair has been referred to as *lower Stackelberg equilibrium* by Loridan and Morgan, who introduced the concept in the context of approximate solutions to pessimistic BOPs (see [32, Section 5]), and as *lower optimal solution* by Dempe, who used this concept in the framework of algorithmic methods in optimistic BOPs (see [14, Section 5.1]).

The aim of the paper is to investigate the concept of lower Stackelberg equilibrium (LSE for short). We will develop the study of this notion from the viewpoint both of optimization and of game theory.

First, in the framework of bilevel optimization, we show results on non-emptiness and closure of the set of LSEs and on connections with the sets of pessimistic and of optimistic equilibria. In particular, the problem of finding an LSE includes those of finding equilibria of both pessimistic and optimistic BOPs and it coincides with them when the set of solutions to the lower-level problem is a singleton. Moreover, differently from the pessimistic and optimistic BOPs, the lower Stackelberg equilibrium problem is variationally stable. In fact, under mild assumptions on the convergence of the perturbed constraints sets and of the perturbed objective functions, the limit of sequences of LSEs under perturbation is proved to be an LSE. Besides showing a solution concept in bilevel optimization for which stability holds, this result extends the previous limit theorems [34, Theorem 3.6], [35, Propositions 13] and [14, Theorem 7.12].

Then, moving to the framework of Stackelberg games, we show that the lower Stackelberg equilibrium notion is strongly connected with the *subgame perfect Nash equilibrium* notion (SPNE for short) introduced by Selten [49], which is the most widely used solution concept in sequential games of perfect information (see, for example, [2, 42]). Indeed, we prove that the set of LSEs coincides with the set of SPNE-outcomes (see, for example, [42, Chapter 3] for the definition of outcome in a game), that is the problems of finding an LSE and an SPNE-outcome are equivalent. This result allows to achieve a final view on the connections among the solution concepts in bilevel optimization and in Stackelberg games. Moreover, on the one hand, it provides a relevant characterization of the set of SPNE-outcomes of a Stackelberg game and it generalizes some results on the asymptotic behavior of SPNEs of perturbed Stackelberg games in Caruso, Ceparano and Morgan [7, 8]. On the other hand, this equivalence gives a new interpretation in terms of game theory for the LSE and, consequently, for the approximation methods and numerical algorithms based on the convergence of sequences of pessimistic or optimistic equilibria of bilevel problems under specific perturbations.

The paper is organized as follows. In Section 2, we discuss the issue of stability in bilevel optimization. More precisely, we recall the known literature results on the asymptotic behavior of pessimistic BOPs under perturbation (Section 2.1) and of optimistic BOPs under perturbation (Section 2.2). In Section 3, the notion of lower Stackelberg equilibrium is presented and investigated. First, we show results on the existence, the closure and the connections with pessimistic and optimistic equilibria (Section 3.1). Secondly, we prove the stability of the LSE problem (Section 3.2). In Section 4, we study the LSE concept in the framework of game theory. After introduced the Stackelberg game setting and presented the notion of subgame perfect Nash equilibrium (Section 4.1), we show the equality between the sets of LSEs and of SPNE-outcomes, together with some related consequences (Section 4.2). Finally, in Section 5 we state conclusions and directions for future research. In particular, we provide an hint on the extension of the whole analysis to the framework of bilevel problems with Nash equilibrium constraints and associated Stackelberg games with multiple followers.

2 Starting point: stability issues in bilevel optimization

In this section we recall the definitions of pessimistic and optimistic BOP, their related solution notions, and some first literature results concerning the behavior of these problems when perturbations occur in the data. Throughout the section, regardless of the nature (pessimistic or optimistic) of the problem, the objective functions of the upper-level and lower-level problems are denoted by $l\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ and $f\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$, respectively, the set of constraints of the upper-level problem is $X \subseteq \mathbb{R}^N$, and the set of constraints of the lower-level problem varies with $x \in X$ and is denoted by $\mathcal{Y}(x) \subseteq \mathbb{R}^M$, where $\mathcal{Y}$ is a set-valued map from $\mathbb{R}^N$ to $\mathbb{R}^M$. We deal with a minimization setting, as usual in optimization literature; nevertheless all the results can be adapted also to a maximization context.

2.1 On pessimistic bilevel optimization under perturbation

Let us consider the pessimistic BOP (or weak Stackelberg problem) defined as follows:

$$(PB)\ \begin{cases} \min\limits_{x\in X} \ \sup\limits_{y\in\Psi(x)} \ l(x,y) \\ \text{where } \Psi(x) = \operatorname*{arg\,min}\limits_{y\in\mathcal{Y}(x)} \ f(x,y); \end{cases}$$

reminding that $\arg\min_{y\in\mathcal{Y}(x)} f(x,y)$ indicates the set $\{y \in \mathcal{Y}(x) \mid f(x,y) \leq f(x,z) \text{ for any } z \in \mathcal{Y}(x)\}$. The related solution concepts and values are recalled below.

Definition 2.1 ([32]) A vector $\bar{x}$ is a *(weak Stackelberg or pessimistic) solution* to (PB) if

$$\bar{x} \in \operatorname*{arg\,min}_{x\in X} \ \sup_{y\in\Psi(x)} \ l(x,y),$$

that is, $\sup_{y\in\Psi(\bar{x})} l(\bar{x},y) = \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y)$.
The (possibly infinite) number $v^P = \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y)$ is called *(pessimistic or security) value* of (PB).
A pair $(\bar{x},\bar{y})$ is a *(pessimistic or weak Stackelberg) equilibrium* of (PB) if $\bar{x}$ is a solution to (PB) and $\bar{y} \in \Psi(\bar{x})$.

Problem (PB) is renowned to have problems of well-posedness. In fact, the existence of pessimistic solutions and equilibria is not guaranteed, even under compactness and continuity assumptions, because of the possible lack of lower semicontinuity of the set-valued map Ψ (see [32, Remark 2.1], [2, Remark 4.3] and also, for example, [10, Section 4.3.2]). Moreover, even the (variational) stability may be missing for (PB). Indeed, if $\bar{x}_n$ is a solution to the perturbed pessimistic BOP

$$(PB)_n\ \begin{cases} \min\limits_{x\in X_n} \ \sup\limits_{y\in\Psi_n(x)} \ l_n(x,y) \\ \text{where } \Psi_n(x) = \operatorname*{arg\,min}\limits_{y\in\mathcal{Y}_n(x)} \ f_n(x,y), \end{cases}$$

for any $n \in \mathbb{N}$ and the sequence $(\bar{x}_n)_n$ converges to $\bar{x}$, then $\bar{x}$ is not ensured to be a solution to (PB), even under strong assumptions on the convergence of $(X_n)_n$, $(\mathcal{Y}_n)_n$, $(l_n)_n$ and $(f_n)_n$ to X,

$\mathcal{Y}$, l and f, respectively. The same bad-behavior may also happen for pessimistic equilibria and values (see [32, 29] and also, for example, [10, Section 4.3.2]).
Despite that, other relevant properties on the asymptotic behavior of $(PB)_n$ have been firstly shown by Loridan and Morgan in [32] and in [34] when constraint sets are unperturbed and the constraint set of the lower-level problem does depend on x.

Before presenting the first result, proved in [32, Sections 4 and 5], let us define the marginal (or value) functions for the lower-level and the upper-level problem in (PB) and $(PB)_n$:

$$\omega^P(x) = \sup_{y \in \Psi(x)} l(x,y), \qquad \omega_n^P(x) = \sup_{y \in \Psi_n(x)} l_n(x,y), \tag{1}$$

$$\tau(x) = \inf_{y \in \mathcal{Y}(x)} f(x,y), \qquad \tau_n(x) = \inf_{y \in \mathcal{Y}_n(x)} f_n(x,y), \tag{2}$$

and let us denote with v_n^P the value of $(PB)_n$. Moreover, we recall that the sequence of (real-valued) functions $(l_n)_n$ *continuously converges* to l if $\lim_{n\to+\infty} l_n(x_n, y_n) = l(x,y)$ for any sequence $(x_n, y_n)_n$ converging to (x,y); see, for example, [20, Chapter 2, §20-VI].

Theorem 2.1 (Propositions 4.1, 4.2 and 5.1 in [32])**.** *Let $X_n = X$ and $\mathcal{Y}_n(x) = \mathcal{Y}(x) = Y$ for each $n \in \mathbb{N}$ and $x \in X$. Assume that Y is compact and that the sequences $(l_n)_n$ and $(f_n)_n$ are continuously convergent to l and f, respectively. Then:*

i) *for each $(x,y) \in X \times Y$, each sequence $(x_n)_n$ converging to x and each sequence $(y_n)_n$ such that $y_{n_k} \in \Psi_{n_k}(x_{n_k})$ and $(y_{n_k})_k$ converges to y for a subsequence $(n_k) \subseteq \mathbb{N}$, we have $y \in \Psi(x)$;*

ii) *the sequence $(\tau_n)_n$ continuously converges to τ;*

iii) *for each $x \in X$ there exists a sequence $(x_n)_n$ converging to x such that*

$$\limsup_{n\to+\infty} w_n^P(x_n) \le \omega^P(x);$$

iv) $\limsup_{n\to+\infty} v_n^P \le v^P$.

Remark 2.1 The continuous convergence of the sequences $(l_n)_n$ and $(f_n)_n$ to l and f in Theorem 2.1 is assumed just for the sake of readability; the result still holds under weaker convergence hypotheses (see [32, Section 3]). Results of Theorem 2.1 have been extended in [29] to the more general case where the constraints sets are perturbed and the constraints set of the lower-level problem depends on x.

Hence, Theorem 2.1 provides limit results concerning both the lower-level problem and the upper-level problem of $(PB)_n$. In fact, on the one hand, Theorem 2.1 i)-ii) means that the limit of solutions (resp. of the values) of the lower-level problem of $(PB)_n$ is a solution (resp. the value) of the lower-level problem of (PB); and this holds not only for any x fixed, but for any sequence converging to x (in particular, Theorem 2.1 i) means that the sequence of set-valued maps $(\Psi_n)_n$ is upper convergent to Ψ; see, for example, Section 3.2 for the definition). So stability for solutions and values of the lower-level problem of (PB) under perturbation holds. On the other hand, reminding that the stability for the values of (PB) under perturbation is not guaranteed, Theorem 2.1 iii) describes a limit result for the sequence of marginal functions of the upper-level problem of $(PB)_n$ (which is weaker than the continuous convergence achieved for $(\tau_n)_n$) and Theorem 2.1 iv) tells that the value of (PB) is an upper bound for the limit of the sequence of the values of $(PB)_n$.

