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Competitive Provision of Digital Goods

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Abstract

We study the distribution of goods that are freely duplicated and damaged. The monopolist solves a screening problem that is not cost-separable and requires a concave-linear preference specification to generate nontrivial allocations, associated with two interdependent inefficiencies: underacquisition and damaging. In a game where firms acquire market power through an irreversible investment, both monopoly and active competition emerge as equilibria. Despite worsening underacquisition and inducing double-spending, competition may increase welfare because it mitigates the damaging inefficiency by distributing a version for free. We discuss an application to information markets, where experts produce a signal and sell Blackwell-garbled versions of it.

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1 Introduction

This paper studies the distribution of goods with the following characteristics:

1. They are produced along a *single-dimensional quality ranking*: consumers have unit demand and agree on this ranking, but have *heterogenous tastes* for quality;

2. They are *non-rival* but *excludable* and sold through a price-quality menu;[1]

3. The cost of producing a version of the product is increasing and convex in its quality, but sellers can *duplicate and damage* every version at no cost;

4. When multiple firms are active, their products are *homogenous*: same-quality products offered by different sellers are treated as perfect substitutes.

Our analysis is targeted towards the market for digital content – computer software, mobile apps, streaming services – where sellers can freely replicate (and hide some of) the lines of code they have developed, as well as some portions of the market for information – weather forecasts, non-strategic financial consultancy – where experts sell the realization of a signal they have observed, possibly garbled to discriminate for heterogeneous value of information. Until Section 5, that discusses the application to information markets, we abstract from the specifics of any market and refer to goods that have the four features listed above as *digital goods*. This paper explores the role of market power in digital goods markets by characterizing the monopolist and competitive allocations and comparing their properties against each other and against a common efficiency benchmark.

The technological distinctiveness of digital goods (point 3.) is that after developing a product of quality q the seller can offer arbitrary amounts of any quality (weakly) below q. Since consumers have heterogeneous returns from quality (point 1.), a seller with market power (point 2.) acts as classic multiproduct monopolist (Mussa and Rosen (1978), Maskin and Riley (1984)) and might want to damage the good for screening purposes (Deneckere and McAfee (1996)). Contrary to the standard setting, in our model the cost incurred by the seller is not separable in the quality that each agent ends up consuming but depends solely on a statistic – the maximum – of the whole allocation.

[1] An essential non-rivalry arises from free replicability in production, so this property and is one and the same as the free-duplication listed in point 3. The excludability issue is more critical; a literature on so-called "information goods" (e.g. Muto (1986), Varian (2000), Polanski (2007) among others) focuses on the distribution of non-excludable products, including (some) software and books. Those are *not* digital goods according to the definition given in this paper. Admati and Pfleiderer (1986) addresses the partial non-excludability of canonical information products (signals about a payoff relevant state).

Given the special form of non-separability, the monopolist treats the quality-distribution and quality-development as two distinct phases: it is as if she payed an acquisition cost to raise a maximum-quality constraint on a standard screening problem with identically zero cost function. Because the production costs are sunk at the distribution phase, preferences alone must determine the curvature – i.e. dependance on consumers' type – of the monopolist allocation. The multiplicative specification θq, customarily employed in the literature of quality screening, lacks such curvature and induces a trivial allocation where the seller offers a menu with a single item and excludes low valuation types. This implication is empirically implausible because digital goods are often distributed through rich (non-singleton) menus: Most softwares and mobile apps are distributed in several versions where the inferior ones are essentially created by disabling some features of the premium version; movies and other kinds of digital content are also often distributed in versions that differ on video and audio definition, availability of extra content, etc. A specification of consumers' returns from quality (point 1.) that does not preclude rich contracts should therefore be a distinctive feature of digital goods market, on the same level as the cost function. For this reason, we treat the multiplicative specification as a degenerate case of a richer set of preferences obtained by adding a common (i.e. type-independent) decreasing-returns component. A simple microfoundation – detailed in Section 2 and arguably a compelling description of the returns from software consumption– for this type of preferences is that agents use the digital good to perform both a basic and a professional activity with heterogeneous returns only in the accomplishment of the latteractivity. Besides its empirical plausibility, such specification allows to extend the techniques for cost-separable screening problems with multiplicative preferences since the common component of the utility acts essentially like a cost term in determining each type's virtual valuation. Despite such apparent similarity in the setup of the problem, both the shape and the welfare properties of the optimal contract are qualitatively different from those that arise in a cost-separable environment with the same demand primitives. For example, with the digital goods cost structure no type is ever excluded, a flat allocation where every consumer receives the same quality (full bunching) can be optimal, and the classic "efficiency at the top" result is restricted to *distributional* efficiency: even the highest type receives a version below its first best.

With the solution of the monopolist problem at hands, it is easy to characterize the outcomes of a Bertrand game in which homogenous firms (point 4.) compete to distribute an exogenously given set of qualities. Endogenizing the production vector as the (probabilistic) equilibrium of an irreversible investment game we establish existence and study the welfare properties of equilibria with active screening and competition, namely in a setting that is prone to market failures and tractability issues (for a discussion, see Stole (2007)). With this competition structure we are able to match a second empirical regularity of digital goods markets: not only multiple versions of the same good are offered, but one of those is often available for free. An enormous amount of information is available at no monetary cost; the same is true for online services, ranging from e-mail services to document storage and digital contents.[2] Alternative setups that deliver tractable equilibria of competition with screening – most notably the entrant-incumbent and multidimensional spatial (Hotelling) frameworks – are inconsistent with this "free version" regularity that instead emerges in our framework as an immediate consequence of Bertrand competition across homogenous products with sunk (acquisition and replication) costs. We therefore think that the irreversibility of investment is a more plausible source of the market power needed to prevent failure in the competitive provision of digital goods.[3] The analysis of competitive equilibria produces sharp results. We show that, relative to the monopolist benchmark, competition shrinks the set of qualities that are available on the market. This contraction has the counteacting welfare effects of i) exhacerbating the monopolist underprovision of quality, while at the same time ii) reducing the discrimination (damaging) inefficiencies. Depending on the primitives specification either of these forces can dominate, and therefore the question of whether digital goods are a natural monopoly cannot be answered before estimating the demand a production function of the particular market.

The paper proceeds as follows. We conclude this introductory section by describing our resutls in more detail and reviewing the relevant literature. Section 2 formalizes the primitives and establishes the efficiency benchmark. Section 3 studies the problem of a digital goods monopolist under asymmetric information. Section 4 presents a model of competition in digital goods markets and compares the properties of monopolist and competitive equilibria. Section 5 assesses the fitness of our framework to study the market for information. Section 6 concludes. Appendix A presents additional results; all proofs are gathered in Appendix B.

Preview of the Results

Because there are no replication costs damaging is always inefficient, the first-best is a flat allocation where all consumers receive the version that maximizes the average

[2] As for computer software, it is interesting to notice that Open Office was released in 2002, 12 years after Microsoft sold (at positive price) the first Office package, consistent with the implication of this paper that active competition goes hands in hands with some versions of the good being provided for free.

[3] Geographical proximity and transportation costs also don't seem appealing justification for the survival of dominated versions in reasonably transparent and global markets like those of software and information. More in general, whatever in the production function makes the good freely replicable is also likely to make transportation costs negligible. The ex-ante asymmetry across (potential) competitors which is implicit in the incumbent-entrants models, which has indeed been successfully employed to anaylize competition in large incumbent airline and cable industries, seems also less realistic when imposed on digital goods markets.

utility net of development costs. The problem of a monopolist under asymmetric information is equivalent to choosing (at cost) what is the maximum quality that agents can consume in a standard screening problem with zero costs. The constrained allocations are easy to characterize: consumers receive the minimum between the constraint itself and an increasing function of their types – the virtual value maximizer –. We adopt a preference specification that ensures that this maximizer is well-behaved and non trivial (at least for a positive measure of types), thus avoiding the "no-haggling trap" where multiplicative preferences push all constrained allocations. Increasing the constraint shifts up a bunching-at-the-top threshold, raising the allocation (and rent) of high valuation types while leaving the allocation (and rent) of lower types unchanged. In particular, low qualities are optimally distributed in a full-bunching contract, with no inefficient damaging from screening. Notice an important difference with a cost-separable setting where the benefits of changing the allocation of an agent – including information rents for higher types – are traded-off with the cost of producing that specific unit. A digital good monopolist faces instead an acquisition cost that is distinct from the distribution cost: preferences alone determine the distribution of each quality constraint, and the cost only pins down the *level* of the constraint. Two interdependent sources of inefficiency arise as i) the monopolist always acquires a suboptimally low quality and ii) may serve it damaged to some types. In particular, the efficiency at the top property of texbook screening problems is limited to a *distributional* efficiency: although a positive measure of types never receive a damaged quality, even the highest type gets a quality below the efficient level.

We model competition as a two stage game of perfect information, akin to Kreps and Scheinkman (1983) and Champsaur and Rochet (1989): In the first stage firms make a costly and irreversible investment in quality that is observed by everyone before a standard Bertrand pricing game is played. This second stage is easy to solve using the tools developed in the monopolist problem; the owner of the largest quality behaves indeed as an interim monopolist on the quality spectrum she owns exclusively, while competition pushes the price of the second highest among produced versions to zero. In the first stage there are multiple equilibria indexed by n, the number of firms that choose to acquire a positive quality with positive probability – i.e. are *active* –. With $n = 1$ the active firm replicates the monopolist allocation and implicitly commits to wiping out the revenues from any quality below, which is sufficient to deter entry. Although this is the only equilibrium in pure strategies, for any $n \geq 2$ there is a symmetric equilibrium in which n firms randomize investment with full support ranging from zero to the monopolist quality. By shrinking the set of marketed qualities, competition worsens underacquisition but alleviates the damaging inefficiency. It also induces inefficient multiple spending since developing inferior versions is obviously socially wasteful. We show that equilibria with active competition ($n \geq 2$) are Pareto ranked, decreasing in n: The relevant comparison is therefore between the equilibrium with one and two active firms. Their ranking is ambiguous as different shapes of the acquisition cost function can shut down almost completely either the positive or the negative impacts of competition. In particular, if the monopolist was not damaging (steep acquisition cost inducing a full bunching contract), then competition unambiguously reduces total welfare. By contrast, if costs are extremely convex – i.e. approaching a fixed cost structure –, then a competitive market induces an allocation that converges (in probability, since competitive allocations are always stochastic) to the flat allocation where everybody receives the quality produced (but not distributed) by a monopolist, and duopoly dominates.

1.1 Related Literature

We review the contributions related to the building blocks of our model: the demand side, the production technology, and competition with screening. Models of information markets are reviewed separately in Section 5.

The digital good monopolist solves a sequence of constrained screening problem (Mussa and Rosen (1978), Maskin and Riley (1984) and Wilson (1993)). We impose a preference specification which ensures that we can disregard ironing and other technical complications *within* each constrained problem – i.e. the focus of the original paper and Rochet and Choné (1998)–, and that we have a tractable revenue comparison *across* different problems.

The idea of damaging a good for screening purposes was introduced in Deneckere and McAfee (1996).[4] The approach in this paper is different both in modeling choice and in the type of questions addressed. From a modeling perspective, beyond preserving a positive marginal cost from distribution, Deneckere and McAfee (1996) take a binary set of qualities as exogenously fixed – thereby excluding an acquisition margin – and assume that the only way to produce the good of low quality is by damaging the high quality good. Distribution costs are always positive and *larger* for the low quality good.[5] They focus on the monopolist problem, identifying conditions under which the possibility to damage is Pareto improving; Appendix A.1 addresses the natural extension of their question to our framework. Product versioning through quality damaging has been explored also in the context of a durable good monopolist. In related papers, Inderst (2008) and Hahn (2006) consider an environment with two consumer types and a monopolist that sells different versions of a product over time

[4] Prior work of Srinagesh and Bradburd (1989) offer a very general analysis for the case where there are two types of customer. A subsequent extension of McAfee (2007) provides an exact characterization in terms of marginal revenues of when damaging is profitable.

[5] Their motivating examples include processors, printers and other technological products. Clearly, under a positive cost of damaging it is more startling that a monopolist is (sometimes) willing to engage in screening.

and faces the Coasian commitment problem. Availability of a damaged quality may also result from illegal activities such as piracy; Takeyama (1994) shows that the seller may benefit from being copied since this reduces the commitment problem.[6]

The analysis of competition in markets with asymmetric information has produced a vast body of literature initiated by the seminal contribution of Rothschild and Stiglitz (1976) on insurance markets. They invoke a notion of stability, justified by a free entry condition, which can be interpreted as the Nash equilibrium of a contract posting game among many (ex-ante) symmetric firms. Existence is not guaranteed due to the many deviations available to idle firms which can rip incumbents off of their profits by acting *after* they posted their contracts.[7] Garrett et al. (2019) assume that consumers are also imperfectly informed about the offers in the market; this two-sided asymmetric information generates dispersion over price-quality menus in equilibrium, where competition may raise prices for low-quality goods. Market power may alternatively arise because of geographical heterogeneity across consumers, which makes spatially dispersed firms solve interdependent multidimensional (location and preferences) screening problems. Rochet and Stole (2002) show that optimal pricing in this Hotelling framework involves adding a fixed fee to the cost; Hernandez (2011); Lahiri et al. (2021) also consider competition with price discrimination in similar spatial models and find that a duopoly sells solely the high-quality good while a monopolist may serve the damaged good. Such contraction of the pricing schedule under competition also emerges in incumbent-entrant models, employed to study the airline (e.g. Borenstein and Rose (1994); Gerardi and Shapiro (2009)) and cable (e.g. Crawford et al. (2019); Boik and Takahashi (2020)) industries. Johnson and Myatt (2003) assume the entrant and incumbent act simultaneously but that the former can only produce qualities below a certain threshold, thus granting the incumbent an exogenous technological advantage similar to the one that our interim monopolist will earn as a consequence of its investment realization. Neither transaction costs nor the imbalance of an incumbent-entrant framework seem however compelling sources of oligopolistic power in digital goods markets; on the contrary, the separation between production and distribution implicit in the cost structure[8] suggests a two stage game with irreversible investment

[6] Peitz and Waelbroeck (2006) provides a critical overview of the theoretical literature that addresses the economic consequences of end-user copying, though focusing mostly on the non-excludability of low qualities that is induced by the illegal activity, hence drawing a connection with the literature on non-excludable "information goods" referred to in footnote **??**.

[7] For a thorough review, see in the Handbook chapter of Stole (2007). Notable exceptions rely on the multidimensionality of the type space to generate competitive equilibria with profitable contracts in an RS setting. Netzer and Scheuer (2010) extend the basic model to two-dimensional heterogeneity, one exogenous (risk preferences) and one endogenous (wealth); equilibrium contracts that can earn strictly positive profits because any contract that attracts good consumers would also attract bad risk types and become unprofitable. Equilibria with profitable contracts also emerge in the two dimensional screening model of Smart (2000) where both dimensions are exogenous.

[8] The well-known drawback of modeling competition as an extensive-form game is that by choosing the imperfect competition model we make a number of implicit assumptions which make it problematic

as a natural way of modeling imperfect competition, which is the direction we pursue in Section 4.

2 Primitives and Efficiency

Demand

There is a unit mass of consumers that demand (at most) one version of the digital good, which is marketed along a continuum of versions (or qualities); $Q = \mathbb{R}_+$ denotes the quality space. Consumers are characterized by a payoff type θ that parametrizes their cardinal ranking over different qualities. We assume that the returns from quality are given by

$$u(q,\theta) = g(q) + \theta q \tag{1}$$

where is g a concave function that satisfies the Inada conditions $\lim_{q\to 0} g'(q) = \infty, \lim_{q\to\infty} g'(q) = 0$. We will further assume that preference types are uniformly distributed $\theta \sim \mathcal{U}[0,1]$. Section 3.2 considers relaxation of the demand primitives and identifies sufficient conditions to preserve the basic structure of the monopolist allocation (Theorem 1). Consumers have deep pockets and quasilinear preferences in money, so the demand correspondence associated to a quality pricing function $\boldsymbol{p}: Q \to \mathbb{R}$ is given by[9]

$$D_{\boldsymbol{p}}(\theta) = \arg\max_q u(q,\theta) - \boldsymbol{p}(q) \tag{2}$$

Returns from quality: Interpretation and properties

As the specification (1) is non-standard and plays a central role in our analysis we pause a moment to highlight its analytical properties and offer a microfoundation. We need to depart from the linear preference specification[10] $u(\theta,q) = \theta q$ because in the digital goods setting such specification would induce an optimal screening contract that is inconsistent with empirical evidence: A single version of the good would be offered at positive price in any equilibrium (competitive or not). Specification (1) be microfounded in a fairly natural two-tasks setup that we now describe in the context of

to compare results derived in different setups. This makes it difficult to compare our results with those in the literature and identify the effect of the fundamental difference of digital goods market, namely the cost function. For example, Boik and Takahashi (2020) prove (in an incumbent-entrant model) that competition does not affect the highest quality provided though it may force the incumbent to offer medium-quality packages. In our model competition (the mixed equilibrium of an irreversible investment game) induces a highest quality that is stochastic but bounded by the monopolist level which, contrary to their (or any cost-separable) setup is *not* efficient.

[9] Whenver necessary to avoid confusion with the realization values p, q, we use boldface notation $\boldsymbol{p}, \boldsymbol{q}$ to denote functions.