The second result, shown in [34, Section 3], is a limit theorem for the sequence of equilibria of $(PB)_n$ when the lower-level problem is perturbed via the Tikhonov regularization [53].

Theorem 2.2 (Theorem 3.6 in [34]). *Let $X_n = X$ and $\mathcal{Y}_n(x) = \mathcal{Y}(x) = Y$ for each $n \in \mathbb{N}$ and $x \in X$, and*

$$l_n(x,y) = l(x,y) \quad \text{and} \quad f_n(x,y) = f(x,y) + \alpha_n \|y\|^2$$

for each $(x,y) \in X \times Y$, where the sequence $(\alpha_n)_n \subseteq]0,+\infty[$ converges to 0. Assume that X is compact, Y is compact and convex, l and f are continuous, and $f(x,\cdot)$ is convex for each $x \in X$. If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(PB)_n$ for each $n \in \mathbb{N}$, then each accumulation point $(\bar{x}, \bar{y})$ of the sequence $(\bar{x}_n, \bar{y}_n)_n$ satisfies

$$\bar{y} \in \Psi(\bar{x}) \quad \text{and} \quad l(\bar{x},\bar{y}) \leq \inf_{x \in X} \sup_{y \in \Psi(x)} l(x,y). \tag{3}$$

Remark 2.2 The original result [34, Theorem 3.6] requires assumptions weaker than continuity on l and f. Moreover, analogous results hold also when the least-norm regularization [52] and the Molodtsov regularization [43] are used in the lower-level problem of $(PB)_n$; see [34, Section 2] and [35], respectively. It is worth highlighting that further results on the use of the Tikhonov regularization in bilevel problems have been provided by Dempe in [12].

2.2 On optimistic bilevel optimization under perturbation

Now we deal with stability issues in the optimistic BOP (or strong Stackelberg problem):

$$(OB) \begin{cases} \min\limits_{x \in X} \inf\limits_{y \in \Psi(x)} l(x,y) \\ \text{where } \Psi(x) = \arg\min\limits_{y \in \mathcal{Y}(x)} f(x,y). \end{cases}$$

Let us recall the concepts of solution and equilibrium associated to problem (OB).

Definition 2.2 ([27, 14]) A vector $\bar{x}$ is a *(strong Stackelberg or optimistic) solution* to (OB) if

$$\bar{x} \in \arg\min_{x \in X} \inf_{y \in \Psi(x)} l(x,y),$$

that is, $\inf_{y \in \Psi(\bar{x})} l(\bar{x},y) = \inf_{x \in X} \inf_{y \in \Psi(x)} l(x,y)$.
The (possibly infinite) number $v^O = \inf_{x \in X} \inf_{y \in \Psi(x)} l(x,y)$ is called *(optimistic) value* of (OB).
A pair $(\bar{x}, \bar{y})$ is an *(optimistic or strong Stackelberg) equilibrium* of (OB) if $\bar{x}$ is a solution to (OB) and $\bar{y} \in \arg\min_{y \in \Psi(\bar{x})} l(\bar{x},y)$.

Differently from (PB), existence of solutions to (OB) can be ensured under mild assumptions on the data (see, in the case of constraints sets defined by inequalities, [27, Proposition 2.2] for conditions of minimal character and [14, Theorem 5.2] for a differentiable setting). Instead, variational stability of (OB) may fail under "nice" assumptions. The limits of sequences of solutions (resp. values, or equilibria) to perturbed optimistic BOPs

$$(OB)_n \begin{cases} \min\limits_{x\in X_n} \inf\limits_{y\in\Psi_n(x)} l_n(x,y) \\ \text{where } \Psi_n(x) = \arg\min\limits_{y\in\mathcal{Y}_n(x)} f_n(x,y), \end{cases}$$

when varying $n \in \mathbb{N}$, may be not solutions (resp. values, or equilibria) to (OB), see for example [27, Example 4.1] and [14, Section 7.2].

Although the possible lack of stability for (OB), some first interesting limit results have been achieved on the asymptotic behavior of $(OB)_n$ by Lignola and Morgan [27] and by Dempe [12]. In order to present them, we set the framework where the constraints sets of the upper and lower level of (OB) and $(OB)_n$ are identified by a finite numbers of inequalities. Let X and $\mathcal{Y}$ in (OB) be defined as follows:

$$X = \{x \in \boldsymbol{X} \mid g(x) \le 0\} \quad \text{and} \quad \mathcal{Y}(x) = \{y \in \boldsymbol{Y} \mid h(x,y) \le 0\},$$

where $\boldsymbol{X} \subseteq \mathbb{R}^N$, $\boldsymbol{Y} \subseteq \mathbb{R}^M$, $g\colon \boldsymbol{X} \to \mathbb{R}^P$ with $g(x) = (g^1(x), \dots, g^P(x))$ and $h\colon \boldsymbol{X}\times\boldsymbol{Y} \to \mathbb{R}^Q$ with $h(x,y) = (h^1(x,y), \dots, h^Q(x,y))$. Analogously, in $(OB)_n$ let

$$X_n = \{x \in \boldsymbol{X} \mid g_n(x) \le 0\} \quad \text{and} \quad \mathcal{Y}_n(x) = \{y \in \boldsymbol{Y} \mid h_n(x,y) \le 0\},$$

where $g_n\colon \boldsymbol{X} \to \mathbb{R}^P$ with $g_n(x) = (g_n^1(x), \dots, g_n^P(x))$ and $h_n\colon \boldsymbol{X}\times\boldsymbol{Y} \to \mathbb{R}^Q$ with $h_n(x,y) = (h_n^1(x,y), \dots, h_n^Q(x,y))$. Moreover, the continuous convergence of the sequence of (vector-valued) functions $(g_n)_n$ to g has to be understood as the continuous convergence of $(g_n^i)_n$ to g^i for each $i \in \{1, \dots, P\}$.

The first limit result, proved in [27, Section 4], involves the sequences of set-valued maps $(\Psi_n)_n$, the sequence of values $(v_n^O)_n$ of $(OB)_n$, and the marginal functions of the lower-level and the upper-level problem in (OB) and $(OB)_n$:

$$\begin{aligned} \omega^O(x) &= \inf_{y\in\Psi(x)} l(x,y), & \omega_n^O(x) &= \inf_{y\in\Psi_n(x)} l_n(x,y), \\ \tau(x) &= \inf_{y\in\mathcal{Y}(x)} f(x,y), & \tau_n(x) &= \inf_{y\in\mathcal{Y}_n(x)} f_n(x,y). \end{aligned} \tag{4}$$

Theorem 2.3 (Theorem 4.1 in [27]). *Assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are compact, that for each $x \in \boldsymbol{X}$ there exists $y \in \boldsymbol{Y}$ such that $\max_{j=1,\dots,Q} h^j(x,y) < 0$, that the function $h_n^j(x,\cdot)$ is semistrictly quasiconvex for any $x \in \boldsymbol{X}$, $j \in \{1,\dots,q\}$ and $n \in \mathbb{N}$, and that the sequences $(g_n)_n$, $(h_n)_n$, $(l_n)_n$, $(f_n)_n$ are continuously convergent to g, h, l, f, respectively. Then:*

i) *for each $(x,y) \in \boldsymbol{X}\times\boldsymbol{Y}$, each sequence $(x_n)_n$ converging to x and each sequence $(y_n)_n \subseteq \boldsymbol{Y}$ such that $y_{n_k} \in \Psi_{n_k}(x_{n_k})$ and $(y_{n_k})_k$ converges to y for a subsequence $(n_k) \subseteq \mathbb{N}$, we have $y \in \Psi(x)$;*

ii) *the sequence $(\tau_n)_n$ continuously converges to τ;*

iii) *for each $x \in \mathbb{R}^N$ and each sequence $(x_n)_n$ converging to x, we have*

$$\omega^O(x) \le \liminf_{n \to +\infty} w_n^O(x_n);$$

iv) $v^O \le \liminf_{n \to +\infty} v_n^O$.

Remark 2.3 We assumed the continuous convergence of the sequences $(g_n)_n$, $(h_n)_n$ and $(l_n)_n$ to g, h and l just for the sake of readability. It is worth to highlight that, in [27, Theorem 4.1], weaker convergence hypotheses are assumed. Moreover, we recall that the semistrict quasiconvexity assumption (referred also as strict quasiconvexity in [41, Chapter 9]) means that $h_n^j(x, \lambda y' + (1 - \lambda) y'') < \max\{h_n^j(x, y'), h_n^j(x, y'')\}$ for any $\lambda \in]0,1[$ and any y', y'' such that $h_n^j(x, y') \neq h_n^j(x, y'')$; see, for example, [6, Definition 2.3.1].

The answers on the lower-level problem of (OB) under perturbation shown in Theorem 2.3 i)-ii) are the same of the ones obtained for (PB) in Theorem 2.1: the sequence of set-valued functions $(\Psi_n)_n$ upper converges to Ψ and the sequence $(\tau_n)_n$ continuously converges to τ. Concerning the upper-level problem, Theorem 2.3 iii) provides a kind of "half" continuous convergence of $(\omega_n^O)_n$ to ω^O which is stronger than the pessimistic counterpart obtained in Theorem 2.1 iii). Instead, Theorem 2.3 iv) states that the value of (OB) is a lower bound for the limit of the sequence of values of $(OB)_n$, which is the mirror condition of Theorem 2.1 iv).

The second limit result, shown in [14, Section 7.2], regards a property of the limit of sequences of equilibria of $(OB)_n$.

Theorem 2.4 (Theorem 7.12 in [14])**.** *Let $\boldsymbol{X} = \mathbb{R}^N$ and $\boldsymbol{Y} = \mathbb{R}^M$. Assume that the set $\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M \mid h(x, y) \le 0\}$ is non-empty and compact, that for each (x, y) with $x \in X$ and $y \in \mathcal{Y}(x)$ there exists a direction $d \in \mathbb{R}^N$ satisfying $\nabla_x h^j(x, y) \cdot d < 0$ for each j such that $h^j(x, y) = 0$, and that the sequences $(g_n)_n$, $(h_n)_n$, $(l_n)_n$, $(f_n)_n$ are continuously convergent to g, h, l, f, respectively. If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(OB)_n$ for each $n \in \mathbb{N}$, then each accumulation point $(\bar{x}, \bar{y})$ of the sequence $(\bar{x}_n, \bar{y}_n)_n$ satisfies*

$$\bar{y} \in \Psi(\bar{x}) \quad \text{and} \quad l(\bar{x}, \bar{y}) \le \inf_{x \in X} \sup_{y \in \Psi(x)} l(x, y). \tag{5}$$

Remark 2.4 We highlight that in [14, Theorem 7.12] the set $\mathcal{Y}(x)$ is defined as the set of solutions to a finite number of inequalities and equalities and the usual Mangasarian-Fromowitz constraint qualification is assumed. Moreover, in the original statement of the theorem the "pointwise convergence" is assumed, which coincides to what we defined in this paper as "continuous convergence". These slight changes are made just for uniforming the presentation of the results in the section.

3 Lower Stackelberg equilibrium

By looking at Theorem 2.2 and Theorem 2.4, the limits of sequences of equilibria both in particular perturbed pessimistic bilevel problems and in general perturbed optimistic bilevel problems satisfy identical conditions, namely (3) and (5). In the original statements, such limit points are denoted with two different terms:

- *lower Stackelberg equilibrium* in [34, Theorem 3.6], first introduced by Loridan and Morgan in the framework of the approximation of pessimistic BOPs by means of approximate solutions (see [32, Remark 5.3]);
- *lower optimal solution* in [14, Theorem 7.12], presented by Dempe as a surrogate solution concept in order to manage numerical difficulties arising in optimistic BOPs (see [14, p. 126, Equation (5.8)]).

In any way, these limit points are solutions to the following problem:

$$(LS)\begin{cases}\text{find } (\bar{x},\bar{y}) \text{ such that}\\ \bar{x}\in X,\ \ \bar{y}\in\Psi(\bar{x}) \text{ and } l(\bar{x},\bar{y})\leq \inf\limits_{x\in X}\sup\limits_{y\in\Psi(x)} l(x,y),\\ \text{where } \Psi(x)=\operatorname*{arg\,min}\limits_{y\in\mathcal{Y}(x)} f(x,y).\end{cases}$$

In this section we investigate some key-issues on the solutions to (LS), as their existence, their closure, their stability and their connections with solutions or equilibria of problems (PB) and (OB).
Since (LS) can be recognized as an equilibrium problem corresponding to a sequential interaction between two players where one player chooses x and the other player chooses y, we continue to use the denomination *lower Stackelberg equilibrium* to indicate a solution to (LS).

Definition 3.1 A pair $(\bar{x},\bar{y})$ is a *lower Stackelberg equilibrium* if it solves problem (LS).

Remark 3.1 An LSE is well-defined provided that $\inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y) > -\infty$; so in the sequel we will consider this condition always satisfied. Moreover, note that if $(\bar{x},\bar{y})$ is an LSE, then $\inf_{x\in X}\inf_{y\in\Psi(x)} l(x,y)\leq l(\bar{x},\bar{y})\leq \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y)$.

3.1 Existence, closure and connections with pessimistic and optimistic equilibria

First the non-emptiness and the closure of the set of solutions to (LS), denoted with $\mathcal{E}_{LS}$, is proved.

Proposition 3.1. *Assume that X is compact, $\mathcal{Y}$ is a continuous set-valued map with compact values, l is lower semicontinuous and f is continuous. Then:*

i) $\mathcal{E}_{LS}\neq\emptyset$*;*

ii) $\mathcal{E}_{LS}$ *is closed.*

Proof. First note that, since f is continuous and $\mathcal{Y}$ is continuous and compact-valued, then the set-valued map Ψ is upper semicontinuous and has non-empty compact values by the Berge maximum theorem (see, for example [5, Theorem 12.1]).
Proof of i). Being Ψ upper semicontinuous with compact values and l lower semicontinuous, the marginal function $\inf_{y\in\Psi(\cdot)} l(\cdot,y)$ is lower semicontinuous over the compact set X (see, for example,

[1, Theorem 1.4.16]). Thus, the set $\arg\min_{x\in X}\inf_{y\in\Psi(x)} l(x,y)$ is non-empty, that is there exists $\bar{x}\in X$ satisfying

$$\inf_{y\in\Psi(\bar{x})} l(\bar{x},y) \le \inf_{y\in\Psi(x)} l(x,y) \quad \text{for any } x\in X. \tag{6}$$

Moreover, since l is lower semicontinuous and $\Psi(\bar{x})$ is non-empty and compact, then there exists $\bar{y}\in\arg\min_{y\in\Psi(\bar{x})} l(\bar{x},y)$. Let us show that the pair $(\bar{x},\bar{y})$ is an LSE. In fact, by definition of $\bar{y}$ and (6), we have $\bar{y}\in\Psi(\bar{x})$ and

$$l(\bar{x},\bar{y}) = \inf_{y\in\Psi(\bar{x})} l(\bar{x},y) \le \inf_{y\in\Psi(x)} l(x,y) \le \sup_{y\in\Psi(x)} l(x,y) \quad \text{for each } x\in X.$$

Hence $l(\bar{x},\bar{y}) \le \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y)$ and the proof of $i)$ is complete.
Proof of ii). Let $(\bar{x}_n,\bar{y}_n)_n$ be a sequence converging to $(\bar{x},\bar{y})$ such that $(\bar{x}_n,\bar{y}_n)$ is a solution to (LS) for each $n\in\mathbb{N}$. We need to prove that $(\bar{x},\bar{y})$ is an LSE.
First, $\bar{x}\in X$ by compactness of X. Moreover, since Ψ is an upper semicontinuous set-valued map with compact values, then Ψ is closed (see, for example, [1, Proposition 1.4.8]). So, being $\bar{y}_n\in\Psi(\bar{x}_n)$ for each $n\in\mathbb{N}$ by definition of LSE, it follows that $\bar{y}\in\Psi(\bar{x})$.
Finally, since l is lower semicontinuous and $(\bar{x}_n,\bar{y}_n)$ is a solution to (LS) for each $n\in\mathbb{N}$, then

$$l(\bar{x},\bar{y}) \le \liminf_{n\to+\infty} l(\bar{x}_n,\bar{y}_n) \le \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y),$$

so $(\bar{x},\bar{y})$ is an LSE. □

The next result shows the connections among the set of equilibria of (PB) and of (OB), denoted by $\mathcal{E}_{PB}$ and $\mathcal{E}_{OB}$, respectively, and the set of lower Stackelberg equilibria.

Proposition 3.2. *The following relations hold:*

i) $\mathcal{E}_{PB}\cup\mathcal{E}_{OB}\subseteq\mathcal{E}_{LS}$*;*

ii) $\mathcal{E}_{PB}=\mathcal{E}_{OB}=\mathcal{E}_{LS}$ *if* Ψ *is single-valued.*

Proof. Proof of i). Note that, in the proof of Proposition 3.1 i), we have proved that a pair $(\bar{x},\bar{y})$ satisfying $\bar{x}\in\arg\min_{x\in X}\inf_{y\in\Psi(x)} l(x,y)$ and $\bar{y}\in\arg\min_{y\in\Psi(\bar{x})} l(\bar{x},y)$ is an LSE. So the existence of an LSE has been obtained by showing the existence of an optimistic equilibrium. Moreover, we have implicitly show that, regardless of the assumptions, the set of optimistic equilibria is included in the set of LSEs. Hence, now it is sufficient to prove the inclusion $\mathcal{E}_{PB}\subseteq\mathcal{E}_{LS}$. Let $(\bar{x},\bar{y})\in\mathcal{E}_{PB}$. By definition of equilibrium of (PB) we have $\bar{y}\in\Psi(\bar{x})$ and

$$l(\bar{x},\bar{y}) \le \sup_{y\in\Psi(\bar{x})} l(\bar{x},y) = \inf_{x\in X}\sup_{y\in\Psi(x)} l(x,y),$$

so $(\bar{x},\bar{y})\in\mathcal{E}_{LS}$.
Proof of ii). Suppose that Ψ is single-valued and let $\psi\colon\mathbb{R}^N\to\mathbb{R}^M$ be the function such that $\Psi(x)=\{\psi(x)\}$ for any $x\in\mathbb{R}^N$. In this case $\psi(x)$ is the unique solution to the lower-level problem

for each $x \in X$ in (PB), (OB) and (LS). So, the questions of finding an equilibrium of (PB), an equilibrium of (OB) and an LSE collapse in the problem:

$$\begin{cases} \text{find } (\bar{x}, \bar{y}) \text{ such that} \\ \bar{x} \in X, \ \bar{y} = \psi(\bar{x}) \text{ and } l(\bar{x}, \bar{y}) = \inf_{x \in X} l(x, \psi(x)), \\ \text{where } \{\psi(x)\} = \arg\min_{y \in \mathcal{Y}(x)} f(x, y), \end{cases}$$

corresponding to the problem of finding a *(classical) Stackelberg equilibrium* (see [33, Section 2] or, for example [14, Section 5] and [10, Section 4.2.1]). Hence, the sets $\mathcal{E}_{PB}$, $\mathcal{E}_{OB}$ and $\mathcal{E}_{LS}$ coincide. □

Remark 3.2 The following example shows that the inclusion in Proposition 3.2 i) cannot be reversed and that $\mathcal{E}_{LS}$ may be a non-convex set.