[10] We discuss thorughout the paper how our results would change under multiplicative preferences, i.e. if the g function were identically zero. To save on notation, we avoid specifying the a class of utility functions that includes the linear preferences as a special case and refer instead to that case directly.

software consumption. Think of quality q as the computational power of an OS, which consumers use to perform two tasks, one basic and one advanced. Everyone performs the basic task – say, simple calculations, text editing and access to online services – in the same way and measures returns to quality according to a common decreasing returns function; both the fact that everyone

agents however have heterogeneous returns θ in the accomplishment of their professional activity which can be more or less computationally intensive.

Technically, the key effect of the common component g is to make it more profitable to screen type $\theta > \theta'$ when the quality level is large. To see why, notice that because of the Inada conditions the source of heterogeneity θ ranges from being irrelevant to being the dominant factor in $u_q = g'(q) + \theta$ as we climb the quality ladder.[11] Although the *difference* in the marginal utilities $\theta - \theta'$ remains constant in q, their *ratio*

$$\frac{u_q(\cdot,\theta)}{u_q(\cdot,\theta')} = \frac{g'(q)+\theta}{g'(q)+\theta'}$$

is not constant – as it would be under multiplicative preferences –, but increasing in q: as q grows larger, there is effectively more heterogeneity in marginal utilities, i.e. larger returns from screening.

Production and Sale

A producer creates a version of the good of quality q at cost $c(q)$, increasing and convex and measured in the same units as revenues.[12] Then she can supply an arbitrary quantity of version q as well as all versions dominated by q. Formally, firms operate the production set[13]

$$\mathrm{Y} = \{\mathbb{I}\{q' \leq q\}, -c(q)\}_{q \in \mathrm{Q}}. \tag{3}$$

[11] In the two-tasks interpretation detailed above, local to $q = 0$ quality increments are infinitely more valuable for the accomplishment of the basic task, so there is little heterogeneity in the *relative* marginal utilities. At large q, on the contrary, marginal returns in the basic task are negligible and quality increments are used almost exclusively in the accomplishment of the professional task (where heterogeneity matters).

[12] Again, using the re-definition of the quality spectrum from footnote **??**, the cost reads $c_1(q) = c\left(g_2^{-1}(q)\right)$, and we require that $c \circ g_2^{-1}$ is convex.

[13] Consumers are in unit measure and demand at most one version, so a supply of 1 is indeed "arbitrarily large".

After producing q the seller quotes a feasible pricing function $\boldsymbol{p} : [0,q] \to \mathbb{R}$.[14] Her profit maximization problem therefore reads

$$\max_{q,\boldsymbol{p}:[0,q]\to\mathbb{R}} \int_0^1 \boldsymbol{p}\left(\mathrm{D}_{\boldsymbol{p}}(\theta)\right) \mathrm{d}\theta - c(q) \tag{4}$$

2.1 The First-Best

As our efficiency benchmark, we consider the problem of a planner that chooses the quality consumed by each agent to maximize aggregate utility net of the cost of service. Her problem reads

$$\mathrm{W}^\star = \max_{q:[0,1]\to\mathrm{Q}} \int_0^1 u(\rho(\theta), \theta)\, \mathrm{d}\theta - c\left(\sup_\theta \rho(\theta)\right) \tag{5}$$

Proposition 1. *The efficient allocation $\boldsymbol{q}^\star : [0,1] \to \mathrm{Q}$ is the constant function*

$$\boldsymbol{q}^\star(\theta) = q^\star,\ \forall\theta$$

where $q^\star$ is the unique solution to

$$g'(q) + \frac{1}{2} = c'(q) \tag{6}$$

The efficient allocation has singleton image: as replication is free and everyone's utility in monotonic in q the planner never uses her ability to damage. Equation (6) is the first order condition of problem (5) restricted to constant allocations. Both the level $q^\star$ and the fact it is distributed undamaged to every agent in the economy will therefore constitute our efficiency benchmark. Per standard arguments, a monopolist that is not subject to information frictions, namely that observes θ and can charge different prices to different costumers (first-degree price discrimination) will induce the efficient allocation rule $\boldsymbol{q}^\star$ and extract all the surplus $\mathrm{W}^\star$ in the form of profits. In the remainder of the paper we assume that θ is not observable and compare the resulting allocations with $\boldsymbol{q}^\star$.

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[14] An equivalent restatement we will sometimes use is to let the seller choose only a pricing function $\boldsymbol{p} : \mathrm{Q} \to \mathbb{R} \cup \{\infty\}$ and define the cost function $\bar{c}$ on the space of pricing functions as:

$$\bar{c}(\boldsymbol{p}) = c(\sup\{q : \boldsymbol{p}(q) < \infty\})$$

The seller's problem reads

$$\max_{\boldsymbol{p}:\mathrm{Q}\to\mathbb{R}} \int_0^1 \boldsymbol{p}\left(\mathrm{D}_{\boldsymbol{p}}(\theta)\right) \mathrm{d}\theta - \bar{c}(\boldsymbol{p}).$$

3 Monopolist

When θ is private information of the consumer the efficient allocation with full surplus extraction is not implementable and the seller must rely on an incentive compatible menu to allocate different qualities to different types. We set up the problem as a multi-agent mechanism design problem and appeal to the revelation principle to write the monopolist problem (4) as choosing a pair of allocation and transfer rules[15]

$$(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}) : [0,1] \to Q \times \mathbb{R}$$

to maximize profits under sorting and rationality constraints. What is new compared to a standard quality screening problem is that the cost of an providing allocation $\tilde{\boldsymbol{q}} : [0,1] \to Q$ no longer takes an additively separable form: We cannot compute it by summing up the costs to produce the versions that each agent ends up consuming, i.e. there is no primitive cost $\widetilde{c} : Q \to \mathbb{R}$ such that

$$\overline{c}(\tilde{\boldsymbol{q}}) = \int_0^1 \widetilde{c}(\tilde{\boldsymbol{q}}(\theta))\, d\theta \tag{7}$$

By contrast, production cost $\overline{c}(\boldsymbol{q})$ depends on the allocation $\boldsymbol{q}$ only through a single statistic of the allocation – the maximum – and can be written as[16]

$$\overline{c}(\boldsymbol{q}) = c\left(\max_{\theta} \boldsymbol{q}(\theta)\right) \tag{8}$$

where c is the acquisition cost function defined in Section 2. With these observations at hand, we can write the monopolist problem as

$$\begin{array}{lcc} \max_{(\boldsymbol{q},\boldsymbol{p}):[0,1]\to Q\times\mathbb{R}} & & \int_0^1 \boldsymbol{p}(\theta)\, d\theta - c\left(\max_\theta \boldsymbol{q}(\theta)\right) \\ & \text{s.t.} & \\ & \text{IC} & u(\theta, \boldsymbol{q}(\theta)) - \boldsymbol{p}(\theta) \geq u(\theta, \boldsymbol{q}(\theta')) - \boldsymbol{p}(\theta') \ \forall \theta, \theta' \in [0,1]^2 \\ & \text{IR} & u(\theta, \boldsymbol{q}(\theta)) - \boldsymbol{p}(\theta) \geq 0 \ \forall \theta \in [0,1] \end{array} \tag{9}$$

[15]The steps for rewriting the monopolist problem (4) as the design of a direct mechanism are standard (the unconventional cost specification is at this stage immaterial) and therefore omitted. It should not create confusion that the pricing function p has domain the type space Θ rather than the quality space Q.

[16]A natural extension of our analysis is the general investigation of screening problems with non-separable cost functions of the form

$$\overline{c}(\boldsymbol{q}) = \int_\Theta \widetilde{c}(q(\theta), \rho)\, dF(\theta),$$

flexible enough to describe less extreme versions of economies of scale, learning, or nontrivial cost of quality replication and versioning. Such extension is beyond the scope of this paper which instead focuses on the extreme at the polar opposite of the standard linear aggregator.

Because of the non-separability of the digital goods cost function (8), problem (9) cannot be solved by type-wise maximization of a virtual valuation function. However, the simple form of non-separability characterizing (8) suggests that problem (9) can be safely divided in two stages. First, the optimal allocation for each given quality constraint is computed, then the revenues associate to it are compared with the acquisition costs to pin down the optimal level of the constraint.

Lemma 1. *Let* $\mathrm{V} : \mathrm{Q} \to \mathbb{R}$ *the quality-constrained revenue function*

$$\begin{array}{cccc} \mathrm{V}(q) & \longmapsto & \max_{(\boldsymbol{q},\boldsymbol{p}):[0,1]\to \mathrm{Q}\times\mathbb{R}} \int_0^1 \boldsymbol{p}(\theta)\, d\theta & \\ & s.t. & IC\ ,\ IR & \\ & \textit{Quality constraint} & \boldsymbol{q}(\theta) \leq q, & \forall \theta \in \Theta \end{array} \tag{10}$$

and $\boldsymbol{q}_q : [0,1] \to \mathrm{Q}$ *the associated policy. The solution to problem (9) is characterized by the allocation*

$$\boldsymbol{q}^{\mathbf{M}}(\cdot) = \boldsymbol{q}_{q^{\mathrm{M}}}(\cdot)$$

where

$$q^{\mathrm{M}} \in \arg\max_{q \in \mathrm{Q}} \mathrm{V}(q) - c(q) \tag{11}$$

Given the value function V of problem (10), the solution of the acquisition problem (11) is straightforward: we will show V that is concave, so q^{M} is unique and characterized by a first order condition. The challenge to determine the monopolist allocation $\boldsymbol{q}^{\mathbf{M}}(\cdot)$ is therefore to characterize *all* constraint-conditional allocation rules $\boldsymbol{q}_{\bar{q}} : [0,1] \to \mathrm{Q}$, where $\boldsymbol{q}_{\bar{q}}(\theta)$ denotes the quality allocated to type θ when the quality cap is $\bar{q}$.

Towards this characterization we preliminarily define the virtual valuation function

$$v(\theta, q) = g(q) + (2\theta - 1)q$$

and let $\beta : [0,1] \to \mathrm{Q} \cup \{\infty\}$ be its maximizer[17]

$$\beta(\theta) = \begin{cases} (g')^{-1}(2\theta - 1) & \text{if } \theta < \frac{1}{2} \\ \infty & \text{if } \theta > \frac{1}{2} \end{cases}$$

Three properties of β, plotted in Figure 1,[18] are relevant: First,

$$q^{\mathrm{FB}} := \beta(0) = (g')^{-1}(0) > 0$$

[17] With the abuse $\arg\max v(\theta, q) = \infty$ whenever $v(\theta, \cdot)$ is monotone increasing.

[18] All the plots use the specification $g(x) = \sqrt{x}$, under which $q^{\mathrm{FB}} = 0.25$.

because of the Inada condition at $q = 0$; the superscript FB stands for Full Bunching for reasons that will be apparent shortly. Second, β is monotonically increasing in $\left[0, \frac{1}{2}\right)$ diverging in the limit because of strict concavity and the Inada condition at $q = \infty$; Third, β takes value infinity in the interior of the type space since $\theta > \frac{1}{2}$ implies the multiplier of the linear part is positive so $v(\cdot, \theta)$ is monotonically increasing. Notice that if g were absent from the specification (1), i.e. if preferences were linear, then the virtual value is always maximized at the extrema of the quality space and $\beta(\theta)$ would be 0 for $\theta < \frac{1}{2}$ while it would remain infinity for $\theta > \frac{1}{2}$.

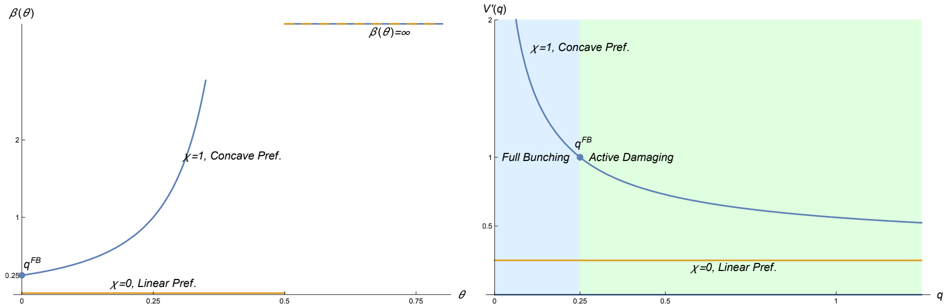

Figure 1: Left: Virtual Value Maximizer $\beta(\theta)$ for standard (1) (blue) and their linear restriction ($g \equiv 0$, yellow) Preferences. Right: Marginal Revenues.

We are now ready to characterize the allocation of a digital good monopolist.

Theorem 1. *[**Monopolist Allocation**] i) The constrained allocations are given by*

$$\boldsymbol{q}_{\bar{q}}(\theta) = \min\{\bar{q}, \beta(\theta)\} \tag{12}$$

ii) V *is concave and continuously differentiable. Its derivative is given by*

$$\mathrm{V}'(q) = \begin{cases} g'(q) & q < q^{\mathrm{FB}} \\ \left(\frac{1+g'(q)}{2}\right)^2 & q \geq q^{\mathrm{FB}} \end{cases} \tag{13}$$

iii) The monopolist produces a quality q^{M} that is strictly below $q^{\star}$.

Point i) provides a simple characterization of the constrained allocations: for each value of the constraint $\bar{q}$, we obtain $\boldsymbol{q}_{\bar{q}}$ by "capping" the β function (which we derived independently of the constraint) at $\bar{q}$. This result appears natural once we realize that our preference specification allows to solve the problem of distributing $\bar{q}$ as a a standard screening problem with multiplicative preferences where the common utility component simply adjusts the returns from production. Indeed, since $g(q)$ is a deterministic function of quality (independent of individuals' types), it can be lumped in with the payment so that it has exactly the same role as standard cost (with sign flipped) in determining the virtual value of an agent θ that is now left with only the multiplicative part $q\theta$ of his preferences.[19] Therefore, the seller allocates to type θ the quality $\boldsymbol{q}_{\bar{q}}(\theta)$ that maximizes $v(\theta, q)$ subject to $q \leq \bar{q}$: if the unconstrained maximizer $\beta(\theta)$ is feasible – i.e. below $\bar{q}$ – then it coincides with the allocation, else the allocation is capped at $\bar{q}$.

Notice that the inverse β function, $b : \mathrm{Q} \to \left[0, \frac{1}{2}\right)$ given by

$$b(q) = \beta^{-1}(q) = \begin{cases} 0 & q < q^{\mathrm{FB}} \\ \frac{1-g'(q)}{2} & q \geq q^{\mathrm{FB}} \end{cases} \tag{14}$$

returns a "bunching at the top" threshold that characterizes all distribution problems: when the cap is q types $\theta \in [b(q), 1]$ receive the good undamaged while the types $\theta \in [0, b(q)]$ receive $\beta(\theta)$. At an intermediate step of the derivation of the marginal revenue (13) we get

$$\mathrm{V}'(q) = (1 - b(q))[g'(q) + b(q)]$$

which has an intuitive explanation. The term $g'(q) + b(q)$ is the marginal utility of the "marginally bunched" type $b(q)$, while $(1 - b(q))$ is the mass of types above him. By increasing the quality cap above q the monopolist does not change the revenues she makes from selling all qualities (weakly) below q, since she allocates those qualities to the same types (and therefore at the same price). The extra revenues from the increment come exclusively from selling such increment to types $[b(q), 1]$, who all increase their transfer by $u_q(q, b(q)) = g'(q) + b(q)$. When $b(q) = \underline{\theta}$, or equivalently $q < q^{\mathrm{FB}}$, then the cap is so low that it is suboptimal to discriminate any consumer and $\mathrm{V}'(q) = g'(q)$, delivering the first branch in (13). Substitution of the nontrivial expression of b delivers the second branch, which we show is smoothly pasted.

By point *iii*) the quality q^{M} produced – and therefore the highest quality distributed – by a monopolist is below the efficient level $q^{\star}$ characterized by (6). As a consequence, the "efficiency at the top" result of screening problems does not hold in this setting and is instead limited to distributional efficiency. Comparing points *i*) and *iii*) in Theorem 1 with Proposition 1 we notice that a digital good monopolist induces two interdependent sources of inefficiency: an *allocative inefficiency* due to the fact that a positive measure of types receive a damaged version (this inefficiency exists only if $q^{\mathrm{M}} > q^{\mathrm{FB}}$), and a *production inefficiency* due to suboptimal investment in quality. Although associated to different stages of the monopolist problem, these inefficiencies are interdependent as (1) the screening allocation is constrained by the quality acquired and (2) the benefits from quality acquisition are a function of the constrained distribution. Expression (13)

[19] Rather than deriving utility $u(\theta) = g(q(\theta)) + \theta q(\theta)$ and paying price $p(\theta)$, agents in the "modified" economy have multiplicative preferences $\tilde{u}(\theta) = \theta q(\theta)$ but pay an adjusted price $\tilde{p}(\theta) = p(\theta) - g(q(\theta))$.

incorporates the optimal distribution of each maximal quality, which generally (in the second branch of (13)) entails damaging.

3.1 Properties of the Monopolist Contract

Allocations

Before exploring in more details the properties of the monopolist contract, we make an important observation.

Remark 1. (Triviality of allocations under linear preferences) Suppose $g \equiv 0$. The constraint-conditional allocations are given by

$$\boldsymbol{q}_{\bar{q}}(\theta) = \begin{cases} 0 & \theta \le \tilde{\theta} \\ \bar{q} & \theta > \tilde{\theta} \end{cases}$$

and the monopolist has constant marginal revenue $V'\left(q^{M}\right) = \tilde{\theta}\left(1-\tilde{\theta}\right)$.

With linear preferences, irrespectively of the acquisition cost, a monopolist would sell a trivial menu containing a single positive quality (bottom panel of Figure 2).[20] Cost-separable screening problems with linear preferences generate a non-trivial allocation by exploiting variations in marginal costs; since the problem of distributing a digital good lacks by construction such variation, in order to generate a rich – i.e. non singleton – allocation we must rely on the preference specification alone. Because rich contracts are often observed in digital goods markets, we focus on a specification like (1) that does not rule them out. Indeed, as an immediate consequence of point i) in Theorem 1, we obtain

Corollary 1. i) ***[No Exclusion]*** *In the monopolist contract, all types receive a positive quality* $\boldsymbol{q}_{q^{M}}(\theta) > 0 \quad \forall\theta$.

ii) ***[Full Bunching at low caps]*** *If* $q^{M} < q^{FB}$*, then no damaging is optimal:* $\boldsymbol{q}_{q^{M}}(\theta) = q^{M} \quad \forall\theta$.

iii) ***[Distributional efficiency at the top]*** *Types* $\theta \in \left[\frac{1}{2}, 1\right]$ *always receive an undamaged quality.*

Points i) and ii) present a contrast with the standard screening model, where the monopolist may exclude some types but never does full bunching. For a fixed quality constraint, both results are immediate consequences of the Inada condition of the

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Remark 2. Notice that any multiplicatively separable utility specification $u(q,\theta) = u_1(q) \cdot u_2(\theta)$ would yield the same "no-haggling" result as the ratio of marginal utilities, key to determine the optimal *distribution* contract, does not depend on the quality acquired.

common component g around zero: by giving a marginal quality to the excluded types the seller would get unbounded marginal revenues, which she can distribute as information rents to make sure the sorting constraints for higher types still hold, proving point *i*). Moreover, if the quality constraint is low enough there will be little variation in *relative* marginal utilities across the available spectrum, so it will not be optimal to screen any type (top panel of Figure 2).

By point *iii*), there is a set of types that are bunched at the top irrespectively of the quality cap. The two-task interpretation of the concave specification provides an effective way of explaining this result. As we climb the quality ladder agents' marginal returns come almost exclusively from the accomplishment of the advanced task (as $g'(q) \to \infty$) and hence coincide with those of the linear specification. The seller optimally distributes increments from a large constraint to a set of types that shrinks to $\left[\frac{1}{2}, 1\right]$ (measure $\frac{1}{2}$) and at a price that grows with the marginal utility of the marginally bunched type, which also converges to $\frac{1}{2}$. As the allocation of the marginal quality coincides is the same as under linear preferences, marginal revenues also coincide and $\lim_{q\to\infty} \mathrm{V}'(q) = \frac{1}{4}$ provides a lower bound for the (limit) marginal cost that ensures a solution to the acquisition problem.

Figure 2 below shows a graphical derivation of the optimal contract under different preferences and cost specifications: First, we use the constrained allocations to derive the marginal revenue function (13) and intersect it with the marginal cost to determine q^{M} (left graphs, with axes flipped); Then, we carry this level on the graph of β and obtain the optimal allocation by simply capping β at q^{M} (right graphs).

Welfare

We now turn to the welfare properties of the monopolist allocation and provide an analytic expression of the underacquisition and damaging inefficiencies (yellow and blue shaded regions in the right panels of Figure 2). If q^{M} is below q^{FB} (top panel), then per Corollary **??**-*i*), only the former is active: the seller serves all types, makes revenues $\mathrm{V}(q) = g(q)$ and leaves consumers with surplus $w^{\mathrm{C}}(q) = \frac{1}{2}q$. If instead $q^{\mathrm{M}} > q^{\mathrm{FB}}$, then some low types receive a damaged version. Through simple manipulations we get that consumer surplus in this region grows at rate

$$w^{\mathrm{C}}(q) = \frac{1}{2}\left(\frac{1+g'(q)}{2}\right)^2 = \frac{1}{2}\mathrm{V}'(q) \tag{15}$$

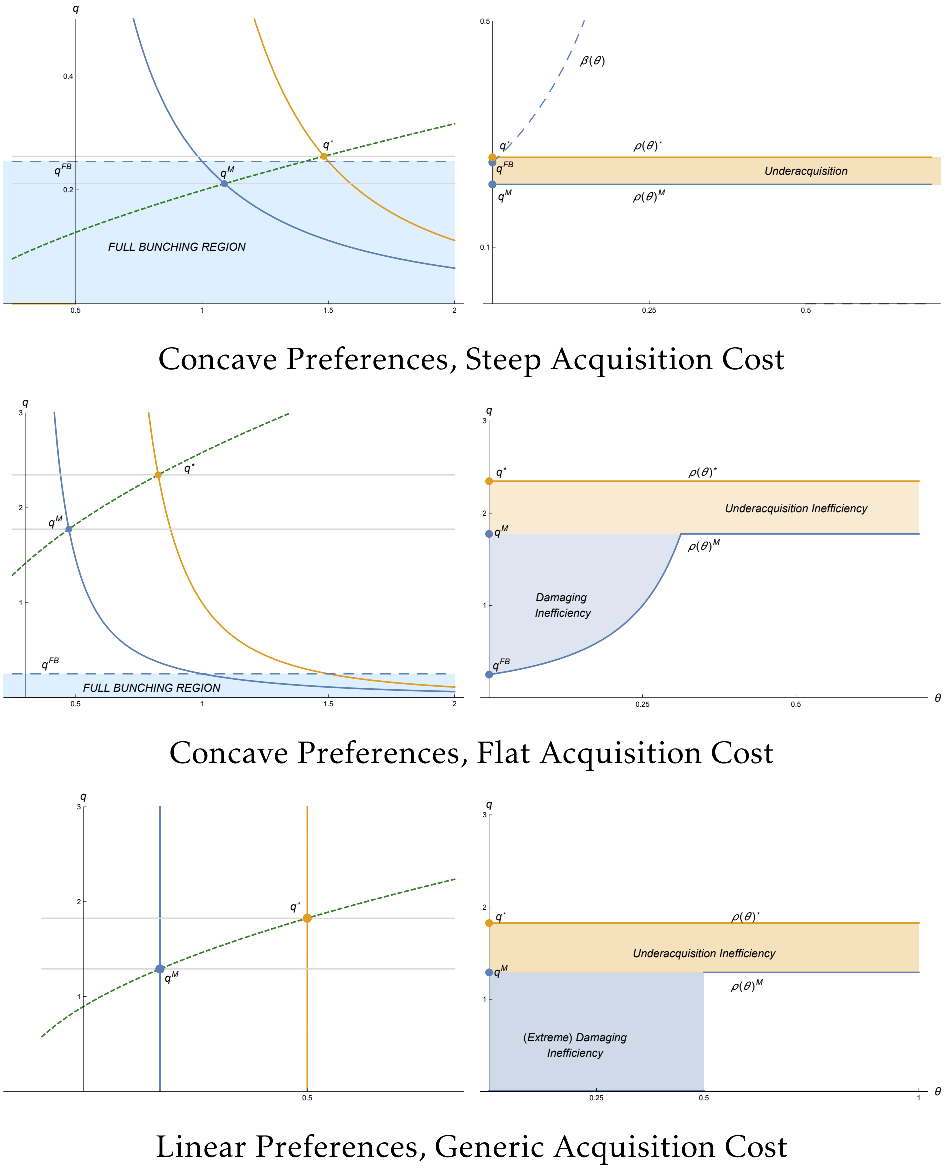

Figure 2: Determination of Monopolist Quality (left) and Allocation with Second Best Inefficiencies (right).

And therefore above q^{FB} the aggregate surplus grows at rate

$$w^{\mathrm{M}}(q) = \underbrace{\left(\frac{1+g'(q)}{2}\right)^2 - c'(q)}_{\text{marginal profits}} + \underbrace{\frac{1}{2}\left(\frac{1+g'(q)}{2}\right)^2}_{\text{marginal rents}} = \frac{3}{2}\left(\frac{1+g'(q)}{2}\right)^2 - c'(q) \tag{16}$$

Subtracting $w^{\mathrm{M}}(q)$ from $w^{\star}(q) = \frac{1}{2} + g'(q) - c'(q)$, the efficient marginal surplus, we obtain the following expression for the marginal inefficiencies from damaging

$$d(q) = \frac{1}{8}\left(1 + (2 - 3g'(q))\, g'(q)\right) \tag{17}$$

We can now provide an analytical decomposition of the monopolist inefficiencies.

Proposition 2.

The monopolist allocation entails a welfare loss $W^{\star} - W^{M}$ *given by*

$$\int_{q^{FB}}^{q^{M}} \underbrace{d(q)}_{\text{damaging ineff. (17)}} \mathrm{d}q + \int_{q^{M}}^{q^{\star}} \overbrace{\frac{1}{2} + g'(q) - c'(q)}^{\text{underacquisition ineff.}} \mathrm{d}q \tag{18}$$

with the convention that $\int_{q^{FB}}^{q^{M}} d(q)\mathrm{d}q = 0$ *if* $q^{FB} > q^{M}$.

Notice that in the linear case ($g \equiv 0$), damaging would take the extreme form of excluding a constraint-independent set of types so $d(q)$ is constant at $\frac{1}{8} = \frac{1}{2}\mathbb{E}\left[\theta \middle| \theta < \frac{1}{2}\right]$. In the concave specification, (17) is valid only if $g'(q) < 1$, in which case it is a positive hump-shaped function in q. Below that level there were no damaging inefficiencies. Although (18) provides a decomposition of the monopolist inefficiencies associated with damaging and underacquisition, it is important to remember that these two inefficiencies are interdependent. The optimality of damaging depends on the quality produced, which in turn depends on the extent of damaging. Appendix A.1 clarifies this interdependence by characterizing the monopolist allocation in a setting where it is impossible to damage versions, thereby exogenously eliminating one inefficiency. Understanding the role of competition in the provision of digital goods, which is the focus of the Section 4, will also require evaluating its impact on both sources of inefficiency. Before turning to that, we address the robustness of our predictions in the monopolist setting to a generalization of our demand primitives.

3.2 Discussion and Demand Primitives Relaxation

At the distribution stage, the common concave utility component g plays the same role in the type-wise objective as a convex separable cost with sign flipped, inducing smooth allocations given each quality constraint. In each of these constrained problems, unboundedness of the common marginal returns around zero[21] drives full bunching at low caps and no exclusion – Corollary **??**, $i) - ii)$ –. Recall from the discussion following Lemma 1 that such constraint-conditional arguments are relevant only because the digital goods cost function creates a separation between the acquisition and distribution problem. This clarifies the limits of the "cost-like" role of common returns component g to a discipline of the constrained allocations (Proposition 1), leaving the selection of the constraint to the "real" cost function c.

[21]Which is fairly natural both in the example of software consumption with simple but essential activities that require access to a basic version of the software and in the binary location problem (invest or not in a fixed income security) presented in the context of information markets Section 5.

The demand primitives can be relaxed and leave unaltered the qualitative properties of the monopolist allocation presented in Theorem 1. For example, it is straightforward to extend the analysis beyond the uniform distribution and modify allow for a regular (i.e. which generates increasing virtual value ϕ) distribution F. The full bunching threshold q^{FB} would be given by $q^{FB} = (g')^{-1}(\phi(\underline{\theta}))$ while the "marginal bunching" b function used to calculate the marginal revenues (13) would be implicitly defined by $b(\cdot) = \max\left\{0, \widetilde{h}^{-1} \circ g'\right\}$ where

$$g'(q) = (h(b(q)) - b(q)) = \widetilde{h}(b(q)).$$

Let $\widetilde{\theta}$ the zero of the function $\theta - h(\theta)$ ($\widetilde{\theta} = \frac{1}{2}$ in the uniform case). All types above $\widetilde{\theta}$ are always bunched at the top, giving constant marginal revenues under linear preferences $V'(q) = \widetilde{\theta}\left[1 - F\left(\widetilde{\theta}\right)\right]$.

The preference specification (1) is instead more substantial. A slight but important generalization – that we will permit the embedding of the value of information from the decision problems presented in Section 5 into our framework – is the following. As quality does not have a natural metric, preferences of the type

$$u(q, \theta) = g_1(q) + \theta g_2(q) \tag{19}$$

for a pair of increasing functions $g_1, g_2 : Q \to \mathbb{R}$ fall into specification (1) once we re-define quality $x = g_2(q)$ with associated cardinal rankings

$$\widetilde{u}(x, \theta) = g_1\left(g_2^{-1}(x)\right) + \theta x$$

provided that the common component $g_1 \circ g_2^{-1} : \mathbb{R} \to \mathbb{R}$ is concave-Inada.[22] The substantive assumption in (1) is therefore the additive separability between a common and a type-dependent component and that the common component is "more concave" than the type dependent one.

4 Competition

In this section we develop a model of competition in digital goods markets, solve for its equilibria, and study their welfare properties. In Appendix A.2, we discuss the issue of equilibrium existence and stability, central to the literature of competition with screening (Stole, 2007).

We augment the model with countably infinite replica of the seller. The set of firms is denoted by $\mathbb{N}$, with typical element i. Firms produce a *homogeneous* range of goods:

[22]The acquisition cost function c, redefined in a similar fashion over the new quality domain, should also remain convex.

versions of the good produced by different firms are treated as perfect substitutes by the consumers, who use the lower envelope of competitors' pricing functions to determine their demand. Firms play a two stage game: at the first stage, they independently make an irreversible investment in quality whose outcome becomes common knowledge before the pricing game is played. In this second stage, each firm quotes a pricing function that is feasible – i.e. on a domain bounded by the quality she has invested in – and treats production costs as sunk. Competitive equilibria are the subgame perfect equilibria of this dynamic game of perfect information without discounting.

We now contrast our approach with the two classic references most relevant to our analysis.[23] Kreps and Scheinkman (1983) make firms commit to a quantity level before Bertrand competing (without screening) on the realized investments. The two stage equilibrium yields Cournot competition outcomes. Contrary to their setting, in our model the value of aggregate production is the *maximum of the qualities* produced, not the *sum of the quantities*; since production along multiple lines is wasteful (because of a multiple spending inefficiency), us finding conditions under which competition is be beneficial (Theorem 5-*ii*)) is to some degree surprising. Champsaur and Rochet (1989) analyze a MR duopoly where each competitor costlessly commits to a subset of qualities and then chooses a pricing function on the selected domain (paying the distribution costs at this stage).[24] In committing to a quality range firms face a trade-off: they want a broad range to discriminate among consumers, but they also want to differentiate their products as price competition lowers profit margins on neighboring qualities. In a Nash equilibrium where each firm makes positive profits, the quality sets to which firms commit are disjoint. In our investment game firms commit to a quality range of the type $[0, q]$, so it is technologically impossible to have an empty intersection. Indeed in all of our competitive equilibria only one firm realizes positive revenues (Proposition 3), and first stage equilibria with active competition are only mixed (Proposition 4-*ii*)).

4.1 The Pricing Game

Fix a vector of realized qualities $\mathbf{q} \in \mathrm{Q}^{\mathbb{N}}$; entries are denoted by q_i, q_j, while $q^{(i)}$ denotes the i^{th} order statistic of $\mathbf{q}$. Firm i chooses a pricing function p_i from the feasible set $\mathrm{A}_i(\mathbf{q}) = \mathbb{R}^{[0,q_i]}$ since at this stage she can sell *arbitrary amounts* of *any version* in $[0, q_i]$. Letting $p_i(q) = \infty$ whenever $q > q_i$, we can define the market pricing function

[23] For a more comprehensive discussion of alternative models of competition with screening, refer to Section 1.1.

[24] Our quality commitment stage is not cheap talk but requires a real and costly investment that becomes sunk at the distribution phase. The relative plausibility of the two setups depends on the length and transparency of R&D processes and patenting, entry regulations, etc. Arguably, costly investment/free distribution is more realistic in markets for software and information.

$m_{\mathbf{p}}(q) := \min_j p_j(q)$ from which we associate each type with the version and the identity of the firm he buys from[25]

$$\begin{aligned} \mathrm{D}_{\mathbf{p}}(\theta) &= \arg\max_{q\in\mathrm{Q}} u(q,\theta) - m_{\mathbf{p}}(q) \\ \iota_{\mathbf{p}}(\theta) &= \min\left\{\arg\min_i \left\{p_i\left(\mathrm{D}_{\mathbf{p}}(\theta)\right)\right\}\right\} \end{aligned} \tag{20}$$

Firm i makes revenues

$$\overline{\mathrm{R}}_i(\mathbf{p}) = \int_{\{\theta:\iota_{\mathbf{p}}(\theta)=i\}} p_i\left(\mathrm{D}_{\mathbf{p}}(\theta)\right)\mathrm{d}\theta. \tag{21}$$

The pricing game is $\Gamma(\mathbf{q}) = \left\langle \mathbb{N}, \left(\mathrm{A}_i(\mathbf{q}), \overline{\mathrm{R}}_i(\cdot)\right)_{i\in\mathbb{N}}\right\rangle$.