Example 3.1 Let $X = [0, 2]$, $\mathcal{Y}(x) = [-1, 1]$ for any $x \in X$,

$$l(x, y) = -x - y \quad \text{and} \quad f(x, y) = \begin{cases} 0, & \text{if } x \in [0, 1] \\ (x - 1)y, & \text{if } x \in]1, 2]. \end{cases}$$

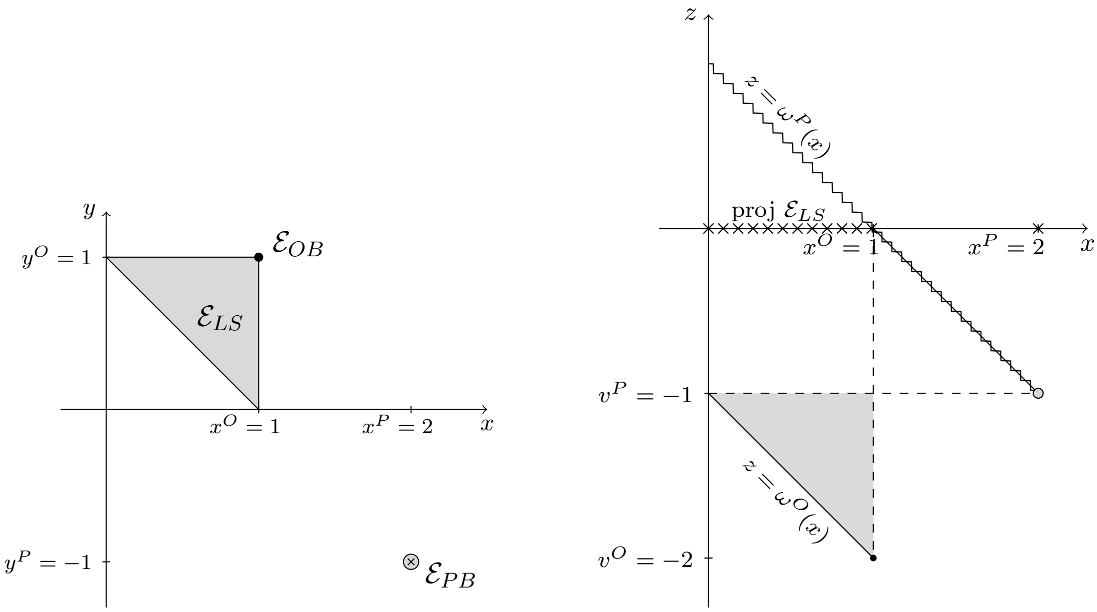
The set-valued map Ψ is defined by:

$$\Psi(x) = \begin{cases} [-1, 1], & \text{if } x \in [0, 1] \\ \{-1\}, & \text{if } x \in]1, 2]. \end{cases}$$

There is a unique equilibrium of (PB) and a unique equilibrium of (OB), namely $(2, -1)$ and $(1, 1)$, respectively. Instead, the set of LSEs is $\{(x, y) \in [0, 1]^2 \mid y \geq -x + 1\} \cup \{(2, -1)\}$.
Graphical representations are provided in Figure 1. More precisely, Figure 1a depicts the sets $\mathcal{E}_{PB} = \{(x^P, y^P)\}$ (cross), $\mathcal{E}_{OB} = \{(x^O, y^O)\}$ (black dot) and $\mathcal{E}_{LS}$ (in gray). So it is immediate to see that there exist LSEs that are neither equilibria of (PB) nor equilibria of (OB) and that the set $\mathcal{E}_{LS}$ is not convex. Figure 1b shows the graphs of the marginal functions ω^P (zigzag line) and ω^O (straight line) defined in (1) and (4), the projection of the set of LSEs on X (crosses) and the set $\{(x, v) \in X \times \mathbb{R} \mid x \in \text{proj}(\mathcal{E}_{LS}) \text{ and } \inf_{x \in X} \omega^O(x) \leq v \leq \inf_{x \in X} \omega^P(x)\}$ in gray. This emphasizes that the set of values achievable by the function l when (x, y) varying in $\mathcal{E}_{LS}$ is $[v^O, v^P]$, which agrees with Remark 3.1.
Finally, we observe also that the set $\{(\bar{x}, \bar{y}) \mid \bar{x} \text{ is a solution to } (OB) \text{ and } \bar{y} \in \Psi(\bar{x})\}$ is not a subset of $\mathcal{E}_{LS}$, i.e. the fact that $\bar{x}$ is an optimistic solution and $\bar{y}$ solves the lower-level problem parametrized by $\bar{x}$ does not guarantee that $(\bar{x}, \bar{y})$ is an LSE. In fact, consider the pair $(\bar{x}, \bar{y}) = (1, -1)$. Then:

$$1 \in \arg\min_{x \in [0,2]} \sup_{y \in \Psi(x)} l(x, y) \quad \text{and} \quad -1 \in \Psi(1),$$

but $l(1, -1) = 0 > -1 = \inf_{x \in [0,2]} \sup_{y \in \Psi(x)} l(x, y)$.



(a) Representation of $\mathcal{E}_{PB}$, $\mathcal{E}_{OB}$ and $\mathcal{E}_{LS}$. (b) Representation of ω^P, ω^O and values of l.

Figure 1: Graphical representations of Example 3.1.

3.2 Stability and related results

Now we deal with the issue of stability of lower Stackelberg equilibria. For any $n \in \mathbb{N}$, consider the perturbed problem

$$(LS_n) \begin{cases} \text{find } (\bar{x}_n, \bar{y}_n) \text{ such that} \\ \bar{x}_n \in X_n,\ \bar{y}_n \in \Psi_n(\bar{x}_n) \text{ and } l_n(\bar{x}_n, \bar{y}_n) \le \inf\limits_{x \in X_n} \sup\limits_{y \in \Psi_n(x)} l_n(x, y), \\ \text{where } \Psi_n(x) = \operatorname*{arg\,min}\limits_{y \in \mathcal{Y}_n(x)} f_n(x, y). \end{cases}$$

Given a sequence $(\bar{x}_n, \bar{y}_n)_n$ where $(\bar{x}_n, \bar{y}_n)$ is a solution to $(LS)_n$ for each $n \in \mathbb{N}$, we will investigate whether the limit of such a sequence of LSEs of perturbed problems is an LSE of the original problem (LS). First, it is worth to recall the Painlevé-Kuratowski notions of convergence for sets and set-valued maps that will be used in the sequel of the section (see, for example, [20, Ch. 2, §29], [1, Sect. 1.1] and [23, Sect. 2.2] for definitions).
Let $(A_n)_n$ be a sequence of sets with $A_n \subseteq \mathbb{R}^N$ for each $n \in \mathbb{N}$.

- The *lower limit* of $(A_n)_n$ is defined by

$$\operatorname*{Liminf}_{n \to +\infty} A_n \coloneqq \{a \in \mathbb{R}^N \mid \exists\, (a_n)_n \text{ converging to } a \text{ s.t. } a_n \in A_n\, \forall n \in \mathbb{N}\},$$

that is, $a \in \operatorname{Liminf}_{n \to +\infty} A_n$ if and only if a is the limit of a sequence $(a_n)_n$ with $a_n \in A_n$ for each $n \in \mathbb{N}$.

- The *upper limit* of $(A_n)_n$ is defined by

$$\operatorname*{Limsup}_{n\to+\infty} A_n := \{a \in \mathbb{R}^N \mid \exists\,(a_k)_k \text{ converging to } a \\ \text{s.t. } a_k \in A_{n_k} \text{ for a subsequence } (n_k)_k \subseteq \mathbb{N}\},$$

  that is, $a \in \operatorname{Limsup}_{n\to+\infty} A_n$ if and only if a is a cluster point of a sequence $(a_n)_n$ with $a_n \in A_n$ for each $n \in \mathbb{N}$.

Let $\mathcal{K}$ and $\mathcal{K}_n$ be set-valued maps from $\mathbb{R}^N$ to $\mathbb{R}^M$ for any $n \in \mathbb{N}$.

- The sequence $(\mathcal{K}_n)_n$ is *lower convergent* to $\mathcal{K}$ if $\mathcal{K}(a) \subseteq \operatorname{Liminf} \mathcal{K}_n(a_n)$ for each $a \in \mathbb{R}^N$ and each sequence $(a_n)_n$ converging to a.

- The sequence $(\mathcal{K}_n)_n$ is *upper convergent* to $\mathcal{K}$ if $\operatorname{Limsup} \mathcal{K}_n(a_n) \subseteq \mathcal{K}(a)$ for each $a \in \mathbb{R}^N$ and each sequence $(a_n)_n$ converging to a.

Before showing the main result on the stability of (LS), we prove two preliminary lemmas. The first one concerns the convergence of the sequence of set-valued maps $(\Psi_n)_n$, and it is a generalization of results already obtained when $\mathcal{Y}_n(x)$ is fixed, i.e. it does not depend on n and on x ([32, Proposition 4.1]), or when $\mathcal{Y}_n(x)$ is defined by inequalities ([27, Proposition 4.1]).

Lemma 3.1. *Assume that the sequence of set-valued maps $(\mathcal{Y}_n)_n$ is lower convergent to $\mathcal{Y}$ and that the sequence of functions $(f_n)_n$ is continuously convergent to f. Then, the sequence of set-valued maps $(\Psi_n)_n$ is upper convergent to Ψ.*

Proof. Let $(x_n)_n$ be a sequence converging to x and $\bar{y} \in \operatorname{Limsup}_{n\to+\infty} \Psi_n(x_n)$. We have to prove that $\bar{y} \in \Psi(x)$, namely $f(x,\bar{y}) \leq f(x,y)$ for any $y \in \mathcal{Y}(x)$.
Let $y \in \mathcal{Y}(x)$. By the lower convergence of the sequence of set-valued maps $(\mathcal{Y}_n)_n$ to $\mathcal{Y}$, there exists a sequence $\widetilde{y}_n$ converging to y with $\widetilde{y}_n \in \mathcal{Y}_n(x_n)$ for each $n \in \mathbb{N}$. Moreover, $\bar{y} \in \operatorname{Limsup}_{n\to+\infty} \Psi_n(x_n)$ implies that there exists a subsequence of integers $(n_k)_k \subseteq \mathbb{N}$ and a sequence $(\bar{y}_k)_k$ such that $(\bar{y}_k)_k$ converges to $\bar{y}$ and $\bar{y}_k \in \Psi_{n_k}(x_{n_k})$ for each $k \in \mathbb{N}$. Hence, since $(f_n)_n$ continuously converges to f, we get

$$f(x,\bar{y}) = \lim_{k\to+\infty} f_{n_k}(x_{n_k},\bar{y}_k) \leq \lim_{k\to+\infty} f_{n_k}(x_{n_k},\widetilde{y}_{n_k}) = f(x,y),$$

where the inequality holds because $\bar{y}_k \in \Psi_{n_k}(x_{n_k})$ and $\widetilde{y}_{n_k} \in \mathcal{Y}_{n_k}(x_{n_k})$. So $\bar{y} \in \Psi(x)$ and the proof is complete. □

The second lemma involves the marginal functions ω_n^P and ω^P defined in (1).

Lemma 3.2. *Assume that the hypotheses of Lemma 3.1 hold, that the sequence $(l_n)_n$ continuously converges to l, that $X \subseteq \operatorname{Liminf}_{n\to+\infty} X_n$ and that there exists a compact set Ω such that $\mathcal{Y}_n(x) \subseteq \Omega$ for each $(n,x) \in \mathbb{N} \times \mathbb{R}^N$. Then, for any $x \in X$ with $\omega^P(x) \in \mathbb{R}$, there exists a sequence $(x_n)_n$ converging to x such that $x_n \in X_n$ for each $n \in \mathbb{N}$ and $\omega_n^P(x_n) < +\infty$ for n sufficiently large.*

Proof. Let $x \in X$ such that $\omega^P(x) \in \mathbb{R}$. Since $X \subseteq \operatorname{Liminf}_{n\to+\infty} X_n$, there exists a sequence $(x_n)_n$ converging to x with $x_n \in X_n$ for each $n \in \mathbb{N}$.
If $\omega_n^P(x_n) = -\infty$ for n sufficiently large, then the lemma is proved.