Proposition 3. *For each $\mathbf{q}\in\mathrm{Q}^{\mathbb{N}}$, $\Gamma(\mathbf{q})$ has an essentially unique Nash equilibrium in pure strategies. It induces allocations*

$$\boldsymbol{q}(\theta,\mathbf{q}) = \begin{cases} q^{(2)} & \text{if } \beta(\theta) < q^{(2)} \\ \beta(\theta) & \text{if } q^{(2)} \leq \beta(\theta) < q^{(1)} \\ q^{(1)} & \text{if } \beta(\theta) \geq q^{(1)} \end{cases} \tag{22}$$

and revenues

$$\mathrm{R}_i(\mathbf{q}) = \max\left\{\mathrm{V}(q_i) - \max_{j\neq i}\mathrm{V}\left(q_j\right), 0\right\} \tag{23}$$

The first and second order statistic of the realized qualities are sufficient to determine of the outcome of the game $\Gamma(\mathbf{q})$. We avoid carrying order statistics notation and denote $x_{\mathbf{q}} = q^{(1)}, y_{\mathbf{q}} = q^{(2)}$, dropping also the subscript $\mathbf{q}$ when it causes no confusion. The competitive allocation (22) is denoted compactly as

$$\boldsymbol{q}^{x,y}(\theta) = \max\{\min\{\beta(\theta), x\}, y\}. \tag{24}$$

Proposition 3 has a fairly straightforward interpretation: since costs are sunk, Bertrand competition drives to zero the revenues in the quality spectrum $[0, y]$ where there is active competition. Therefore, only the owner of the highest quality can make positive revenues in equilibrium by distributing the set of qualities $[y, x]$ over which she has market power. She behaves as an interim monopolist that faces the distribution problem (10) under the additional constraint that every agent consumes (at least) the

[25] The purchasing correspondence embeds a tie-breaking rule in favor of firms with lower indices, which will be instrumental in ruling out competitive equilibria with strictly positive lower bound of the acquisition support. Notice ι is measurable under the assumption that firms quote an increasing function, that will be satisfied in equilibrium.

version y that *at least one* competitor can offer.[26] The set of IR read now

$$u(\boldsymbol{q}(\theta),\theta) - p(\theta) \geq u(y,\theta)$$

which, under the requirement that $p \geq 0$, is equivalent to constraining the allocation ρ to have image contained in $[y,x]$. The solution to this problem has again a simple structure, since the allocations are now obtained by slicing the β function both from below (at y) and from above (at x) – see Figure 3 –. All types below $b(y)$ receive y for free, the others get the same allocation as under monopolist with quality x but pay less. If y is below q^{FB} the lower bound constraint on the interim monopolist is not binding and the competitive allocation coincides with that of an x–monopolist and transfers are uniformly reduced by $g(y)$. This is clearly never the case under the linear specification where $q^{\mathrm{FB}} = 0$ and

Example (*Competitive Equilibrium with linear preferences*)**.** Under the linear specification $g \equiv 0$ competitive allocations and transfers are given by

$$(\boldsymbol{q},p)^{x,y}(\theta) = \begin{cases} (y,0) & \theta \in \left[0,\frac{1}{2}\right] \\ \left(x,\frac{1}{2}(x-y)\right) & \theta \in \left[\frac{1}{2},1\right] \end{cases} \tag{25}$$

Low types $\left[0,\frac{1}{2}\right]$ who were excluded in the monopolist contract now receive the free version y; High types still receive the best available quality x but pay only $\frac{1}{2}(x-y)$. The interim monopolist earns revenues $\frac{1}{4}(x-y)$.

We now endogenize x,y as the outcome of the acquisition game.

4.2 The Acquisition Stage

At the investment stage, firms trade off the acquisition cost with the revenues that will realize in the second stage as the equilibrium utility of the pricing game $\Gamma(\mathbf{q})$. Per Proposition 3, those revenues depend on the competitive environment only through the highest quality z that the opponents possess; in particular, the net returns of acquiring q against a pool of competitors characterized by z are given by

$$\Pi(q,z) = \mathrm{R}(q,z) - c(q) = \max\{\mathrm{V}(q) - \mathrm{V}(z), 0\} - c(q), \tag{26}$$

If z was known, a potential entrant would be solving the monopolist problem of Section 3 augmented with a fixed cost that can only affect his participation decision. Therefore,

[26] The equilibrium is only essentially unique because it is not determined who between the producer of the highest and second highest quality ends up distributing y. Although this indeterminacy impacts neither the firms' payoff (and therefore first-stage play) nor welfare, it still has some empirical content: in a competitive digital goods market we may observe multiple providers of the free version, but only one seller of premium versions.

the best response to any entry vector belongs to the doubleton set $\left\{0, q^{\mathrm{M}}\right\}$.[27] As z is (possibly) stochastic in equilibrium, firms take its distribution as given and choose q to solve $\max_{q \in \mathrm{Q}} \mathbb{E}\left[\Pi\left(q, \boldsymbol{z}\right)\right]$, where boldface notation is now used to denote random variables (and $\mathbb{E}$ integrates those under equilibrium play).[28] To ensure the existence of an equilibrium with active competition we allow firms to randomize among the maximizers. Therefore, the symmetric game

$$\Upsilon = \langle \mathbb{N}, \Delta\left(\mathrm{Q}\right), \Pi\left(\cdot,\cdot\right)\rangle$$

represents the strategic interaction at the acquisition stage. Notice that Υ has a structure similar to an all-pay auction since the cost of participating is paid by every player, even those that earn no revenues in the second stage. The "bid" of others, however, not only affects the probability of winning, but has also a direct impact on the payoff of the winner $\mathrm{R}\left(q, z\right)$ as it wipes out all revenues from the versions that are shared. We now present and discuss the equilibria of Υ, first introducing some terminology.

A firm that plays $\delta_0 \in \Delta\left(\mathrm{Q}\right)$ is called **idle**. All other firms, i.e. those that play a positive quality with positive probability, are called **active**. Equilibria are parametrized by the number n of active firms: In an n–equilibrium, firms $\{1, 2, \ldots, n\}$ are active; the idle firms $m > n$ still play a role as we need to ensure they do not have a profitable entry. An n–equilibrium is called symmetric if all active firms choose the same action. Equilibria with $n \geq 2$ are called competitive.

Proposition 4. *For each number of active firms $n \geq 1$, there is a unique and symmetric n–equilibrium of Υ.*

i) With $n = 1$, the active firm plays $\delta_{q^{\mathrm{M}}}$ and makes profits Π^{M}. This is also the only equilibrium in pure strategies.

ii) For $n \geq 2$, active firms play a mixed action with full support $\left[0, q^{\mathrm{M}}\right]$ and continuously differentiable distribution given by

$$\mathrm{H}^{\mathrm{EQ}(n)}\left(q\right) = \left[\frac{c'\left(q\right)}{\mathrm{V}'\left(q\right)}\right]^{\frac{1}{n-1}} \tag{27}$$

and make zero (expected) profits.

It is immediate to notice that there are no competitive equilibria in pure actions, since whomever commits to a dominated positive quality can save on acquisition costs by becoming idle. All firms being idle cannot be an equilibrium as well, since everyone

[27] Although the returns of the interim monopolist are reduced by $\mathrm{V}\left(z\right)$, the *marginal* returns $\frac{\partial}{\partial q}\mathrm{R} = \mathrm{V}'\left(q\right)\mathbb{I}\left[q > z\right]$ coincide with those of the unconstrained monopolist. Therefore, for any given x the unique candidate for an interior optimum remains q^{M}, but inactivity is preferred when the realization of z is large.

[28] Notation $\mathbf{x}$ denotes a vector; notation $\boldsymbol{x}$ denotes a random variable.

would deviate to q^{M}. Therefore the only candidate equilibrium in pure actions is one firm playing q^{M}, everyone else being idle. Point *i*) establishes that this is indeed an equilibrium. The active firm optimizes since the monopolist is (tautologically) the best responder to an idle environment. Idle firms do not want to acquire a quality below q^{M} and make zero revenues; they can become interim monopolist by acquiring a quality $q > q^{\mathrm{M}}$, but they would make negative profits as[29]

$$\begin{aligned} \Pi\left(q, q^{\mathrm{M}}\right) &= \mathrm{V}(q) - \mathrm{V}\left(q^{\mathrm{M}}\right) - c(q) \\ &= \int_{q^{\mathrm{M}}}^{q} \underbrace{\mathrm{V}'(q') - c'(q')}_{\leq 0 \text{ above } q^{\mathrm{M}}} \mathrm{d}q' - c\left(q^{\mathrm{M}}\right) \;<\; 0. \end{aligned}$$

We now turn to the probabilistic competitive equilibria. By standard arguments from the construction of equilibria in the all-pay auction, each firm plays an atomless distribution with full support that must include the lower bound 0,[30] which implies they make zero profits $\mathbb{E}[\Pi(q, \boldsymbol{z})] = 0$. The indifference condition $\frac{\partial}{\partial q}\mathbb{E}[\Pi(q, \boldsymbol{z})] = 0$ pins down the distribution of $\boldsymbol{z}$ in the interior of the support

$$\mathrm{H}(z) = \frac{c'(z)}{\mathrm{V}'(z)} \tag{28}$$

Notice that H is differentiable since V″ is continuous per Proposition 1-*ii*). Expression (28) gives the distribution of the highest quality among $(n-1)$ competitors in any n–competitive equilibrium; the number of active firms remains indeterminate: for any n, individual play (27) generates the market distribution (28). Sufficiency is easily checked and relies again on the irreversibility of the investment at the pricing stage that allows incumbents to retaliate and make potential entries by idle firms not profitable.[31]

We put aside for a moment the full characterization (27) and notice an important property of the competitive allocation that results from the support restriction alone, combined with Proposition 3.

Corollary 2. *All competitive equilibria distribute, with probability 1*

1. *A highest quality strictly below q^{M}, and*

2. *A strictly positive quality for free*

[29] Our timing assumption gives the incumbent monopolist the possibility to fight back at the interim stage against a deviator; models in the spirit of Rothschild and Stiglitz (1976) allow instead the potential entrant to make revenues approximately close to those of an "interim idle" incumbent. It is therefore not surprising that the monopolist allocation can result as an equilibrium of the competition game.

[30] Equilibria with strictly positive lower bound that is played with strictly positive probability by active firms (that hope to share revenues in the case all other competitors also realize the lower bound) are ruled out by the tie-breaking rule (20).

[31] Entries at a quality in the support of the equilibrium induce a payoff lower than $\mathbb{E}_{\mathrm{X}}[\Pi(q, \boldsymbol{x})] = 0$ – since $\boldsymbol{x}$ is now the maximum of n rather than $n-1$ identical entry decisions – while entries above q^{M} still ensure an interim monopolist position but still add negative marginal profits.

A valuable version is distributed for free if (and only if, since the monopolist never offers free versions) there is active competition. In this case, high valuation types receive a quality that is lower than their second-best allocation. This gives a qualitative idea, represented graphically in Figure 3, of the impact that active competition has on welfare. By forcing the distribution of a free quality (point 2.), competition effectively prevents the interim monopolist from doing as much damaging as he would find optimal, hence reducing the distributive inefficiencies (17). The underacquisition inefficiency, however, is worsened (point 1.).

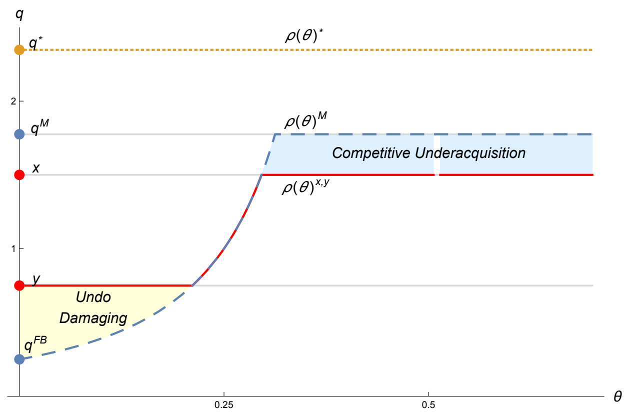

Figure 3: Competitive Allocation (x, y stochastic) and Welfare Impact.

4.3 Properties of the Competitive Allocations

We now address the welfare properties of the equilibria derived in Proposition 4. In particular, we study how welfare changes across competitive equilibria (i.e. compare n, m both larger than 2) using welfare in the monopolist equilibrium as a common benchmark (i.e. compare $n = 1$ with $m > 1$, quantifying the forces described in Corollary 2).

Conditional on the realized market (x, y), type θ gets surplus

$$W^{x,y}(\theta) = g(y) + \int_0^\theta \boldsymbol{q}^{x,y}(\theta')\, d\theta' \tag{29}$$

using the $\mathbb{E}_n$ operator to integrate out the market statistics $(\boldsymbol{x}, \boldsymbol{y})$ under the distribution induced by the n–competitive equilibrium (27) we get

$$W_n(\theta) := \mathbb{E}_n [W^{\boldsymbol{x},\boldsymbol{y}}(\theta)]$$

Since producers make zero expected profits in any competitive equilibrium, total surplus in an n–competitive equilibrium is simply

$$\mathrm{W}_n = \int_0^1 \mathrm{W}_n(\theta)\,\mathrm{d}\theta$$

Proposition 5. i) *Competitive equilibria are Pareto-ranked, decreasing in n. Moreover, improvement is uniform in types, that is*

$$\mathrm{W}_n(\theta) \geq \mathrm{W}_m(\theta), \qquad \forall \theta \in [0,1],\ 2 \leq n \leq m.$$

ii) *The comparison between monopolist and duopoly welfare is ambiguous and depends on the shape of the cost function. In particular, if $q^{\mathrm{M}} \leq q^{\mathrm{FB}}$ (full bunching monopolist), then competition reduces welfare. However, for any $q^{\mathrm{M}} > q^{\mathrm{FB}}$ it is possible to specify a cost function under which the competitive allocation is arbitrarily close to q^{M} and the double spending inefficiency vanish, meaning that the complete undoing of damaging inefficiency makes the duopoly dominate.*

The type-dependent surplus (29) is increasing in x and y: higher x gives extra surplus to high types without affecting the allocation (and rent) of low types; higher y increases the allocation (and rent) of low types and reduces the payment of high types. We show that the *joint* distribution of $(\boldsymbol{x}, \boldsymbol{y})$ is decreasing in n along the FOSD order,[32] from which point i) follows. Therefore, the stochastic ranking of market statistics gives a conclusive comparison across competitive equilibria, establishing that more intense competition is detrimental.

Point ii) shows that W_2 can exceed or fall short of the monopolist surplus W^{M}, depending on the shape of the cost function. In general, the difference between duopoly and monopolist welfare can be written – maintaining the convention that $\int_a^b f(x)\,\mathrm{d}x = 0$ whenever $a > b$ – as

$$\mathrm{W}_2 - \mathrm{W}^{\mathrm{M}} = \mathbb{E}_2\Big[\underbrace{\int_{q^{\mathrm{FB}}}^{\boldsymbol{y}} d(q)\,\mathrm{d}q}_{\text{undo screening}} - \underbrace{c(\boldsymbol{y})}_{\text{double spending}} \underbrace{\int_{\boldsymbol{x}}^{q^{\mathrm{M}}} w^{\mathrm{M}}(q)\,\mathrm{d}q}_{\text{underacquisition}}\Big] \tag{30}$$

where w^{M} is the marginal monopolist surplus (16) and d is the marginal damaging inefficiencies (17). The decomposition (30) highlights that welfare gains in duopoly result solely from realizations of $\boldsymbol{y}$ above q^{FB}: realizations below that level have no effects on welfare as they only imply a one-for-one transfer of surplus from the seller to (all) consumers in the form of lower payments (left panel of Figure 4). If $q^{\mathrm{M}} < q^{\mathrm{FB}}$,

[32] In particular, the distribution of $\boldsymbol{y}$ *conditional on* $\boldsymbol{x} = x$ is independent of n for each x, and the *marginal* distribution of $\boldsymbol{x}$ is ranked in n according to FOSD.

i.e. the monopolist is not damaging for screening purposes, then then $\boldsymbol{y}$ never realizes above q^{FB} and there is no channel through which competition can improve welfare: (17) has only negative summands and monopoly dominates.

Guaranteeing that $q^{\mathrm{M}} > q^{\mathrm{FB}}$ is clearly not sufficient to turn (17) positive. For that to happen both $\boldsymbol{x}$ and $\boldsymbol{y}$ should put significant weight on high realizations: the former must be close (in distribution) to q^{M} to reduce the integration domain of the underprovision inefficiency, while the latter should be well above q^{FB} to imply a substantial undoing of damaging inefficiencies (right panel of Figure 4). High realizations of $\boldsymbol{y}$ come with the detrimental effect of worsening the double spending inefficiency. Intuitively, by the Jensen's inequality we can increase $\mathbb{E}_2[\boldsymbol{y}]$ at limited costs $\mathbb{E}_2[c(\boldsymbol{y})]$. In particular, for any target quality $\tilde{q} > q^{\mathrm{FB}}$, the class of acquisition cost functions $c_\alpha(q) = \left(\frac{q}{\tilde{q}}\right)^\alpha$ induce (irrespectively of the demand primitive) large α limits that satisfy

$$q^{\mathrm{M}}_\infty = \lim_{\alpha\to\infty} q^{\mathrm{M}}(\alpha) = \tilde{q}, \quad c^{\mathrm{M}}_\infty = \lim_{\alpha\to\infty} \left(q^{\mathrm{M}}(\alpha)\right)^\alpha = 0$$

Moreover, substituting marginal cost into (27) we get that competitive equilibrium strategies converge in probability to q^{M}_∞ and that $\mathbb{E}_2\left[c\left(\boldsymbol{y}^{\mathrm{EQ}(2)}(\alpha)\right)\right] \le c\left(q^{\mathrm{M}}(\alpha)\right) \to 0$. Plugging these results in (30) we conclude that

$$\lim\nolimits_{\alpha\to\infty} \mathrm{W}_2 - \mathrm{W}^{\mathrm{M}} \quad = \quad \int_{q^{\mathrm{FB}}}^{\tilde{q}} d(q)\,\mathrm{d}q \quad > \quad 0 \tag{31}$$

in the limit only the "undo screening" effect is active, and is also complete: all types receive the quality that a monopolist would have produced (but not distributed), double spending and underacquisition are both shut down. This is not surprising, as the limit approaches a fixed cost structure with little adjustment on the acquisition margin and substantial cost savings due to "extreme" convexity.