Thus, assume it is not true that $\omega_n^P(x_n) = -\infty$ for n sufficiently large. So there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\omega_{n_k}^P(x_{n_k}) > -\infty$ for each $k \in \mathbb{N}$. Let us prove that $\omega_{n_k}^P(x_{n_k}) < +\infty$ for k sufficiently large.
By contradiction, suppose that there exists a subsequence $(x_{n_{k_h}})_h$ of $(x_{n_k})_k$ such that $\omega_{n_{k_h}}^P(x_{n_{k_h}}) = +\infty$. Let $M > 0$. By definition of ω_n^P, for each $h \in \mathbb{N}$ there exists $\widetilde{y}_{n_{k_h}} \in \Psi_{n_{k_h}}(x_{n_{k_h}})$ such that $l_{n_{k_h}}(x_{n_{k_h}}, \widetilde{y}_{n_{k_h}}) > M$. In particular, we have $\widetilde{y}_{n_{k_h}} \in \mathcal{Y}_{n_{k_h}}(x_{n_{k_h}}) \subseteq \Omega$ for each $h \in \mathbb{N}$. Thus, by the compactness of Ω, there exists a subsequence of $(\widetilde{y}_{n_{k_h}})_h$ converging to $\widetilde{y} \in \Omega$ and, in light of Lemma 3.1, it follows that $\widetilde{y} \in \Psi(x)$. To avoid heavy notations, we still denote by $(\widetilde{y}_{n_{k_h}})_h$ such a subsequence. Since $(l_n)_n$ continuously converges to l and $\widetilde{y} \in \Psi(x)$, then

$$M \leq \lim_{h \to +\infty} l_{n_{k_h}}(x_{n_{k_h}}, \widetilde{y}_{n_{k_h}}) = l(x, \widetilde{y}) \leq \omega^P(x).$$

This implies that $\omega^P(x) = +\infty$ (as M is arbitrary), which is in conflict with the assumption $\omega^P(x) \in \mathbb{R}$. Hence $\omega_{n_k}^P(x_{n_k}) < +\infty$ for k sufficiently large and the statement is proved. □

Now, let us show the stability for lower Stackelberg equilibria of problem (LS).

Theorem 3.1. *Assume that:*

$\mathcal{H}_1$) $\operatorname{Liminf}_{n\to+\infty} X_n = X$;

$\mathcal{H}_2$) *there exists a compact set* Ω *such that* $\mathcal{Y}_n(x) \subseteq \Omega$ *for each* $(n, x) \in \mathbb{N} \times \mathbb{R}^N$;

$\mathcal{H}_3$) *the sequence of set-valued maps* $(\mathcal{Y}_n)_n$ *is lower convergent to* $\mathcal{Y}$;

$\mathcal{H}_4$) *the sequence of functions* $(l_n)_n$ *continuously converges to* l;

$\mathcal{H}_5$) *the sequence of functions* $(f_n)_n$ *continuously converges to* f.

Let $(\bar{x}_n, \bar{y}_n)$ *be a solution to* $(LS)_n$ *for each* $n \in \mathbb{N}$. *If the sequence* $(\bar{x}_n, \bar{y}_n)_n$ *converges to* $(\bar{x}, \bar{y})$, *then* $(\bar{x}, \bar{y})$ *is a solution to* (LS).

Proof. Let $(\bar{x}, \bar{y})$ be the limit of the sequence $(\bar{x}_n, \bar{y}_n)_n$. In light of assumption $\mathcal{H}_1$), we have $\bar{x} \in X$. Moreover, since $\bar{y} \in \operatorname{Liminf}_{n\to+\infty} \Psi_n(\bar{x}_n)$, by Lemma 3.1 we get $\bar{y} \in \Psi(\bar{x})$. Now, let us show that $l(\bar{x}, \bar{y}) \leq \inf_{x\in X} \sup_{y\in\Psi(x)} l(x, y)$.
If $\inf_{x\in X} \sup_{y\in\Psi(x)} l(x, y) = +\infty$ the theorem is proved. Taking into account that $\inf_{x\in X} \sup_{y\in\Psi(x)} l(x, y) > -\infty$ (see Remark 3.1), we assume $\inf_{x\in X} \sup_{y\in\Psi(x)} l(x, y) \in \mathbb{R}$. We split the rest of the proof in three steps, recalling that v_n^P and v^P are defined in Section 2.1.

Claim 1: for any $x \in X$ there exists a sequence $(x_n)_n$ such that $x_n \in X_n$ for each $n \in \mathbb{N}$, the sequence $(x_n)_n$ converges to x and $\limsup_{n\to+\infty} \omega_n^P(x_n) \leq \omega^P(x)$.
Let $x \in X$. If $\omega^P(x) = +\infty$, *Claim 1* holds. If $\omega^P(x) = -\infty$, then $v^P = \inf_{x\in X} \omega^P(x) = -\infty$, which contradicts what stated before *Claim 1*. So we assume that $\omega^P(x) \in \mathbb{R}$. In light of Lemma 3.2, there exists a sequence $(\widetilde{x}_n)_n$ converging to x such that $\widetilde{x}_n \in X_n$ for each $n \in \mathbb{N}$ and $\omega_n^P(\widetilde{x}_n) < +\infty$ for n sufficiently large. Let $\epsilon > 0$. By definition of ω_n^P, for each $n \in \mathbb{N}$ there exists $y_n^\epsilon \in \Psi_n(\widetilde{x}_n)$ such that

$$\omega_n^P(\widetilde{x}_n) - \epsilon < l_n(\widetilde{x}_n, y_n^\epsilon), \tag{7}$$

for n sufficiently large. In particular $y_n^\epsilon \in \mathcal{Y}_n(\widetilde{x}_n)$ and so, by assumption $\mathcal{H}_2$), $y_n^\epsilon \in \Omega$ for each $n \in \mathbb{N}$. Thus there exists a subsequence $(y_{n_k}^\epsilon)_k$ of $(y_n^\epsilon)_n$ converging to $y^\epsilon \in \Omega$. Lemma 3.1 implies that $y^\epsilon \in \Psi(x)$. Hence, by (7) and assumption $\mathcal{H}_4$), we have

$$\limsup_{k\to+\infty} \omega_{n_k}^P(\widetilde{x}_{n_k}) \leq \limsup_{k\to+\infty} l_{n_k}(\widetilde{x}_{n_k}, y_{n_k}^\epsilon) + \epsilon = l(x, y^\epsilon) + \epsilon \leq \omega^P(x) + \epsilon,$$

where the last inequality holds as $y^\epsilon \in \Psi(x)$ and by definition of ω^P. Since ϵ is arbitrarily chosen, then $\limsup_{k\to+\infty} \omega_{n_k}^P(\widetilde{x}_{n_k}) \leq \omega^P(x)$ and so *Claim 1* is proved.

Claim 2: $\limsup_{n\to+\infty} v_n^P \leq v^P$.
Let $\epsilon > 0$. By definition of v^P and reminding that $v^P \in \mathbb{R}$ (see the beginning of the proof), there exists $x^\epsilon \in X$ such that

$$\omega^P(x^\epsilon) < v^P + \epsilon. \tag{8}$$

Moreover, in light of *Claim 1*, there exists a sequence $(\widetilde{x}_n^\epsilon)_n$ such that $\widetilde{x}_n^\epsilon \in X_n$ for each $n \in \mathbb{N}$, the sequence $(\widetilde{x}_n^\epsilon)_n$ converges to x^ϵ and $\limsup_{n\to+\infty} \omega_n^P(\widetilde{x}_n^\epsilon) \leq \omega^P(x^\epsilon)$. Hence, by (8) and definitions of v_n^P and v^P, we have

$$\limsup_{n\to+\infty} v_n^P \leq \limsup_{n\to+\infty} \omega_n^P(\widetilde{x}_n^\epsilon) \leq \omega^P(x^\epsilon) < v^P + \epsilon.$$

Being ϵ arbitrary, *Claim 2* is shown.

Claim 3: $l(\bar{x}, \bar{y}) \leq v^P$.
Since $(\bar{x}_n, \bar{y}_n)$ is a solution to $(LS)_n$, then $l_n(\bar{x}_n, \bar{y}_n) \leq v_n^P$ for each $n \in \mathbb{N}$. So, by assumption $\mathcal{H}_4$) and *Claim 2*, it follows that

$$l(\bar{x}, \bar{y}) = \lim_{n\to+\infty} l_n(\bar{x}_n, \bar{y}_n) \leq \limsup_{n\to+\infty} v_n^P \leq v^P.$$

Therefore *Claim 3* is proved and the proof is complete. □

Remark 3.3 Assumption $\mathcal{H}_2$) is connected with conditions used in previous works [28, 23, 7]. More precisely, $\mathcal{H}_2$) implies the *sequential subcontinuity*, as considered by Lignola and Morgan in [23, Definition 2.1.5], of each set-valued map $\mathcal{Y}_n$. In fact, Theorem 3.1 still holds if $\mathcal{H}_2$) is replaced by the sequential (*equi-*)subcontinuity of the sequence $(\mathcal{Y}_n)_n$:

$\widetilde{\mathcal{H}}_2$) for any $x \in \mathbb{R}^N$, any sequence $(x_n)_n$ converging to x and any sequence $(y_n)_n$ with $y_n \in \mathcal{Y}_n(x_n)$ for each $n \in \mathbb{N}$, there exists a subsequence (y_{n_k}) of $(y_n)_n$ convergent to $y \in \mathbb{R}^M$.

We preferred to state the stronger assumption $\mathcal{H}_2$) just for the sake of readability. Moreover, $\mathcal{H}_2$) collapses in hypothesis $(\mathcal{F}_3)$ in [7] when $\mathcal{Y}_n(x) = Y_n$ for any x and n.

Theorem 3.1 shows that the limit of sequences of LSEs of the perturbed problems $(LS)_n$ is an LSE of the original problem (LS). Hence, differently from what happens for problem (PB) and problem (OB), variational stability holds for problem (LS). It is worth noting that Theorem 3.1 implies also the following limit results on the asymptotic behavior of problems (PB) and (OB).

Corollary 3.1. *Assume $\mathcal{H}_1$)-$\mathcal{H}_5$). If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(PB)_n$ for each $n \in \mathbb{N}$, then each accumulation point of the sequence $(\bar{x}_n, \bar{y}_n)_n$ is an LSE.*

Corollary 3.2. *Assume $\mathcal{H}_1$)-$\mathcal{H}_5$). If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(OB)_n$ for each $n \in \mathbb{N}$, then each accumulation point of the sequence $(\bar{x}_n, \bar{y}_n)_n$ is an LSE.*

Therefore, on the one hand, Corollary 3.1 extends Theorem 2.2 where the constraints sets are not perturbed and a specific perturbation on the objective functions is applied. On the other hand, Corollary 3.2 extends Theorem 2.4 by accommodating possible non-differentiable frameworks.

4 From bilevel optimization to game theory

In this section we introduce the game theoretical framework associated to the BOPs. First, we present the Stackelberg games and the notion of subgame perfect Nash equilibrium, that is the traditional solution concept related to Stackelberg games. Then, we show the relation between the subgame perfect Nash equilibrium and the lower Stackelberg equilibrium concepts.

4.1 Stackelberg games and the subgame perfect Nash equilibrium

Consider a situation where two agents have to take their decisions sequentially in two periods: the agent acting in period 1, called *leader*, chooses an action x in her action set X; then, the agent acting in period 2, called *follower*, observes x and chooses an action y in his set of available actions $\mathcal{Y}(x)$, which depends on the leader's choice. Once both agents acted, the leader receives the payoff $l(x, y)$ and the follower receives the payoff $f(x, y)$. Such a strategic interaction represents, in game theory, the so-called *Stackelberg game* (due to H. von Stackelberg who first described this situation for sequential duopolies) or *one-leader one-follower two-stage game* or also *single-leader single-follower game*, that we denote with $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$. The restriction of the set-valued map Ψ to the set X is called *follower's best reply correspondence*, since it assigns to each action x of the leader the set of follower's best reactions to x, that is the set of actions available to the follower that maximize the follower's payoff function $f(x, \cdot)$ parameterized by the leader's choice.
In light of the game theoretical framework just presented, the pessimistic BOP represents the problem faced in a Stackelberg game by an "pessimistic" leader who believes that the follower will choose the worst action for her in his set of best reactions. Analogously, the optimistic BOP has an opposite interpretation (see [10, Section 4.2] for further discussion).
Now, let us recall the definition of a renowned solution concept for Stackelberg games and, in general, for sequential games.

Definition 4.1 ([49]) A pair $(\bar{x}, \bar{\varphi})$ with $\bar{x} \in X$ and $\bar{\varphi}\colon X \to \mathbb{R}^m$ is a *subgame perfect Nash equilibrium* of $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$ if the following two conditions are satisfied:

SE1) $\bar{\varphi}(x) \in \Psi(x)$ for each $x \in X$;

SE2) $\bar{x} \in \arg\min_{x \in X} l(x, \bar{\varphi}(x))$.

Moreover, a pair $(\bar{x}, \bar{y})$ is a *subgame perfect Nash equilibrium outcome* (SPNE-outcome for short) of Γ if the following condition is satisfied:

SEout) there exists $\bar{\varphi}\colon X \to \mathbb{R}^m$ such that $(\bar{x}, \bar{\varphi})$ is an SPNE of Γ and $\bar{\varphi}(\bar{x}) = \bar{y}$.

By looking at Definition 4.1, it is immediate to note that the SPNE is a pair made of a vector and a function, differently from the equilibrium concepts of (OB) and of (PB). The function $\bar{\varphi}$ is a *strategy of the follower* and states an action for the follower in each point of the game where he is required to make a decision, i.e. for each action the leader may choose. This notion is crucial for the *principle of sequential rationality*, which founds the SPNE concept: the players have to behave optimally from any point of the game onwards, so even with regard to off the equilibrium path actions. Conditions SE1) and SE2) formalize the sequential rationality for the follower and the leader, respectively. The SPNE-outcome is the pair of actions actually taken by the leader and the follower once the path described by the SPNE is realized. We recall that the SPNE is the most known refinement of the Nash equilibrium solution concept (in fact, the players' strategies in an SPNE represent a Nash equilibrium when restricted to each *subgame* of the original game) and it is the most used solution in the framework of sequential games with perfect information; see, [42, 2] for further discussion on definitions, properties and applications.

Remark 4.1 The existence of SPNEs is guaranteed when X and $\mathcal{Y}(x)$ are compact and l and f are continuous (see [17]). Moreover, we mention that specific selection procedures for SPNEs in Stackelberg games has been proposed in [45, 9, 40] by exploiting the Tikhonov regularization, the proximal point algorithm and the Shannon entropy, respectively. Recently, the variational stability for SPNEs and related outcomes has been investigated in [7] and a general method to construct an SPNE starting from sequences of SPNEs of perturbed Stackelberg games has been defined in [8].

Given the game-theoretical interpretations of BOPs, it is reasonable to think the pessimistic equilibrium and the optimistic equilibrium as solution concepts also in a Stackelberg game framework. So one can legitimately use expressions like *pessimistic equilibrium of the game* Γ and *optimistic equilibrium of the game* Γ. In particular, we mention that pessimistic and optimistic equilibrium notions allow to define specific *selections* of SPNEs motivated according to the optimistic and to the pessimistic attitudes of the leader, as presented in [10, Sections 4.2.5 and 4.3.5] (see [18] for the general theory of selection in games).

4.2 Lower Stackelberg equilibria and SPNE-outcomes

Now we investigate how the problem (LS) enters in the framework of Stackelberg games. The next result shows the connection between the concept of LSE and the concept of SPNE. Let us denote with $\mathcal{O}_{SPN}$ the set of SPNE-outcomes of a Stackelberg game $\Gamma = \langle X, \mathcal{Y}, l, f\rangle$.

Theorem 4.1. *Assume that the set-valued map $\mathcal{Y}$ is continuous and compact-valued and that the payoff functions l and f are continuous. Then the set of lower Stackelberg equilibria and the set of SPNE-outcomes coincide, that is $\mathcal{E}_{LS} = \mathcal{O}_{SPN}$.*

Proof. We prove separately the two inclusions $\mathcal{O}_{SPN} \subseteq \mathcal{E}_{LS}$ and $\mathcal{E}_{LS} \subseteq \mathcal{O}_{SPN}$.
Proof of $\mathcal{O}_{SPN} \subseteq \mathcal{E}_{LS}$. Let $(\bar{x}, \bar{y})$ be an SPNE-outcome. By Definition 4.1, there exists $\bar{\varphi}\colon X \to \mathbb{R}^M$ such that $(\bar{x}, \bar{\varphi})$ is an SPNE and $\bar{\varphi}(\bar{x}) = \bar{y}$.
First we have $\bar{y} \in \Psi(\bar{x})$ by SE1). Moreover, in light of SE2) and SE1), we get

$$l(\bar{x}, \bar{y}) = l(\bar{x}, \bar{\varphi}(\bar{x})) \le l(x, \bar{\varphi}(x)) \le \sup_{y \in \Psi(x)} l(x, y)$$

for any $x \in X$. Hence, $l(\bar{x}, \bar{y}) \leq \inf_{x \in X} \sup_{y \in \Psi(x)} l(x, y)$ and $(\bar{x}, \bar{y})$ is an LSE.
Proof of $\mathcal{E}_{LS} \subseteq \mathcal{O}_{SPN}$. Note that the continuity and the compact-valuedness of $\mathcal{Y}$ together with the continuity of f guarantee that Ψ has non-empty compact values, by the Berge maximum theorem. Let $(\bar{x}, \bar{y})$ be an LSE and $\bar{\varphi} \colon X \to \mathbb{R}^M$ be a function satisfying

$$\bar{\varphi}(\bar{x}) = \bar{y} \quad \text{and} \quad \bar{\varphi}(x) \in \arg\max_{y \in \Psi(x)} l(x, y) \text{ for any } x \in X \setminus \{\bar{x}\},$$

which is well-defined since Ψ has non-empty compact values and l is continuous. Let us show that $(\bar{x}, \bar{\varphi})$ is an SPNE.
By the definitions of $\bar{\varphi}$ and of LSE, condition SE1) holds.
In order to show SE2), it is sufficient to prove that $l(\bar{x}, \bar{y}) \leq l(x, \bar{\varphi}(x))$ for each $x \in X \setminus \{\bar{x}\}$, as $\bar{y} = \bar{\varphi}(\bar{x})$.
Let $x \in X \setminus \{\bar{x}\}$. Since $\bar{\varphi}(x) \in \arg\max_{y \in \Psi(x)} l(x, y)$, then $l(x, \bar{\varphi}(x)) \geq l(x, y)$ for any $y \in \Psi(x)$ and so

$$\sup_{y \in \Psi(x)} l(x, y) \leq l(x, \bar{\varphi}(x)).$$

In light of the above and by definition of LSE, we get

$$l(\bar{x}, \bar{y}) \leq \inf_{z \in X} \sup_{y \in \Psi(z)} l(z, y) \leq \inf_{z \in X \setminus \{\bar{x}\}} \sup_{y \in \Psi(z)} l(z, y) \leq \inf_{z \in X \setminus \{\bar{x}\}} l(z, \bar{\varphi}(z)) \leq l(x, \bar{\varphi}(x)).$$

Since x is arbitrarily chosen in $X \setminus \{\bar{x}\}$, then SE2) holds. Hence, being $\bar{y} = \bar{\varphi}(\bar{x})$, then SEout) is satisfied and the pair $(\bar{x}, \bar{y})$ is an SPNE-outcome. □

Therefore, Theorem 4.1 makes it legitimate referring to the solutions to (LS) as the *lower Stackelberg equilibria of the game* Γ. Hence, in light of Theorem 4.1 and Proposition 3.2, we can obtain a comprehensive picture on the connections among the considered equilibrium concepts in Stackelberg games (and in bilevel optimization), as represented in Figure 2.