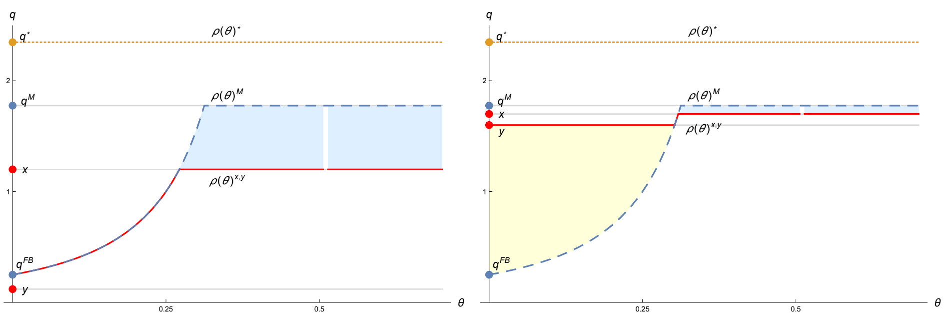

Figure 4: Monopoly (left) and Duopoly (right) Domination for different realizations of (x, y)

We now complete the analysis of equilibria under linear preferences by studying their welfare properties.

Example (*Linear Competitive Equilibrium, Cont'd.*). By substituting the allocation rule (25) we get that welfare in a competitive equilibrium is given by

$$\mathrm{W}_n = \mathbb{E}_n\left\{\int_0^{\frac{1}{2}} \theta \boldsymbol{y} \mathrm{d}\theta + \int_{\frac{1}{2}}^1 \left[\boldsymbol{x}\theta - (\boldsymbol{x}-\boldsymbol{y})\frac{1}{2}\right]\mathrm{d}\theta\right\} = \frac{1}{8}\mathbb{E}_n\left[\boldsymbol{x}+3\boldsymbol{y}\right]$$

If the acquisition cost is quadratic $c(q) = \frac{1}{2}q^2$, then the monopolist produces $q^{\mathrm{M}} = \frac{1}{4}$ and makes profit equal to $\frac{1}{32}$ that coincide with consumer surplus, so $\mathrm{W}^{\mathrm{M}} = \frac{1}{16}$. Per Proposition 4, in an n–competitive equilibrium active firms play

$$\mathrm{H}^{\mathrm{EQ}(n)}(q) = \mathbb{I}\left[0 \leq q \leq \frac{1}{4}\right] \cdot (4q)^{\frac{1}{n-1}}.$$

Integrating, we obtain $\mathrm{W}_n = \frac{5}{64}\frac{n}{2n-1}$, which is decreasing in n — conforming with Proposition 5-*i*) –. Moreover, notice that

$$\mathrm{W}^{\mathrm{M}} = \frac{1}{16} > \frac{5}{64}\cdot\frac{2}{3} = \mathrm{W}_2 > \mathrm{W}_3 > \cdots > \lim_{n\to\infty}\mathrm{W}_n = \frac{5}{128} > \frac{1}{32} = \mathrm{W}^{\mathrm{C}},$$

namely, under moderately convex cost the monopolist dominates duopoly in terms of total surplus but competition of any intensity makes consumers better off.

The class of convex cost function $c(q) = q^\alpha$ induce a large α limit (fixed-cost structure) of $q^{\mathrm{M}}_\infty = 1$ and monopolist welfare

$$\mathrm{W}^{\mathrm{M}} = \int_{\frac{1}{2}}^1 \left(\theta q^{\mathrm{M}} - \frac{1}{2}q^{\mathrm{M}}\right)\mathrm{d}\theta + \int_{\frac{1}{2}}^1 \frac{1}{2}q^{\mathrm{M}}\mathrm{d}\theta - c\left(q^{\mathrm{M}}\right) \longrightarrow 1\int_{\frac{1}{2}}^1 \theta\mathrm{d}\theta - 0 = \frac{3}{8}$$

Equilibrium play under duopoly is

$$\mathrm{H}^{\mathrm{EQ}(2)}_\alpha(q) = \mathbb{I}\left\{q \in \left[0, \left(\frac{1}{4\alpha}\right)^{\frac{1}{\alpha-1}}\right]\right\}\cdot(4\alpha q)^{\alpha-1}$$

which converges in probability to 1. Therefore $\mathrm{W}_2 \to \frac{1}{8}(1+3) = \frac{1}{2}$, exceeding the limit monopoly surplus by $\frac{1}{8}$; this conforms with equation (31), as $\frac{1}{8}$ is exactly the (limit) damaging inefficiency induced by excluding types $\left[0, \frac{1}{2}\right]$

$$\int_0^{q^{\mathrm{M}}_\infty} d(q)\mathrm{d}q = \int_0^1 \left(\int_0^{\frac{1}{2}} \mathrm{d}\theta\right)\mathrm{d}q = \frac{1}{8}$$

5 Application: The Market for "Hard" Information

This section explores information markets as an application of our previous analysis. We emphasize the implicit assumptions regarding demand and production technology, illustrating why our framework is more suitable than alternative approaches (which are briefly reviewed) for studying certain phenomena in these markets. We then introduce a straightforward yet precise microfoundation where producers observe increments of a common Brownian motion at a cost and sell its realization—potentially damaged by hiding some bits—to agents. These agents then solve a two-asset investment problem, resulting in a value of information that falls within the generalized specification (19).

5.1 General Features of the Market

We use our model to study a market for information that has the following features.

Demand A consumer θ is identified with a bayesian decision problem[33] (action set, prior, utility) defined over a common one-dimensional state space Ω. Consumers can not create their own information structure and must rely on those offered by a set of profit maximizing experts. This contrasts with models of unrestricted information acquisition with statistical pricing of information structures (e.g. Shannon entropy) which is the approach taken in (most of) the rational inattention literature.[34] Agents have no difficulty in understanding the signal they pay for, to do the proper (bayesian) updating, and assign to each structure the appropriate value based on their decision problem. Marketable signal structures are ranked according to a single-dimensional Blackwell order Q, and the class of decision problems faced by consumers (i.e. the set of decision problems $\theta \in [0,1]$) is such that returns from information fall within the generalized preference (19). In a similar setting, Bergemann et al. (2018a) derive a (piecewise) linear value of information when agents' types are their prior beliefs over a finite dimensional state space. The monopolist seller never damages information by reducing precision – conforming with the "no-haggling" result stated in this paper as Corollary **??** – but she degrades and sells non-trivial screening packages that reveal only a portion of the available data to the buyer – a deterioration margin along which margin consumers' valuation are not linear –.[35]

[33]We do not allow for strategic interaction to avoid that the value of information is endogenous – inconsistent with specification (1) – as it would depend on equilibrium acquisition and use of information. Several contributions (Hellwig and Veldkamp (2009), Myatt and Wallace (2011), and Colombo et al. (2014)) show that in the presence of strategic externalities also the information acquisition game has strategic complementarities.

[34]Sims (1998, 2003) proposed the idea that decision makers are finite capacity information channels, unable to process all the information available. Their information acquisition problem is equivalently rewritten by having they pay an attention cost that is linear in the reduction of Shannon entropy, where the (per-bit) price emerges as the Lagrange multiplier on the attention constraint. In this interpretation, information is floating around agents that grasp it costly bit by costly bit, and agents with deep pockets (large capacity constraint) can obtain any state-action correlation.

[35]As an example of damaging through partial revelation occurring in information markets, Bergemann et al. (2018b) point at the "Undisclosed Debt Monitoring" packages sold by Equifax in which the data broker offers individual rating reports to financial firms considering application for loans in three

Production Experts (the firms) are endowed with a technology to produce primary information structures (signals about the state) and a free technology to replicate and Blackwell-garble those structures. Both production and damaging occurs along a single dimension, exogenously given, and signals produced by different firms are perfect substitutes.[36] Firms maximize the expected revenues from the sale of (their menu of) information structures, net of production cost. Notice that the objective function clearly differs from that of a Bayesian Persuasion (e.g. Kamenica and Gentzkow (2011)) and, more in general, Information Design (e.g. Bergemann and Morris (2017)) setting, where the principal tailors the information transmission to influence the action of agents that are payoff-relevant for him. Also, the market of newspapers, cable tv and opinion outlets that are chosen based on consumers' ideology rather than information value and make revenues from advertising rather than from subscription fees are not examples of information markets whose functioning is likely captured by this paper's model.[37] Finally, both reputation issues and problems related to non-excludability of the information product are neglected.[38]

5.2 A microfoundation

We present a class of investment problems and information structures such that the value of information falls within specification (19). Derivations and additional details are in Appendix C.3.

Investors

Agents make two investment decisions, and both of the returns depend on the realization of the same unknown state $\omega \in \mathbb{R}$, commonly known to be distributed $\mathcal{N}\left(0, \tau_p^{-1}\right)$ and broadly interpreted as the fundamentals of an economy. The first is a binary

different versions differing in the number of "red flags" that the lender receives if the borrowers' history includes some negative events.

[36]Free replication and damaging is a somehow innocuous assumption in this setting, as it simply requires that sellers have no disutility in sharing multiple times, say, a report (or any partial version of it) about the state of an economy. As for the perfect substitutes assumption, section C.3 discusses its empirical content and considers an extension.

[37]Galperti and Trevino (2020) endogenize the supply of information as the outcome of competition among potential information sources that choose where to locate on the accuracy/clarity space in a Myatt and Wallace (2011) setting. In a different setting Perego and Yuksel (2018) study competitive provision and endogenous acquisition of political information with horizontal differentiation of potential consumers. In both those papers firms compete for the *attention* of their consumers, which is justified as many information companies make most revenues from advertising.

[38]Reputation issues in information transmission are studied, among others, by Wang (2009) and Ottaviani and Sørensen (2006). The issue of non-excludability is particularly relevant for information markets: beyond prohibiting re-selling of opinions, private information may be "leaked" through aggregate variables (this channel is explored in Admati and Pfleiderer (1986)). Many financial information packages include agent-specific information (as the rating of potential borrowers in the Equifax example of Bergemann et al. (2018b)), and live prices (Bloomberg vs Reuters), which somehow reduce the concern of non-excludability.

decision $A_1 = \{-1, 1\}$, where agents have to guess *just the sign* of the state (buying or not buying a fixed income security that defaults if $\omega < 0$):

$$u_1(a_1, \omega) = \text{sign}(a_1 \cdot \omega)$$

The second decision is a standard tracking problem in which investors have to guess *the exact location* of the state, $A_2 = \mathbb{R}$ (fine-tuning the investment in the stock market to match the exact state of the economy), and face quadratic losses

$$u_2(a_2, \omega) = -(a_2 - \omega)^2$$

Investors have heterogeneous and exogenous relative exposure $\theta \sim \mathcal{U}[0,1]$ to the second problem (risk aversion, risky-investment endowment), meaning that their overall utility is given by

$$U(a_1, a_2, \omega) = u_1(a_1, \omega) + \theta u_2(a_2, \omega)$$

Investors evaluate information structures $\mathcal{S} \in \Delta(S)^{\Omega}$ by firstly associating with each realization s the policies $a_i(s)_{i \in \{1,2\}}$ and then computing

$$\begin{aligned} v(\mathcal{S}, \theta) &= v_1(\mathcal{S}) + \theta v_2(\mathcal{S}) \\ \text{where} \quad v_i(\mathcal{S}) &= \mathbb{E}_{S \times \Omega}\left[u_i(a_i(s), \omega)\right] - \mathbb{E}_{\Omega}\left[u_i\left(a_i^p, \omega\right)\right] \end{aligned} \tag{32}$$

The class of information structures that are produced and marketed under technology (33) (next section) take the form $\mathcal{S}(q) = \mathcal{N}\left(\omega, q^{-1}\right)$, from which it is immediate to check that for any q and realization $s \sim \mathcal{S}(q)$, investors choose $a_1 = \text{sign}(s), a_2 = \frac{sq}{q+\tau_p}$. Notice that information structures are Blackwell-ordered along the precision parameter $q \in \mathbb{R}$ so $v(\mathcal{S}(q), \theta)$ is increasing in q for any θ; moreover, the value (32) falls within the generalized specification (19) described in Section 3.2: Although returns in both problems are concave in q, there exists a concave transformation g such that $v_1(q) = g(v_2(q))$ for all $q < \tau_p$, which becomes a bound on our quality space.[39] The intuition is that the first bits of the signal are infinitely more informative about the sign of the state than about its exact location; once the sign is (approximately) known, the extra bits of information are valuable almost exclusively as they allow to better target such location.

[39]Clearly, a cautious specification of the cost function ensures that the constraint is satisfied. This is anyways a substantial restriction since it means that the amount of information that is distributed can not exceed the amount that is already possessed by agents in the form of common knowledge. A reasonable interpretation requires therefore to take an "incremental" perspective and think of information about perturbations of economies whose fundamentals are relatively well-known. Notice in this respect that the zero prior mean is completely immaterial: the fixed-income problem corresponds to a bet on whether the perturbation ameliorates such fundamentals or not while the tracking problem corresponds to a quantification of its effects.

Consultants

There is a set $\mathbb{N}$ of experts that decide how much of their costly time to spend observing a Brownian motion $(X_t)_{t\geq 0}$ with drift ω, reflecting that the more time an expert spends staring at the process, the more precisely they learn about ω. Formally, X_t that evolves according to

$$dX_t = \omega dt + dW_t \tag{33}$$

After staring at X for q_j units of time,[40] expert j has effectively produced a signal

$$s_j = \frac{1}{\sqrt{q_j}} X_{q_j} \sim \mathcal{N}\left(\omega, \frac{1}{q_j}\right)$$

about the state ω, which is sold to investors in the following way. Expert j quotes a menu of contracts $m_j = \left\{\left(X_q, p_q\right)\right\}_{q\leq q_j}$ meaning they commit to reveal the realization of X at a time $q \leq q_j$ that they have observed once they are paid price p_q. Two remarks are in order. First, it is a substantive assumption that damaged information coincides with "non-terminal" realizations of X: in principle, experts could report whether X belongs to a certain region, whether it is closer to point A or B, or in general commit to any garbling. We are imposing that both production *and* damaging occur along the same Blackwell order. Second, that inferior qualities can be equivalently obtained by adding independent normal noise, i.e. reporting $X_q = X_{q_j} + \epsilon$ with $\epsilon \sim \mathcal{N}\left(0, \sigma\left(q, q_j\right)\right)$ for an appropriately defined function σ. If agents do not care about correlation (e.g. they play independent decision problems) then how the signal is damaged is immaterial.[41]

Discussion and Extensions

The market for financial information described above fits the general framework of this paper: Investors make across-experts comparison to determine the market pricing function m, then choose the signal with precision that maximize their value (32) net of price. Since the (common) fixed income investment is "solved" with the first bits, the heterogeneity in values – and therefore the motives to damage the signal for screening purposes – grows with the precision of the signal. The full bunching threshold is given by $q^{\text{FB}} = \frac{2\pi\left(\pi+\sqrt{\pi^2-\tau_p^2}\right)}{\tau_p} - \tau_p$, which is inside the feasible quality space if and only

[40] Consistency with the preference specification requires that marketed qualities, which corresponds in the acquisition setting to redefining the cost function over the domain $v_2\left(\left[0, \tau_p\right]\right)$. Since it just complicates the exposition, we keep this transformation implicit.

[41] In a strategic setting, the correlation induced by signal damaging affects agents' value. In particular, in presence of strategic substitutability, the desire for anti-coordination implies that a monopolist may damage the best signal she possesses by adding valuable investor-uncorrelated noise. This is interesting as it offers an explanation for imperfect principal's disclosure alternative to that of Admati and Pfleiderer (1986) (reduce the dissemination through aggregate variables) and that embedded in the obedience constraints of a classic sender-receiver problem.

if $\tau_p < \frac{\pi}{2}$. Else, the monopolist induces full bunching and the effect of competition is only to distribute (to all types) a lower quality signal at below-monopolist price, which is unambiguously welfare reducing per Proposition 5-*ii*). Another implication of the model, that only one firm sells positive qualities and makes profits (Proposition 3), seems to be counterfactual in the market for financial consultancy. The result is driven by the combination of Bertand competition and product homogeneity, which arises from all firms observing the same Brownian Motion (33): whomever stares at it the longer can push out of the market everyone else by undercutting any positive price competitors charge for the realizations that they can also offer. A natural way to introduce product heterogeneity in this application is to add a degree of "originality" in the experts' signal formation process. To this end, suppose that expert j observes the Brownian motion X^j

$$\begin{aligned} dX_t^j &= \omega dt + dW_t^j \\ \text{where} \quad dW_t^j &= \rho dW_t + \sqrt{1-\rho^2} dZ_t^j \end{aligned} \tag{34}$$

where dZ_t^j is a firm-specific process[42] and $\rho \in [0,1]$ is a measure of the correlation in experts' conjecturing effort; one can easily check that $Cov\left(X_{q_j}, X_{q_i}\right) = \rho \min\left\{q_i, q_j\right\}$ for all i, j. If $\rho = 1$ then we are back to homogenous setting presented in this paper, while if $\rho = 0$ (signals are uncorrelated) generates a model of additive social value of production as in Kreps and Scheinkman (1983). In general, products are no more homogeneous and a inferior production effort $q_j < q_i$ creates a signal X_{q_j} contains now information about ω *even after conditioning on* X_{q_i}. In Appendix C.3 we argue that, as preferences admit a quality aggregator, it is straightforward to extend the analysis of first and second best presented in this paper to the setting with heterogeneous values. The second best, however, cannot be decentralized as a pricing stage equilibrium among competitive firms as we did for the case of homogenous products. The technical complications to solve for a competitive equilibrium with screening and heterogeneous products are illustrated by (Stole, 2007), and a generalization of the competitive equilibria we derived for the homogenous goods case has proven elusive.