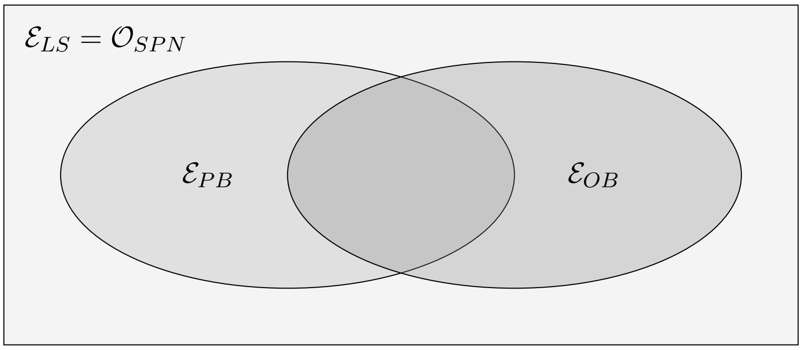


Figure 2: Relation between equilibrium concepts of a Stackelberg game $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$.

This achievement has interesting implications, both in game theory and in bilevel optimization frameworks.

From a game-theoretical point of view, Theorem 4.1 first provides a characterization of the set of SPNE-outcomes; more precisely, finding an SPNE-outcome is equivalent to find a solution to

problem (LS).
Moreover, the proof of the inclusion $\mathcal{E}_{LS} \subseteq \mathcal{O}_{SPN}$ tells how to identify SPNEs from LSEs, as stated below.

Corollary 4.1. *Assume that the set-valued map $\mathcal{Y}$ is continuous and compact-valued and that l and f are continuous. Let $(\bar{x}, \bar{y})$ be an LSE of Γ. Then, each pair $(\bar{x}, \bar{\varphi})$ satisfying $\bar{\varphi}(\bar{x}) = \bar{y}$ and $\bar{\varphi}(x) \in \arg\max_{y \in \Psi(x)} l(x, y)$ for any $x \in X \setminus \{\bar{x}\}$ is an SPNE of Γ.*

Note that, in light of the inclusion $\mathcal{E}_{PB} \cup \mathcal{E}_{OB} \subseteq \mathcal{E}_{LS}$ shown in Proposition 3.2 i), Corollary 4.1 applies also to obtain SPNEs starting from pessimistic or optimistic equilibria.
Furthermore, Theorems 3.1 and 4.1 imply the stability of Stackelberg games with respect to SPNE-outcomes.

Corollary 4.2. *Assume $\mathcal{H}_1$)-$\mathcal{H}_5$). Let $(\bar{x}_n, \bar{y}_n)$ be an SPNE-outcome of the perturbed Stackelberg game $\Gamma_n = \langle X_n, \mathcal{Y}_n, l_n, f_n \rangle$ for each $n \in \mathbb{N}$. If the sequence $(\bar{x}_n, \bar{y}_n)_n$ converges to $(\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y})$ is an SPNE-outcome of the original Stackelberg game $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$.*

In particular, Corollary 4.2 extends the stability result in [7, Theorem 3] where $\mathcal{Y}(x) = Y$ and $\mathcal{Y}_n(x) = Y_n$ for any x, i.e. the set of available actions for the follower does not depend on the leader's choice.

As regards to the bilevel optimization side, Theorem 4.1 gives a new insight on the lower Stackelberg equilibria in terms of game theory that allows to reinterpret the limit results Corollaries 3.1 and 3.2 on the equilibria of perturbed problems $(PB)_n$ and $(OB)_n$.

Corollary 4.3. *Assume $\mathcal{H}_1$)-$\mathcal{H}_5$).*

i) *If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(PB)_n$ for each $n \in \mathbb{N}$, then each accumulation point of the sequence $(\bar{x}_n, \bar{y}_n)_n$ is an SPNE-outcome of the Stackelberg game $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$.*

ii) *If $(\bar{x}_n, \bar{y}_n)$ is an equilibrium of $(OB)_n$ for each $n \in \mathbb{N}$, then each accumulation point of the sequence $(\bar{x}_n, \bar{y}_n)_n$ is an SPNE-outcome of the Stackelberg game $\Gamma = \langle X, \mathcal{Y}, l, f \rangle$.*

Therefore, in all the algorithms for approximating equilibria of (PB) and of (OB) where special perturbations are applied to the objective functions (like Tikhonov regularization, proximal methods, penalty methods; see [16, Section 20.6] for a review of the solution algorithms in bilevel optimization), the related limit points are guaranteed to be SPNE-outcomes of the associated Stackelberg game.

5 Conclusions

In this paper we investigated the notion of lower Stackelberg equilibrium, an equilibrium concept for bilevel optimization first appearing just as a limit point in the analysis of the behavior of pessimistic and of optimistic BOPs under perturbation. On the optimization side, we showed results of existence, connections with pessimistic and optimistic BOPs, closure of the set of LSEs and stability of the LSE concept under perturbations of both the constraints sets and of the objective functions. On the game theory side, considering Stackelberg games (i.e. the natural game-theoretical environment associated to BOPs), we show that the problems of finding an LSE and an SPNE-outcome are equivalent and so the LSE concept acquires (even more) relevance also beyond the bilevel optimization context.

In order to enrich the study, it would be interesting to compare the lower Stackelberg equilibrium also with other equilibrium concepts in bilevel optimization and Stackelberg games, as the *intermediate Stackelberg equilibrium* (as defined in [39, 37]) or the equilibria associated to the *viscosity solutions* (as defined in [30, 26]). To not overload the paper, we postpone such an investigation to a future research.

The notion of lower Stackelberg equilibrium is not to be understood as confined to the classical bilevel optimization framework. In fact, we present an hint on how to extend the LSE concept to the *bilevel problems with generalized Nash equilibrium constraints*, i.e. hierarchical problems consisting of an optimization problem in the upper level and a Nash equilibrium problem in the lower level (see Morgan and Raucci [47, 46], Lignola and Morgan [22, 30] for existence and approximation results on the pessimistic and optimistic versions of these problems; see also [24] and [10, Section 4.4.3] for the wider context of optimization problems with equilibrium constraints and [36] for the connected MPECs). Before defining the new concept, let us introduce the notation.
In the lower level, let $I = \{1, \dots, S\}$ be the set of players of the generalized Nash equilibrium problem, $f_i \colon \mathbb{R}^N \times \mathbb{R}^{M_1} \times \dots \times \mathbb{R}^{M_S} \to \mathbb{R}$ be the objective function of player $i \in I$, and $\mathcal{Y}_i$ be the set-valued map from $\mathbb{R}^N \times \mathbb{R}^{M_{-i}}$ to $\mathbb{R}^{M_i}$ where $\mathbb{R}^{M_{-i}} := \Pi_{j \in I \setminus \{i\}} \mathbb{R}^{M_j}$ and $\mathcal{Y}_i(x, y_{-i})$ is the constraint set of the player i once the upper-level player chooses x and the lower-level players other than i choose $y_{-i} := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_S)$. In the upper level, let $X \subseteq \mathbb{R}^N$ be the constraint set of the upper-level optimization problem and $l \colon \mathbb{R}^N \times \mathbb{R}^{M_1} \times \dots \times \mathbb{R}^{M_S} \to \mathbb{R}$ be the related objective function. The extension of problem (LS) and of the LSE notion to the framework described above are the following:

$$
(LSN)\ \begin{cases}
\text{find } (\bar{x}, \bar{y}_1, \dots, \bar{y}_S) \text{ such that} \\
\bar{x} \in X,\ (\bar{y}_1, \dots, \bar{y}_S) \in \Upsilon(\bar{x}) \\
\text{and } l(\bar{x}, \bar{y}_1, \dots, \bar{y}_S) \leq \inf\limits_{x \in X} \sup\limits_{(y_1, \dots, y_S) \in \Upsilon(x)} l(x, y_1, \dots, y_S), \\
\text{where} \\
\Upsilon(x) = \{(y_1, \dots y_S) \mid f_i(x, y_i, y_{-i}) \leq f_i(x, z_i, y_{-i})\ \forall z_i \in \mathcal{Y}_i(x, y_{-i}), \forall i \in I\}.
\end{cases}
$$

Definition 5.1 A vector $(\bar{x}, \bar{y}_1, \dots, \bar{y}_S)$ is a *lower Stackelberg-Nash equilibrium* (LSNE for short) if it solves problem (LSN).

A research direction could be the study of the properties of the lower Stackelberg-Nash equilibrium both from the optimization and the game theory points of view. On the one hand, the issues of existence, closure, stability and the connections of LSNEs with the pessimistic and with the optimistic equilibria of bilevel problems with generalized Nash equilibrium constraints could be addressed. On the other hand, one could investigate how the LSNE enters in the framework of *Stackelberg-Nash games* or *single-leader multi-follower games* (introduced by Sherali, Soyster and Murphy [50]).

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References

[1] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Boston, MA: Birkhäuser, 2009.

[2] T. Başar and G. Olsder. *Dynamic Noncooperative Game Theory*. Society for Industrial and Applied Mathematics, 1998.

[3] J.F. Bard. "An algorithm for solving the general bilevel programming problem". In: *Math. Oper. Res.* 8 (1983), pp. 260–272. DOI: 10.1287/moor.8.2.260.

[4] Y. Beck, I. Ljubić, and M. Schmidt. "A survey on bilevel optimization under uncertainty". In: *Eur. J. Oper. Res.* 311 (2023), pp. 401–426. DOI: 10.1016/j.ejor.2023.01.008.

[5] K.C. Border. *Fixed point theorems with applications to economics and game theory*. Cambridge (UK): Cambridge University Press, 1985.

[6] A. Cambini and L. Martein. *Generalized Convexity and Optimization*. Berlin Heidelberg: Springer, 2009. ISBN: 978-3-540-70875-9. DOI: 10.1007/978-3-540-70876-6.

[7] F. Caruso, M.C. Ceparano, and J. Morgan. "Asymptotic behavior of subgame perfect Nash equilibria in Stackelberg games". In: *Ann. Oper. Res.* (2023). DOI: 10.1007/s10479-023-05422-2.

[8] F. Caruso, M.C. Ceparano, and J. Morgan. "On construction of subgame perfect Nash equilibria in Stackelberg games". In: *Int. Game Theory Rev.* (forthcoming). working paper version available from https://www.csef.it/WP/wp636.pdf.

[9] F. Caruso, M.C. Ceparano, and J. Morgan. "Subgame perfect Nash equilibrium: a learning approach via costs to move". In: *Dyn. Games Appl.* 9 (2019), pp. 416–432. DOI: 10.1007/s13235-018-0277-3.

[10] F. Caruso, M.B. Lignola, and J. Morgan. "Regularization and approximation methods in Stackelberg games and bilevel optimization". In: *Bilevel Optimization: Advances and Next Challenges*. Ed. by S. Dempe and A. Zemkoho. Cham: Springer International Publishing, 2020, pp. 77–138. DOI: 10.1007/978-3-030-52119-6_4.

[11] B. Colson, P. Marcotte, and G. Savard. "An overview of bilevel optimization". In: *Ann. Oper. Res.* 153 (2007), pp. 235–256. DOI: 10.1007/s10479-007-0176-2.

[12] S. Dempe. "A bundle algorithm applied to bilevel programming problems with non-unique lower level solutions". In: *Comput. Optim. Appl.* 15 (2000), pp. 145–166. DOI: 10.1023/A:1008735010803.

[13] S. Dempe. "A simple algorithm for the-linear bilevel programming problem". In: *Optimization* 18 (1987), pp. 373–385. DOI: 10.1080/02331938708843247.

[14] S. Dempe. *Foundations of bilevel programming*. Springer US, 2002.

[15] S. Dempe and H. Schmidt. "On an algorithm solving two-level programming problems with nonunique lower level solutions". In: *Comput. Optim. Appl.* 6 (1996), pp. 227–249. DOI: 10.1007/BF00247793.

[16] S. Dempe and A. Zemkoho. *Bilevel Optimization*. Cham: Springer, 2020. DOI: 10.1007/978-3-030-52119-6.

[17] C. Harris. "Existence and characterization of perfect equilibrium in games of perfect information". In: *Econometrica* 53 (1985), pp. 613–628.

[18] J.C. Harsanyi and R. Selten. *A General Theory of Equilibrium Selection in Games*. Cambridge, MA: The MIT Press, 1988.

[19] T. Kleinert, M. Labbé, I. Ljubić, and M. Schmidt. "A survey on mixed-integer programming techniques in bilevel optimization". In: *EURO J. Comput. Optim.* 9 (2021), p. 100007. DOI: 10.1016/j.ejco.2021.100007.

[20] K. Kuratowski. *Topology*. Vol. 1. New York: Academic Press, 1966.

[21] G. Leitmann. "On generalized Stackelberg strategies". In: *J. Optim. Theory Appl.* 26 (1978), pp. 637–643. DOI: 10.1007/BF00933155.

[22] M.B. Lignola and J. Morgan. "α-Well-posedness for Nash Equilibria and For Optimization Problems with Nash Equilibrium Constraints". In: *J. Global Optim.* 36 (2006), pp. 439–459. DOI: 10.1007/s10898-006-9020-5.

[23] M.B. Lignola and J. Morgan. "Convergences of marginal functions with dependent constraints". In: *Optimization* 23 (1992), pp. 189–213. DOI: 10.1080/02331939208843758.

[24] M.B. Lignola and J. Morgan. "Existence for optimization problems with equilibrium constraints in reflexive Banach spaces". In: *Optimization in Economics, Finance and Industry*. Ed. by G. P. Crispi et al. Milano: Datanova, 2002, pp. 15–35.

[25] M.B. Lignola and J. Morgan. "Existence of solutions to generalized bilevel programming problem". In: *Multilevel Optimization: Algorithms and Applications*. Ed. by A. Migdalas, P.M. Pardalos, and P. Värbrand. Boston, MA: Springer US, 1998, pp. 315–332. DOI: 10.1007/978-1-4613-0307-7_14.

[26] M.B. Lignola and J. Morgan. "Inner regularizations and viscosity solutions for pessimistic bilevel optimization problems". In: *J. Optim. Theory Appl.* 173 (2017), pp. 183–202. DOI: 10.1007/s10957-017-1085-4.

[27] M.B. Lignola and J. Morgan. "Stability of Regularized Bilevel Programming Problems". In: *J. Optim. Theory Appl.* 93 (1997), pp. 575–596. DOI: 10.1023/A:1022695113803.

[28] M.B. Lignola and J. Morgan. "Topological existence and stability for min sup problems". In: *J Math Anal Appl* 151 (1990), pp. 164–180. DOI: 10.1016/0022-247X(90)90248-E.

[29] M.B. Lignola and J. Morgan. "Topological existence and stability for Stackelberg problems". In: *J. Optim. Theory Appl.* 84 (1995), pp. 145–169. DOI: 10.1007/BF02191740.

[30] M.B. Lignola and J. Morgan. "Viscosity solutions for bilevel problems with Nash equilibrium constraints". In: *Far Est Journal of Applied Mathematics* 88 (2014), pp. 15–34.

[31] P. Loridan and J. Morgan. "A theoretical approximation scheme for Stackelberg problems". In: *J. Optim. Theory Appl.* 61 (1989), pp. 95–110. DOI: 10.1007/BF00940846.

[32] P. Loridan and J. Morgan. "Approximate solutions for two-level optimization problems". In: *Trends in Mathematical Optimization: 4th French-German Conference on Optimization 1986*. Ed. by K.-H. Hoffmann, J. Zowe, J.-B. Hiriart-Urruty, and C. Lemarechal. Basel: Birkhäuser Basel, 1988, pp. 181–196. DOI: 10.1007/978-3-0348-9297-1_13.

[33] P. Loridan and J. Morgan. "New results on approximate solution in two-level optimization". In: *Optimization* 20 (1989), pp. 819–836. DOI: 10.1080/02331938908843503.

[34] P. Loridan and J. Morgan. "Regularizations for two-level optimization problems". In: *Advances in Optimization*. Ed. by W. Oettli and D. Pallaschke. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 239–255.

[35] P. Loridan and J. Morgan. "Weak via strong Stackelberg problem: new results". In: *J. Glob. Optim.* (1996), pp. 263–287. DOI: 10.1007/BF00121269.

[36] Z.-Q. Luo, J.-S. Pang, and D. Ralph. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press, 1996.

[37] L. Mallozzi and J. Morgan. "Hierarchical systems with weighted reaction set". In: *Nonlinear Optimization and Applications*. Ed. by G. Di Pillo and F. Giannessi. Boston, MA: Springer US, 1996, pp. 271–282. DOI: 10.1007/978-1-4899-0289-4_19.

[38] L. Mallozzi and J. Morgan. "Oligopolistic markets with leadership and demand functions possibly discontinuous". In: *J. Optim. Theory Appl.* 125 (2005), pp. 393–407. DOI: 10.1007/s10957-004-1856-6.

[39] L. Mallozzi and J. Morgan. "Problema di Stackelberg con risposte pesate". In: *Atti del XIX Convegno AMASES* (1995), pp. 416–425.

[40] L. Mallozzi and P.M. Pardalos. "Entropic regularization in hierarchical games". In: *Oper. Res. Forum* 3 (2022). DOI: 10.1007/s43069-022-00130-2.

[41] O.L. Mangasarian. *Nonlinear programming*. Society for Industrial and Applied Mathematics, 1994. DOI: 10.1137/1.9781611971255.

[42] M. Maschler, E. Solan, and S. Zamir. *Game theory*. Cambridge: Cambridge University Press, 2013.

[43] D.A. Molodtsov and V.V. Fedorov. "Approximation of two-person games with information exchange". In: *USSR Comput. Math. Math. Phys.* 13 (1973), pp. 123–142. DOI: 10.1016/0041-5553(73)90010-4.

[44] J. Morgan. "Constrained well-posed two-level optimization problems". In: *Nonsmooth Optimization and Related Topics*. Ed. by F.H. Clarke, V.F. Dem'yanov, and F. Giannessi. Boston, MA: Springer US, 1989, pp. 307–325. DOI: 10.1007/978-1-4757-6019-4_18.

[45] J. Morgan and F. Parone. "Stackelberg problems: subgame perfect equilibria via Tikhonov regularization". In: *Advances in Dynamic Games: Applications to Economics, Management Science, Engineering, and Environmental Management*. Ed. by A. Haurie, S. Muto, L.A. Petrosyan, and T.E.S. Raghavan. Boston, MA: Birkhäuser Boston, 2006, pp. 209–221. DOI: 10.1007/0-8176-4501-2_12.

[46] J. Morgan and R. Raucci. "Approximate solutions for strong hierarchical saddle point problems". In: *Atti del Seminario Matematico e Fisico dell'Università di Modena* 45 (1997), pp. 395–410.

[47] J. Morgan and R. Raucci. "Continuity Properties of ϵ-Solutions for Generalized Parametric Saddle Point Problems and Application to Hierarchical Games". In: *J. Math. Anal. Appl.* 211 (1997), pp. 30–48. DOI: 10.1006/jmaa.1997.5362.

[48] J.V. Outrata. "On the numerical solution of a class of Stackelberg problems". In: *Z. Oper. Res.* 34 (1990), pp. 255–277. DOI: 10.1007/BF01416737.

[49] R. Selten. "Spieltheoretische behandlung eines oligopolmodells mit nachfrageträgheit". In: *Z. Gesamte Staatswiss.* 121 (1965), pp. 301–324.

[50] H.D. Sherali, A.L. Soyster, and F.H. Murphy. "Stackelberg-Nash-Cournot equilibria: characterizations and computations". In: *Oper. Res.* 31 (1983), pp. 253–276.

[51] K. Shimizu and E. Aiyoshi. “A new computational method for Stackelberg and min-max problems by use of a penalty method”. In: *IEEE T. Automat. Control* 26 (1981), pp. 460–466. DOI: 10.1109/TAC.1981.1102607.

[52] V.F. Sholohovich. “Unstable extremal problems and geometric properties of Banach spaces”. In: *Dokl. Akad. Nauk* 195 (1970), pp. 289–291.

[53] A.N. Tikhonov. “Methods for the regularization of optimal control problems”. In: *Sov. Math. Dokl.* 6 (1965), pp. 761–763.

[54] L. Vicente, G. Savard, and J. Júdice. “Descent approaches for quadratic bilevel programming”. In: *J. Optim Theory Appl.* 81 (1994), pp. 379–399. DOI: 10.1007/BF02191670.

[55] L.N. Vicente and P.H. Calamai. “Bilevel and multilevel programming: a bibliography review”. In: *J. Glob. Optim.* 5 (1994), pp. 291–306. DOI: 10.1007/BF01096458.

[56] H. von Stackelberg. *Marktform und Gleichgewicht.* [Translation: D. Bazin, R. Hill and L. Urch. *Market Structure and Equilibrium.* Springer-Verlag Berlin Heidelberg, 2011]. Berlin: Verlag von Julius Springer, 1934.

[57] J.J. Ye and D.L. Zhu. “Optimality conditions for bilevel programming problems”. In: *Optimization* 33 (1995), pp. 9–27. DOI: 10.1080/02331939508844060.