6 Conclusions

We developed a model of production and distribution of digital goods. The monopolist solves a quality screening problem where the cost of providing an allocation is not separable but depends solely on one statistic of such allocation (the maximum). We discipline the demand side of the market with a preference specification, interpreted in a two-tasks setting, that renders the problem tractable and its solution non-trivial – in

[42]Meaning dW_t, dZ_t^j, dZ_t^i for $i \neq j$ are independent Wiener processes.

the sense that rich menus are offered –. Contrary to a standard screening problem with the same demand primitives, no agent is ever excluded, full bunching can be optimal and efficiency at the top is limited to a distributional efficiency. Market power and asymmetric information induce the interdependent inefficiencies of under-investment in quality and product damaging.

We modeled competition in digital goods markets as two-stage game in which investment in quality is a sunk cost at the pricing stage. Monopolist allocation emerges as one equilibrium of this game – the only one in pure strategies –, which also admits (mixed) equilibria with active competition. Competition induces wasteful double spending and worsens under-acquisition since the highest quality distributed by a competitive market is stochastic but bounded by the monopolist level. However, the distribution of the second-highest quality for free reduces the distributive inefficiency. The welfare comparison between monopoly and duopoly – Pareto dominant among competitive equilibria – is ambiguous and we showed by example that the acquisition cost function can be tailored to completely shut down the channels that favor either of them.

Finally, we presented an application of the general framework to the market for financial information. We motivated some extensions of the model by discussing how its structure imposes limitations within this application and sketch, as an example, a natural way of generating product heterogeneity through correlation of the primary information sources.

References

Admati, Anat R and Paul Pfleiderer, "A monopolistic market for information," *Journal of Economic Theory*, 1986, *39* (2), 400–438. 2, 30, 32

Bergemann, Dirk, Alessandro Bonatti, and Alex Smolin, "The design and price of information," *American Economic Review*, 2018, *108* (1), 1–48. 29

_ , _ et al., "Markets for Information: An Introduction," *Cowles Foundation for Research in Economics Discussion paper*, 2018. 29, 30

_ and Stephen Morris, "Information design: A unified perspective," 2017. 30

Boik, Andre and Hidenori Takahashi, "Fighting Bundles: The Effects of Competition on Second-Degree Price Discrimination," *American Economic Journal: Microeconomics*, 2020, *12* (1), 156–87. 7, 8

Borenstein, Severin and Nancy L Rose, "Competition and price dispersion in the US airline industry," *Journal of Political Economy*, 1994, *102* (4), 653–683. 7

Champsaur, Paul and Jean-Charles Rochet, "Multiproduct duopolists," *Econometrica: Journal of the Econometric Society*, 1989, pp. 533–557. 5, 20

Colombo, Luca, Gianluca Femminis, and Alessandro Pavan, "Information acquisition and welfare," *The Review of Economic Studies*, 2014, *81* (4), 1438–1483. 29

Crawford, Gregory S, Oleksandr Shcherbakov, and Matthew Shum, "Quality overprovision in cable television markets," *American Economic Review*, 2019, *109* (3), 956–95. 7

Deneckere, Raymond J and Preston R McAfee, "Damaged goods," *Journal of Economics & Management Strategy*, 1996, *5* (2), 149–174. 2, 6, 37, 38

Galperti, Simone and Isabel Trevino, "Coordination motives and competition for attention in information markets," *Journal of Economic Theory*, 2020, *188*, 105039. 30

Garrett, Daniel F, Renato Gomes, and Lucas Maestri, "Competitive screening under heterogeneous information," *The Review of Economic Studies*, 2019, *86* (4), 1590–1630. 7

Gerardi, Kristopher S and Adam Hale Shapiro, "Does competition reduce price dispersion? New evidence from the airline industry," *Journal of Political Economy*, 2009, *117* (1), 1–37. 7

Hahn, Jong-Hee, "Damaged durable goods," *The RAND Journal of Economics*, 2006, *37* (1), 121–133. 6

Hellwig, Christian and Laura Veldkamp, "Knowing what others know: Coordination motives in information acquisition," *The Review of Economic Studies*, 2009, *76* (1), 223–251. 29

Hernandez, Manuel A, "Nonlinear pricing and competition intensity in a Hotelling-type model with discrete product and consumer types," *Economics letters*, 2011, *110* (3), 174–177. 7

Inderst, Roman, "Durable goods with quality differentiation," *Economics Letters*, 2008, *100* (2), 173–177. 6

Johnson, Justin P and David P Myatt, "Multiproduct quality competition: Fighting brands and product line pruning," *American Economic Review*, 2003, *93* (3), 748–774. 7

Kamenica, Emir and Matthew Gentzkow, "Bayesian persuasion," *American Economic Review*, 2011, *101* (6), 2590–2615. 30

Kreps, David M and Jose A Scheinkman, "Quantity precommitment and Bertrand competition yield Cournot outcomes," *The Bell Journal of Economics*, 1983, pp. 326–337. 5, 20, 33, 58

Lahiri, Surjasama, Munirul Nabin, and AAron Nicholas, "The provision of damaged goods under competition," *mimeo*, 2021. 7

Maskin, Eric and John Riley, "Monopoly with incomplete information," *The RAND Journal of Economics*, 1984, *15* (2), 171–196. 2, 6

McAfee, R Preston, "Pricing damaged goods," 2007. 6

Mussa, Michael and Sherwin Rosen, "Monopoly and product quality," *Journal of Economic theory*, 1978, *18* (2), 301–317. 2, 6

Muto, Shigeo, "An information good market with symmetric externalities," *Econometrica: Journal of the Econometric Society*, 1986, pp. 295–312. 2

Myatt, David P and Chris Wallace, "Endogenous information acquisition in coordination games," *The Review of Economic Studies*, 2011, *79* (1), 340–374. 29, 30

Netzer, Nick and Florian Scheuer, "Competitive screening in insurance markets with endogenous wealth heterogeneity," *Economic Theory*, 2010, *44* (2), 187–211. 7

Ottaviani, Marco and Peter Norman Sørensen, "Reputational cheap talk," *The Rand journal of economics*, 2006, *37* (1), 155–175. 30

Peitz, Martin and Patrick Waelbroeck, "Piracy of digital products: A critical review of the theoretical literature," *Information Economics and Policy*, 2006, *18* (4), 449–476. 7

Perego, Jacopo and Sevgi Yuksel, "Media Competition and Social Disagreement," 2018. 30

Polanski, Arnold, "A decentralized model of information pricing in networks," *Journal of Economic Theory*, 2007, *136* (1), 497–512. 2

Rochet, Jean-Charles and Lars A Stole, "Nonlinear pricing with random participation," *The Review of Economic Studies*, 2002, *69* (1), 277–311. 7

— **and Philippe Choné**, "Ironing, sweeping, and multidimensional screening," *Econometrica*, 1998, pp. 783–826. 6

Rothschild, Michael and Joseph Stiglitz, "Equilibrium in competitive insurance markets: An essay on the economics of imperfect information," *The quarterly journal of economics*, 1976, pp. 629–649. 7, 24, 40, 41

Sims, Christopher A, "Stickiness," in "Carnegie-Rochester Conference Series on Public Policy," Vol. 49 Elsevier 1998, pp. 317–356. 29

— , "Implications of rational inattention," *Journal of monetary Economics*, 2003, *50* (3), 665–690. 29

Smart, Michael, "Competitive insurance markets with two unobservables," *International Economic Review*, 2000, *41* (1), 153–170. 7

Srinagesh, Padmanabhan and Ralph M Bradburd, "Quality distortion by a discriminating monopolist," *The American Economic Review*, 1989, pp. 96–105. 6

Stole, Lars A, "Price discrimination and competition," *Handbook of industrial organization*, 2007, *3*, 2221–2299. 3, 7, 19, 33

Takeyama, Lisa N, "The welfare implications of unauthorized reproduction of intellectual property in the presence of demand network externalities," *The journal of industrial economics*, 1994, pp. 155–166. 7

Varian, Hal R, "Buying, sharing and renting information goods," *The Journal of Industrial Economics*, 2000, *48* (4), 473–488. 2

Wang, Hefei, "Reputation acquisition of underwriter analysts–theory and evidence," *Journal of Applied Economics*, 2009, *12* (2), 331–363. 30

Wilson, Robert B, *Nonlinear pricing*, Oxford University Press on Demand, 1993. 6

A Additional Results

In this appendix we discuss two issues that, while not directly related to our main investigation of the role of market power in digital goods markets, still enhance our understanding of the forces at play. Specifically, we examine how the monopolist allocation would change if damaging were not allowed and the robustness of competitive equilibria to deviations by idle firms as the investment game unfolds.

A.1 No-Screening Economy

We compare the monopolist allocation of Section 3 with a No-Screening economy in which the monopolist cannot sell damaged versions: this can occur either because of a technological constraint (damaging is costly as in the original Deneckere and McAfee (1996) paper, and prohibitively so), or because of a "damaging ban".[43] For convenience we follow the second interpretation and present our results as if we were evaluating the effects of a ban. The exercise is useful for two reasons. First, it represents the natural extension of the Deneckere and McAfee (1996) normative question "when is the possibility of screening beneficial for all types in the economy?". The different specification of the cost function and the fact that available qualities are not pre-determined add different channels through which the (im)possibility of screening affects allocations and welfare. Second, interpreting NS as the result of a regulatory ban, we are evaluating how a policy intervention might affect the inefficiencies of digital goods monopolies, thus providing a useful benchmark of comparison for the welfare implications of competition (Proposition 5).

[43] We have discussed two cases where even an "unconstrained" digital good monopolist offers a single version of the good: Either preferences are linear, a constant mass of agents receives the undamaged quality while others are excluded (Remark **??**), or g is concave but the quality produced is low enough (steep marginal cost) that the optimal distribution entails full bunching (Corollary **??**).

The NS Problem

A No-Screening (NS) monopolist chooses a quality q^{NS} and the threshold consumer $n\left(q^{\mathrm{NS}}\right)$ to serve. Types $\left[n\left(q^{\mathrm{NS}}\right),1\right]$ buy quality q^{NS} at price $g\left(q^{\mathrm{NS}}\right)+n\left(q^{\mathrm{NS}}\right)q^{\mathrm{NS}}$, the others are excluded.[44] The exclusion policy $n : \mathrm{Q} \to [0,1]$ solves the quality-conditional pricing problem

$$\mathrm{V}^{\mathrm{NS}}(q) = \max_{\theta}\left[g(q)+\theta q\right](1-\theta).$$

and the constrained monopolist acquires the quality q^{NS} that solves $\max_{q\in\mathrm{Q}} \mathrm{V}^{\mathrm{NS}}(q) - c(q)$.

Proposition 6. *i) The exclusion policy is given by*

$$n(q) = \max\left\{\frac{q-g(q)}{2q}, 0\right\};$$

$n(q) \le b(q)$, strictly whenever $b(q) > 0$. Moreover, $n(q) = 0$ for some $q > q^{\mathrm{FB}}$.
ii) $q^{\mathrm{NS}} \le q^{\mathrm{M}}$, strictly whenever $q^{\mathrm{M}} > q^{\mathrm{FB}}$.

Consider first – point *i*) – how the constrained and unconstrained monopolists distribute the same quality q. As $n(q) < b(q)$, the NS monopolist will serve it undamaged to a larger portion of types; this is the mechanical positive impact of prohibiting quality damaging. However, the ban may induce the seller to perform an extreme version of price discrimination, namely full exclusion: The types that are "screened" by the NS monopolist receive nothing, while in the unconstrained contract they had strictly positive consumption and value – Corollary **??**-*ii*) –. Proposition 6-*i*) suggests a three-way partition of the quality space, highlighted in the right panel of Figure 5. In Region A, where $q < q^{\mathrm{FB}}$, the constraint is immaterial and the marginal revenue functions coincide. In Region B, where $q > q^{\mathrm{FB}}$ but still $n(q) = 0$, the NS monopolist sells to all types so her marginal revenues are still $g'(q)$, strictly below the branch of (13) associated with positive damaging. In Region C also $n(q) > 0$ and the NS monopolist excludes a positive mass of low types. This partition is useful in evaluating the distributional impact of the ban; for its overall effect we must also take into account that – point *ii*) – the ban strictly worsens the acquisition inefficiency:[45] by depressing the (marginal) revenues, it makes the seller produce an even lower quality. Notice however that by making the cost function arbitrarily steep around the monopolist quality – approaching a fixed cost structure – we can effectively shut down this channel and work in an exogenous supply setup similar to that analyzed in Deneckere and McAfee (1996), focusing on the allocative effects of the ban.

We are now ready to investigate the welfare effects of the ban. It is clear that the low types that are excluded in Region C are harmed by the ban. Likewise, the monopolist is always worse off since she is solving a constrained version of problem (9). By focusing on the case where the unconstrained monopolist (and, a fortiori, the constrained one) produces in Region B we obtain tighter normative implications of the damaging ban.

[44] To ease comparison with the unconstrained monopolist, we present the problem of the NS monopolist as a two stage maximization. An equivalence result similar to Lemma 1 is immediate and omitted.

[45] In particular, it is possible that q^{M} falls in Region C while q^{NS} falls in Region B.

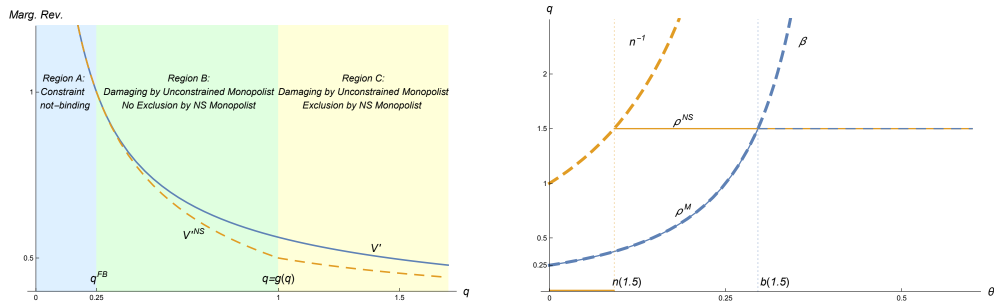

Figure 5: Left: Constrained vs. Unconstrained Marginal Revenues, with Distribution Regions. Right: Exclusion and Damaging Thresholds (dashed), Constrained and Unconstrained distribution of quality 1.5 (Region C).

Proposition 7 (**Welfare impact in Region B**). *If q^{M} belongs to Region B, then*

i) A set of (low) types is better-off under the NS policy.

ii) The net gains from enacting the NS policy can be expressed as

$$\mathrm{W}^{\mathrm{NS}} - \mathrm{W}^{\mathrm{M}} = \int_{q^{\mathrm{FB}}}^{q^{\mathrm{NS}}} d(q)\,\mathrm{d}q - \int_{q^{\mathrm{NS}}}^{q^{\mathrm{M}}} w^{\mathrm{M}}(q)\,\mathrm{d}q \tag{35}$$

where $w^{\mathrm{M}}(q)$ is the marginal monopolist surplus (16) and $d(q)$ are the marginal damaging inefficiencies (17).

iii) If $c''\left(q^{\mathrm{M}}\right)$ is large enough (approaches the fixed cost limit), the NS policy increases total welfare.

The reason for the reversal of the welfare effect on low types from Region B to Region C – point i) – is the following. Since the allocation in Region B is flat, the consumer surplus function is given by $\mathrm{W}^{\mathrm{NS}}(\theta) = \theta q^{\mathrm{NS}}$; in the unconstrained case the types below $b\left(q^{\mathrm{M}}\right)$ get surplus $\mathrm{W}^{\mathrm{M}}(\theta) = \int_0^{\theta} \beta(\theta')\,\mathrm{d}\theta'$. Since $q^{\mathrm{NS}} > \beta(0) = q^{\mathrm{FB}}$, then local to $\theta = 0$ it must hold $\mathrm{W}^{\mathrm{NS}}(\theta) > \mathrm{W}^{\mathrm{M}}(\theta)$. As for the overall effect of the ban, focusing on Region B makes the "complete exclusion" margin non-existent, so point ii) only trades off the positive impact from undoing damaging in the $\left[q^{\mathrm{FB}}, q^{\mathrm{NS}}\right]$ region, and the negative underacquisition impact. Assuming costs are extremely convex around q^{M} ensures marginal cost quickly covers the gap between marginal revenues and so $q^{\mathrm{NS}} \to q^{\mathrm{M}}$. In the fixed cost limit every type receives under the ban the quality that an unconstrained monopolist would produce but fail to distribute: also the underprovision inefficiency is shut down and the ban has only the positive "mechanical" effect of undoing the damaging inefficiencies $\int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} d(q)\,\mathrm{d}q$. A graphical representation of the effects of the ban in Region B under steep and flat marginal cost is offered in the top panel of Figure 6, while the bottom panel performs a specular analysis in the case both monopolists produce in Region C. As the distributional inefficiency from excluding consumers $\left[0, n\left(q^{\mathrm{NS}}\right)\right]$ is present even under exogenous supply (actually, it is made worse as n is increasing), an unambiguous "limit domination" result similar to point iii) does not hold in this case.

Essentially, the specification of the cost function leaves us a degree of freedom which is crucial to determine the impact of a policy in digital goods markets. In this case, the level of the marginal cost curve selects the region that is relevant for the distributional impacts of the ban

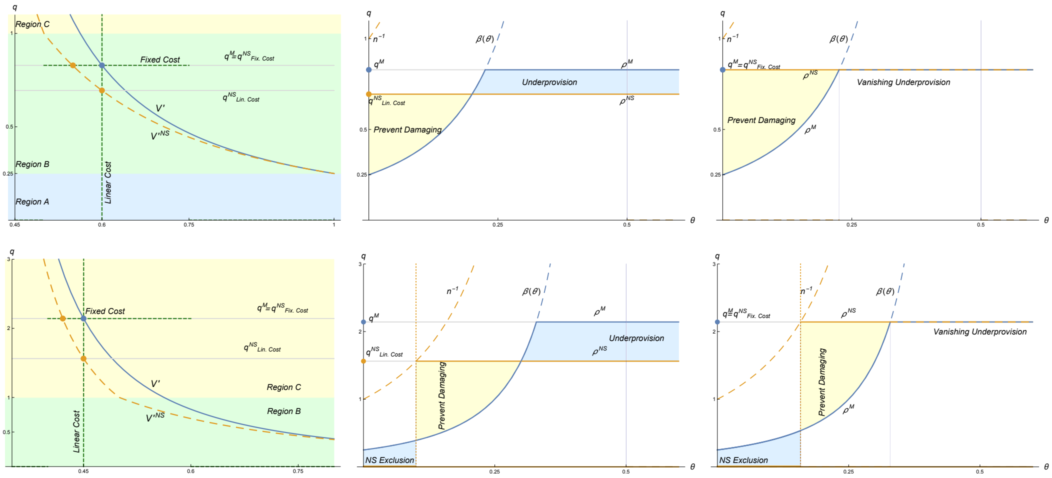

Figure 6: Left: Constrained and unconstrained acquisition, linear vs. fixed cost. Center: Constrained and unconstrained distribution, linear cost. Right: Constrained and unconstrained distribution, fixed cost. Top: Production in Region B. Bottom: Production in Region C.

(i.e. whether or not it induces exclusion), while its curvature around the monopolist quality determines the size of the acquisition gap $q^{\mathrm{M}} - q^{\mathrm{NS}}$. A similar logic applies to the analysis of the competitive environment, which we now introduce.

A.2 Equilibrium Stability

Although it ensures the existence and tractability of competitive equilibria in a setting that is prone to Rothschild and Stiglitz (1976)-like market failure, our timing assumption raises issues of equilibrium stability: As the game unfolds in two stages and there are (infinitely many) idle players during and after the pricing stage, we might worry about off-equilibrium deviations by some firms that – violating the rules of the game – produce a version of the digital good after the first stage. Whether such deviations are plausible clearly depends on the specifics of the application; we abstract from those specifics and investigate the robustness of our equilibria. Depending on when the outsiders make an unexpected move, two notions of stability naturally emerge

- *Interim deviation*: after the first stage, a potential entrant observes the realized entry vector $\boldsymbol{q}$ and chooses whether (and eventually *at which quality*) to enter and play the second stage against $\boldsymbol{q}$.
- *Ex-post deviation:* after the whole game is played, a potential entrant observes the realized market pricing function and chooses whether (and eventually *with which pricing function*) to enter and compete with the realized contract.

Definition. An entry vector is interim (ex-post) stable if it rules out profitable interim (ex-post) deviations. The degree of interim (ex-post) stability of an equilibrium is the probability that the realized entry vector is interim (ex-post) stable.

Notice that neither of the stability notions are associated to the equilibrium of a game in which active firms recognize the threat from outsiders.[46] If firms were playing a pure strategy equilibrium in the first stage game Υ, however, the ex-post stability refinement would collapse to the equilibrium of a Rothschild and Stiglitz (1976) modeling of digital goods markets: firms quote a pricing function $p_i : \mathrm{Q} \to \overline{\mathbb{R}}_+$ at cost $\overline{c}(p_i) = c(\sup\{q : p_i(q) < \infty\})$,[47] all strategic interaction being subsumed in the revenue specification $\overline{\mathrm{R}}_j\left(\{p_i\}_{i\in\mathbb{N}}\right)$ given by (21).

Definition. An ex-post (Rothschild and Stiglitz (1976)) equilibrium is a profile of contracts $\left\{p_i^\star\right\}_{i\in\mathbb{N}}$ such that

$$\overline{\mathrm{R}}_i\left(p_i^\star, p_{-i}^\star\right) - \overline{c}\left(p_i^\star\right) \geq \overline{\mathrm{R}}_i\left(p_i, p_{-i}^\star\right) - \overline{c}(p_i), \qquad \forall i, p_i$$

By the same arguments of Proposition 4-*i*), one firm quoting the monopolist pricing function – and everyone else abstaining at $p_j \equiv \infty$ – is the unique *candidate* ex-post equilibrium. This time, however, an entrant can produce q^{M} and make revenues that are arbitrarily close to $\mathrm{V}\left(q^{\mathrm{M}}\right)$ by granting a small discount that attracts every type against the "idle" monopolist, breaking the equilibrium. Therefore,

Proposition 8.

i) *There is no ex-post equilibrium; the monopolist equilibrium of Proposition 4-i) is interim stable (degree 1), but ex-post unstable (degree 0).*

ii) *All competitive equilibria of Proposition 4-ii) feature intermediate degrees of interim and ex-post stability, and both of them are decreasing in n.*

The intuition for point *i*) is given above, while the other results derive from the following logic. The best response to any *realized* entry vector (x, y) is either to abstain or to choose q^{M} and earn net profits $\Pi\left(x, q^{\mathrm{M}}\right)$ in a market where x becomes the free version. At the ex-post stage, however, the entrant can obtain also the revenues that the interim monopolist makes in the quality spectrum $[y, x]$; therefore y becomes the relevant market statistic to determine the value of the optimal deviation – which is arbitrarily close to $\Pi\left(y, q^{\mathrm{M}}\right)$ –. Since

$$\Pi\left(q^{\mathrm{M}}, q^{\mathrm{M}}\right) < 0 < \Pi\left(q^{\mathrm{M}}, 0\right) = \Pi^{\mathrm{M}}$$

and $\Pi\left(q^{\mathrm{M}}, z\right)$ is continuous and monotonically decreasing in z, there is a unique threshold $z^\star \in \left(0, q^{\mathrm{M}}\right)$ such that $\Pi\left(z^\star, q^{\mathrm{M}}\right) = 0$. An entry vector $\mathbf{q}$ is interim stable if $x_{\mathbf{q}} > z^\star$, and it is ex-post stable if $y_{\mathbf{q}} > z^\star$, so the degrees of stability are pinned down by the equilibrium distributions (evaluated at $z^\star$) of the market statistics (x, y). Point *iii*) then follows from the fact that (x, y) are FOSD ordered across competitive equilibria – as established in the proof of Proposition 5-*i*) –.

[46] Indeed, an interim monopolist that anticipates ex-post entry would not (generally) choose allocation rule (22). Likewise, active firms that anticipate interim and/or ex-post deviations would not evaluate the returns from quality according to (26).

[47] Contrary to the original setting, our production primitives imply that this cost neither depends on competitors' actions nor on consumers' demand.

B Proofs of the Main Results

B.1 Proofs of Section 2

Proof of Proposition 1

As utility is increasing in quality for any type θ, damaging is always inefficient. Therefore, the planner's problem becomes a maximization in a single variable, the quality produced and efficiently allocated to all types

$$\begin{aligned} W^{\star} &= \max_{\bar{q}} \int_0^1 u(\bar{q},\theta)\, \mathrm{d}q - c(\bar{q}) \\ &= \max_{\bar{q}} g(\bar{q}) + \frac{1}{2}\bar{q} - c(\bar{q}) \end{aligned}$$

(6) is the first order condition of this problem (sufficiency is immediate because the objective is concave) and we let $q^{\star}$ be its solution. A seller with perfect information can charge a type-dependent price p_{θ}. Then the profit maximization problem coincides with the social surplus problem, he produces $q^{\star}$, distributes it to all types and he extracts all the surplus.

Proof of Lemma 1

Using the notation introduced in Section 2, the monopolist problem reads

$$\begin{array}{ccc} \max_{\boldsymbol{q},p:[0,1]\to \mathrm{Q}\times\mathbb{R}} & & \int_0^1 p(\theta)\,\mathrm{d}\theta - \overline{c}(\boldsymbol{q}) \\ & \text{s.t.} & \text{IC,IR} \end{array} . \tag{36}$$

Define

$$\omega(x) = \{\boldsymbol{q}, p : [0,1] \to \mathrm{Q}\times\mathbb{R} : \text{ IC, IR hold and } \overline{c}(\boldsymbol{q}) \leq x\},$$

the set incentive compatible and individually rational allocation and pricing function whose cost does not exceed x. Using this constraint sets, problem (36) can be rewritten as

$$\begin{aligned} &\max_{x,\{p:\exists \boldsymbol{q},(\boldsymbol{q},p)\in\omega(x)\}} \int_0^1 p(\theta)\,\mathrm{d}\theta - x \\ &= \max_{x\in\mathbb{R}} \left[\max_{\{p:\exists \boldsymbol{q},(\boldsymbol{q},p)\in\omega(x)\}} \int_0^1 p(\theta)\,\mathrm{d}\theta \right] - x \end{aligned} \tag{37}$$

given the specification of the cost function (8)

$$\begin{array}{ccc} \boldsymbol{q} \in \omega(x) & \text{only if} & c(\max_{\theta} \boldsymbol{q}(\theta)) \leq x \\ & \Longleftrightarrow & \max_{\theta} \boldsymbol{q}(\theta) \leq c^{-1}(x) \end{array}$$

so

$$\omega(x) = \left\{ \boldsymbol{q}, p : [0,1] \to \mathrm{Q}\times\mathbb{R} : \text{ IC, IR hold and } \max_{\theta} \boldsymbol{q}(\theta) \leq c^{-1}(x) \right\}$$

and therefore

$$\max_{\{p:\exists \boldsymbol{q},(\boldsymbol{q},p)\in\omega(x)\}} \int_0^1 p(\theta)\,\mathrm{d}\theta = \mathrm{V}\left(c^{-1}(x)\right)$$

where V is the value of quality defined in (10) as the constraint set of that problem coincides with $\omega\left(c^{-1}(x)\right)$. Redefining the domain of choice to be $\mathrm{Q} = c^{-1}(\mathbb{R})$ – which is possible as c strictly increasing –, Problem (37) becomes

$$\max_{q\in\mathrm{Q}} \mathrm{V}(q) - c(q)$$

as we wanted to show.

B.2 Proofs of Section 3 (Monopolist)

Proof of Theorem 1

Step 0: Properties of β. We first establish the properties of β stated in the text. The virtual value is supermodular as

$$\frac{\partial}{\partial q \partial \theta} v(q,\theta) = \frac{\partial}{\partial q \partial \theta}\left[g(q) + (2\theta - 1)q\right] = 2 > 0$$

and so has an increasing maximizer β. When $\theta > \frac{1}{2}$, v is monotonically increasing and $\beta(\theta) = \infty$; on the contrary the maximizer is characterized by

$$g'(\beta(\theta)) = 1 - 2\theta$$

Putting the two branches together we get

$$\beta(\theta) = \begin{cases} (g')^{-1}(1-2\theta) & \theta < \frac{1}{2} \\ \infty & \theta \geq \frac{1}{2} \end{cases}$$

That $\beta(\theta) > 0$ follows from the Inada condition at 0. Likewise, $\beta(\theta) \to \infty$ as $\theta \to \frac{1}{2}$.

Step 1: Constrained Allocations are $\boldsymbol{q}(q,\theta) = \min\{\beta(\theta), q\}$. Fix a generic quality cap q. The quality constrained problem (10) equivalently reads

$$\begin{array}{lll} \mathrm{V}(q) & \longmapsto & \max_{\boldsymbol{q},p:[0,1]\to\mathrm{Q}\times\mathbb{R}} \int_0^1 p(\theta) - c_\infty(\boldsymbol{q}(\theta))\,\mathrm{d}\theta \\ & \text{s.t.} & \mathrm{IC}\,,\, \mathrm{IR} \end{array}$$

for the extreme cost

$$c_\infty(q') = \begin{cases} 0 & q' \leq q \\ \infty & else \end{cases}.$$

Notice $c_\infty(q')$ is not differentiable, but can be approximated by the continuously differentiable convex function

$$c_n(q') = \left(\frac{q'}{q}\right)^n$$

We can now define the sequence of auxiliary problems

$$\begin{array}{rcc} \mathrm{V}_n(q) & \longmapsto & \max_{\boldsymbol{q},p:[0,1]\to\mathrm{Q}\times\mathbb{R}} \int_0^1 p(\theta) - c_n(\rho(\theta))\,\mathrm{d}\theta \\ & \text{s.t.} & \mathrm{IC}\ ,\mathrm{IR} \end{array}$$

As $\lim_{n\to\infty} c_n(q') = c_\infty(q')$ pointwise, the objective in V_n converges to the objective in V and as policies and values of the auxiliary problems are bounded, the sequence of solutions $(\boldsymbol{q}_n, p_n)$ converges to the solution of the original problem. The auxiliary problem for a generic n can be solved using standard techniques of (cost-separable) monopolist screening.

Firstly, the pairwise comparison of incentive constraints implies that the allocation $\boldsymbol{q}_n$ is monotonically increasing. Through standard arguments, we can rewrite the monopolist's objective as

$$\max_{\boldsymbol{q}_n(\theta)\text{ increasing}} \int_0^1 \left[u(\boldsymbol{q}_n(\theta),\theta) - c_n(\boldsymbol{q}_n(\theta)) - \boldsymbol{q}_n(\theta)(1-\theta)\right]\mathrm{d}\theta.$$

Substituting the preference specification (1) and maximizing the integrand point-wise we obtain a candidate $\boldsymbol{q}_n(\theta)$ as the solution to

$$g'(\boldsymbol{q}_n(\theta)) + (2\theta - 1) - \frac{n}{q^n}(\boldsymbol{q}_n(\theta))^{n-1} = 0 \tag{38}$$

which defines a monotonically increasing function as

$$v_{q,n}(x,\theta) := g(x) + \theta x - x(1-\theta) - \left(\frac{x}{q}\right)^n = v(\theta,x) - \left(\frac{x}{q}\right)^n$$

is still supermodular (and the FOC still characterizes a maximum). Recall that $\beta(\theta)$ solves the cost-free version of (38)

$$g'(\beta(\theta)) + (2\theta - 1) = 0$$

and since

$$\lim_{n\longrightarrow\infty} \frac{n}{q^n} x^{n-1} = \begin{cases} 0 & x < q \\ \infty & x > q \end{cases}$$

the pointwise limit of $\boldsymbol{q}_n(\theta)$ is

$$\boldsymbol{q}_n(\theta) \to \begin{cases} \beta(\theta) & \text{if } \beta(\theta) < q \\ q & \beta(\theta) > q \end{cases}$$

and so $\boldsymbol{q}(q,\theta) = \min\{\beta(\theta), q\}$, which is our desideratum.

Step 2: Marginal Revenues Let $b : \mathrm{Q} \to \left[0, \frac{1}{2}\right]$ the inverse β function, namely

$$b(q) = \beta^{-1}(q) = \max\left\{0, \frac{1 - g'(q)}{2}\right\} \tag{39}$$

so that allocations computed in Step 1 read

$$q(\bar{q}, \theta) = \begin{cases} \beta(\theta) & \theta \le b(\bar{q}) \\ q & \text{else} \end{cases}$$

Now,

$$\begin{array}{rcl} \mathrm{V}(q) & = & \int_0^1 u(\boldsymbol{q}(\theta, q), \theta) - (1-\theta)\boldsymbol{q}(\theta, q)\,\mathrm{d}\theta \\ & = & \int_0^{b(q)} u(\beta(\theta), \theta) - (1-\theta)\beta(\theta)\,\mathrm{d}\theta + \int_{b(q)}^1 u(q, \theta) - (1-\theta) q\,\mathrm{d}\theta \end{array}$$

which is differentiated using Leibniz's rule to obtain

$$\begin{array}{rcl} \mathrm{V}'(q) & = & b'(q)\left[u(q, b(q)) - (1 - b(q))q\right] - b'(q)\left[u(q, b(q)) - (1 - b(q))q\right] + \int_{b(q)}^1 u_q(q, \theta) - (1-\theta)\,\mathrm{d}\theta \\ & = & \int_{b(q)}^1 g'(q) + (2\theta - 1)\,\mathrm{d}\theta = (1 - b(q))(g'(q) - 1) + \left(1 - b(q)^2\right) \\ & = & (1 - b(q))\left[g'(q) - 1 + (1 + b(q))\right] = (1 - b(q))\left[g'(q) + b(q)\right] \end{array} \tag{40}$$

We now just need to substitute the expression (39) for b. In the concave case, if $q < q^{\mathrm{FB}}$ then $g'(q) > 1$ which implies $b(q) = 0$ and $\mathrm{V}'(q) = g'(q)$; substitution of the non-trivial branch of (39) gives that when $g'(q) < 1$ marginal revenues are

$$\mathrm{V}'(q) = \left(\frac{1 + g'(q)}{2}\right)^2$$

which is the expression in the Proposition. Notice that the argument is easily modified to account for the linear case $g \equiv 0$, whence we get $b(q)$ and $\mathrm{V}'(q)$ are constant, respectively at $\frac{1}{2}$ and $\frac{1}{4}$.

Step 3: V is Continuously Differentiable To conclude that V' is $\mathcal{C}^1$ we need to show that the two branches are smoothly pasted at q^{FB} since continuous differentiability inside the two branches is immediate. Notice

$$\left.\left(\frac{1 + g'(q)}{2}\right)^2\right|_{q^{\mathrm{FB}}} = 1 = \left.g'(q)\right|_{q^{\mathrm{FB}}}$$

proves continuity, while

$$\frac{\mathrm{d}}{\mathrm{d}q}\left.\left(\frac{1 + g'(q)}{2}\right)^2\right|_{g'(q)=1} = 2\left.\frac{1 + g'(q)}{2}\frac{g''(q)}{2}\right|_{g'(q)=1} = 2\frac{1+1}{2}\frac{g''(q)}{2} = g''(q) = \frac{\mathrm{d}}{\mathrm{d}q}g'(q)$$

proves continuous differentiability.

Step 4: Monopolist Underprovision To show that $q^{M} < q^{\star}$ it is sufficient to show that

$$V'(q) < w^{\star}(q) = g'(q) + \frac{1}{2} \tag{41}$$

Clearly, $g'(q) + \frac{1}{2} > g'(q)$ so in the full bunching region monopolist underprovision follows from the classic average vs marginal agent targeting. Since

$$x \leq 1 \implies x + \frac{1}{2} > \left(\frac{1+x}{2}\right)^2$$

it follows that $V'(q)$ is strictly below efficient marginal surplus even when $g'\left(q^{M}\right) \leq 1$, proving that (41) is always satisfied and completing the proof.[48]

Proof of Proposition 2

As the monopolist surplus follows from the marginal revenue equation (13), to compute the second best welfare W^{M} we just need to derive consumer surplus.

Lemma. *In the monopolist contract, consumer surplus is given by*

$$W^{C} = \frac{1}{2}\left[q^{FB} + \int_{q^{FB}}^{q^{M}} \left(\frac{1+g'(q)}{2}\right)^2 \mathrm{d}q\right] \tag{42}$$

(of the Lemma). Since in equilibrium $u_{\theta} = \boldsymbol{q}(\theta)$, the surplus of agent θ when the cap is q is given by

$$W^{C}(\theta, q) = u(0) + \int_0^{\theta} u_{\theta}(\boldsymbol{q}(\theta', q), \theta') \mathrm{d}\theta' = \int_0^{\theta} \min\{\beta(\theta'), q\} \mathrm{d}\theta' \tag{43}$$

Total consumer surplus is then given by

$$\begin{array}{rclcl} W^{C}(q) & = & \int_0^1 W(\theta, q)\mathrm{d}\theta & = & \int_0^1 \left(\int_0^{\theta} \min\{\beta(\theta'), q\}\mathrm{d}\theta'\right)\mathrm{d}\theta \\ & = & \int_0^1 \min\{\beta(\theta), q\}(1-\theta)\mathrm{d}\theta & = & \int_0^1 \min\left\{(g')^{-1}(2\theta - 1), q\right\}(1-\theta)\mathrm{d}\theta \\ & = & \int_0^{b(q)} (g')^{-1}(2\theta-1)(1-\theta)\mathrm{d}\theta + \int_{b(q)}^1 q(1-\theta)\mathrm{d}\theta & & \end{array}$$

Now, if $b(q) = 0$ the first term is zero and $W^{C}(q) = \frac{1}{2}q$; if $b(q) > 0$ we use the same steps in the derivations (40) of marginal revenues to show that terms in b' drop and consumer surplus grows at rate

$$\begin{array}{rclcl} w^{C}(q) & = & \int_{b(q)}^1 \frac{\mathrm{d}}{\mathrm{d}q}(q(1-\theta))\mathrm{d}\theta & = & \int_{b(q)}^1 (1-\theta)\mathrm{d}\theta \\ & = & (1-b(q)) - \frac{1}{2}x^2\Big|_{b(q)}^1 & = & \frac{1}{2}(1-b(q))^2 \\ & = & \frac{1}{2}\left(\frac{1+g'(q)}{2}\right)^2 & = & \frac{1}{2}V'(q) \end{array} \tag{44}$$

which is the expression given in the text. Integrating marginal surplus below, and above (44) q^{FB} we obtain (42). □

[48] Again, extension to the linear case is straightforward as it is implied by the inequality $\frac{1}{2} > 0$.

Summing marginal revenues (13) and consumer surplus (44) we get monopolist surplus below q^{FB} grows as first best (has no damaging), while above it grows with slope

$$w^{\mathrm{M}} = \left(\frac{1+g'(q)}{2}\right)^2 - c'(q) + \frac{1}{2}\left(\frac{1+g'(q)}{2}\right)^2 = \frac{3}{2}\left(\frac{1+g'(q)}{2}\right)^2 - c'(q)$$

Monopolist welfare can therefore be written as

$$\mathrm{W}^{\mathrm{M}} = \int_0^{q^{\mathrm{FB}}} \tfrac{1}{2} + g'(q) - c'(q)\,\mathrm{d}q + \int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} \tfrac{3}{2}\left(\tfrac{1+g'(q)}{2}\right)^2 - c'(q)\,\mathrm{d}q \tag{45}$$

First best surplus is given by

$$\begin{aligned}
\mathrm{W}^{\star} &= \tfrac{1}{2}q^{\star} + g\left(q^{\star}\right) - c\left(q^{\star}\right) \\
&= \int_0^{q^{\star}} \underbrace{\frac{1}{2} + g'(q) - c'(q)}_{w^{\star}(q)}\mathrm{d}q
\end{aligned}$$

In the region $\left[q^{\mathrm{FB}}, q^{\mathrm{M}}\right]$ we have marginal damaging inefficiencies

$$\begin{aligned}
d(q) &= \underbrace{\frac{1}{2} + g'(q) - c'(q)}_{w^{\star}} - \underbrace{\left[\frac{3}{2}\left(\frac{1+g'(q)}{2}\right)^2 - c'(q)\right]}_{w^{\mathrm{M}}} \\
&= \tfrac{1}{8}\left(1 + \left(2 - 3g'(q)\right)g'(q)\right)
\end{aligned}$$

which is expression (17) in the text. The inefficiencies decomposition (18) is obtained by splitting the integral representation of $\mathrm{W}^{\star}, \mathrm{W}^{\mathrm{M}}$ in the three regions $\left[0, q^{\mathrm{FB}}\right]$, $\left[q^{\mathrm{FB}}, q^{\mathrm{M}}\right]$ and $\left[q^{\mathrm{M}}, q^{\star}\right]$ as follows

$$\begin{aligned}
\mathrm{W}^{\star} - \mathrm{W}_{\mathrm{M}} &= \int_0^{q^{\mathrm{FB}}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q - \int_0^{q^{\mathrm{FB}}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q \\
&+ \int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q - \int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} \tfrac{3}{2}\left(\tfrac{1+g'(q)}{2}\right)^2 - c'(q)\,\mathrm{d}q \\
&+ \int_{q^{\mathrm{M}}}^{q^{\star}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q - 0 \\
&= 0 \\
&+ \int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} d(q)\,\mathrm{d}q \\
&+ \int_{q^{\mathrm{M}}}^{q^{\star}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q
\end{aligned}$$

B.3 Proofs of Section 4 (Competition)

Proof of Proposition 3

Let x, y generic be the maximum and second order statistics of the realized entry vector $\mathbf{q}$.

Because of Bertrand competition and zero distribution cost, no firm can make revenues by selling qualities $q \leq y$: if that was the case, a competitor can modify his pricing function to copy the revenue-earner in that quality region, not alter the market pricing function (hence his revenues on other qualities) and share those positive revenues. By definition of y, this deviation is feasible for at least one competitor.

We now solve the problem of the interim monopolist.

Using the same steps as for the unconstrained monopolist, we write the problem of the interim monopolist as choosing a type dependent quality allocation rule ρ which is increasing (pairwise comparison of IC) and has image $[y,x]$ (as $p(y)=0$). The interim monopolist revenues are therefore

$$\mathrm{R}_i(x,y) \longrightarrow \max_{\boldsymbol{q}(\theta)\in[y,x],\ \text{increasing}} \int_0^1 [u(\boldsymbol{q}(\theta),\theta) - \boldsymbol{q}(\theta)(1-\theta)]\,\mathrm{d}\theta$$

Pointwise maximization of the objective delivers the candidate allocation

$$\widetilde{\boldsymbol{q}}^{x,y}(\theta) = \arg\max_{q\in[x,y]} g(q) + q(2\theta-1)$$

From the concavity of the objective first order condition characterizes the interior optimum, hence $\widetilde{\boldsymbol{q}}^{x,y}(\theta)=\beta(\theta)$ if $\beta(\theta)\in[y,x]$. The objective is instead strictly decreasing (on the relevant domain) in q if $\beta(\theta)<y$, strictly increasing if $\beta(\theta)>x$. It therefore follows

$$\widetilde{\boldsymbol{q}}^{x,y}(\theta) = \begin{cases} y & y<\beta(\theta) \\ \beta(\theta) & \beta(\theta)\in[y,x] \\ x & x>\beta(\theta) \end{cases}$$

which is a weakly increasing function (in θ), hence the solution of the interim monopolist problem. This proves the allocation rule (22).

We now need to compute the revenue function. Per the discussion above all firms but the interim monopolist make zero revenues. The interim monopolist earns

$$\mathrm{R}_i(x,y) = \mathrm{R}(y,y) + \int_y^x \frac{\partial}{\partial q}\mathrm{R}(q,y)\,\mathrm{d}q \tag{46}$$

Clearly, $\mathrm{R}(y,y)=0$ and, given allocation function (22), marginal revenues for the interim monopolist coincide with those of the unconstrained monopolist: the marginal quality is assigned to types $[b(q),1]$ at marginal price $g'(q)+b(q)$. So

$$\frac{\partial}{\partial q}\mathrm{R}(q,y) = \begin{cases} 0 & y>q \\ \mathrm{V}'(q) & y\le q \end{cases}$$

Therefore,

$$\mathrm{R}_i(x,y) \;=\; \int_y^x \mathrm{V}'(q)\,\mathrm{d}q \;= \mathrm{V}(x)-\mathrm{V}(y)$$

as we wanted to show.

Proof of Proposition 3 (Acquisition Equilibria)

Step 1: Monopolist is the only equilibrium in pure strategies. The argument is given in the text. Notice an entrant at $q > q^{\mathrm{M}}$ makes profits

$$\begin{aligned}\Pi\left(q, q^{\mathrm{M}}\right) &= \mathrm{R}\left(q, q^{\mathrm{M}}\right) - c(q) &&= \mathrm{V}(q) - \mathrm{V}\left(q^{\mathrm{M}}\right) - c(q)\\ &= \int_{q^{\mathrm{M}}}^{q} \left(\mathrm{V}'(q') - c'(q')\right) \mathrm{d}q' - c\left(q^{\mathrm{M}}\right) &&< 0\end{aligned}$$

as $c'(q) > \mathrm{V}'(q)$ above q^{M}.

Step 2: Active firms play an atomless distribution with support including 0. The support of equilibrium play must contain 0: if the support was bounded below by a strictly positive quality $\underline{q}$, then playing the costly $\underline{q}$ gives the firm with larger index zero revenues with probability 1 as the tie-breaking rule (20) excludes revenues sharing, so she deviates to inactivity. The argument why no firm can choose a quality q with positive probability is typical of war of attrition games. If that were the case, all opponents best respond by placing zero probability on an open set including q to get a discrete jump in the probability of winning and (almost) the same profits. But then the firm itself wants to shift the mass away from q, depending on the sign of $\mathrm{V}'(q) - c'(q)$.

Step 3: The distribution of the maximum of opponent's qualities must be $\mathrm{H}(q) = \frac{c'(q)}{\mathrm{V}'(q)}$. From (23), the maximum across competitors' realizations z is sufficient to determine firms' revenues. Let $\mathrm{H}(z)$ be the distribution of such maximum, which from the previous step we know is continuous on $[0, \overline{q}]$ for some $\overline{q} > 0$. By playing q makes expected profits

$$\overline{\Pi}(q) = \mathbb{E}[\Pi(q, \boldsymbol{z})] = \int_{\mathrm{Q}} \mathrm{R}(q, z)\, \mathrm{d}\mathrm{H}(z) - c(q)$$

Using Leibniz's rule on an invariant support and (23), the flat profit condition $\overline{\Pi}'(q) = 0$, necessary for indifference reads, gives

$$\begin{aligned}c'(q) &= \int_{\mathrm{Q}} \frac{\partial}{\partial q} \mathrm{R}(q, z)\, \mathrm{d}\mathrm{H}(z) &&= \int_0^q \mathrm{V}'(q)\, \mathrm{d}\mathrm{H}(z)\\ &= \mathrm{V}'(q) \int_0^q \mathrm{d}\mathrm{H}(z) &&= \mathrm{V}'(q)\, \mathrm{H}(q)\end{aligned}$$

and therefore

$$\mathrm{H}(q) = \frac{c'(q)}{\mathrm{V}'(q)} \tag{47}$$

It holds $\mathrm{H}(0) = 0$ since, by Proposition 1 *ii*), for low q marginal revenues $\mathrm{V}'(q)$ is equal to $g'(q)$ approaching ∞ by the Inada condition. H is increasing as c is assumed convex and V is concave. The right extremum of the support is determined by

$$\mathrm{H}(q) = 1 \implies c'(q) = \mathrm{V}'(q) \implies q = q^{\mathrm{M}}$$

Hence the maximum among $n - 1$ competitors is an absolutely continuous random variable with support $\left[0, q^{\mathrm{M}}\right]$ and distribution (47).

Step 4: An equilibrium candidate with n active firms is (27) The CDF (47) pins down the distribution of the maximal quality among $n-1$ competitors that makes the n^{th} firm indifferent among any quality $q \in \left[0, q^{M}\right]$. So for each n we have one (and only one) candidate equilibrium which has everyone plays

$$H^{EQ(n)}(q) = [H(q)]^{\frac{1}{n-1}}$$

that delivers (27). Notice $H^{EQ(n)}$ admits a positive density

$$h_n(q) = \frac{1}{n-1}[H(q)]^{\frac{2-n}{n-1}} h(q)$$

which is continuous since $h(q) = \frac{d}{dq}H(q)$ is continuous in q in light of continuity of V'' established in Proposition 1 *ii*).

Step 5: Sufficiency By construction, all active firms are indifferent across all qualities in $\left[0, q^{M}\right]$ and they are indifferent with abstaining as 0 is in the support of the equilibrium. We are left to prove that firms do not want to produce more than q^{M}. In that case they would be sure to be the interim monopolist, making profits

$$\overline{\Pi}(q) = \mathbb{E}\left[\Pi\left(q^{M}, \boldsymbol{z}\right)\right] + \int_{q^{M}}^{q} V'(q') - c'(q')\, dq'$$

The first summand is zero by the flat profit condition, while the second term is negative by definition of q^{M}. Inactive firms do not want to deviate either: each of the n firms, competing against $n-1$ opponents makes zero profits in expectation and competing competing against n firms increases (in the sense of FOSD) the distribution of the best competitors' quality.

Proof of Proposition 5

Point i) - Pareto Ranking of Competitive Equilibria.

In any equilibrium with active competition firms make zero profits in expectation and welfare coincides with the consumer surplus. Using the allocation rule of Proposition 3 we obtain the expression (29) in the text for type-dependent welfare in a x, y market.

$$\begin{aligned} W^{x,y}(\theta) &= u(0, x, y) + \int_0^{\theta} u_{\theta}\left(\boldsymbol{q}^{x,y}(\theta'), \theta'\right) d\theta' \\ &= g(y) + \int_0^{\theta} \max\{y, \min\{x, \beta(\theta')\}\} d\theta' \end{aligned}$$

It is immediate to notice

$$\forall \theta, \quad (x, y) \geq_2 (x', y') \text{ implies } W^{x,y}(\theta) \geq W^{x',y'}(\theta)$$

where $\geq_2$ is the standard order in $\mathbb{R}^2$. Given monotonicity of value conditional on the realized qualities, to establish point i) it is sufficient to show that the random vector of marketed qualities $\boldsymbol{x}, \boldsymbol{y}$ has distribution ordered according to first order stochastic dominance (FOSD) in the equilibria with active competition.

Using individual firms' equilibrium play (27) we derive the distribution of the maximal quality $\boldsymbol{x}$ in the n–competitive equilibrium

$$\mathrm{H}_{\boldsymbol{x}}^{\mathrm{EQ}(n)}[x] \quad = \quad \mathrm{P}r^{\mathrm{EQ}(n)}[\max\{q_1, q_2, \ldots q_n\} \leq x] \quad = \quad [\mathrm{H}(q)]^{\frac{n}{n-1}} \ .$$

Since

$$s(n) = v^{\frac{n}{n-1}}$$

is a (strictly) increasing function of n for every $v \in [0,1]$, it follows

$$n > m \implies \mathrm{H}_n[x] > \mathrm{H}_m[x] \ \forall x$$

meaning the highest quality is FOSD decreasing in the intensity of competition. Now we use the following

Fact. *Let* $\mathrm{X}_1, \ldots \mathrm{X}_n$ *be independent observations from a continuous CDF* F. *Then, the conditional distribution of the second order statistic given* $\max_{i \in [n]} \mathrm{X}_i = x$ *is the same as the unconditional distribution of the maximum in a sample of size* $n-1$ *from a new distribution, namely the original* F *truncated at the right at* x*.*

Conditional on $\boldsymbol{x} = x$, the second order statistics $\boldsymbol{y}$ in the n–competitive equilibrium is therefore distributed on $[0, x]$ according to

$$\mathrm{H}_{\boldsymbol{y}|x}^{\mathrm{EQ}(n)}[y] = \left[\left[\frac{\mathrm{H}(y)}{\mathrm{H}(x)}\right]^{\frac{1}{n-1}}\right]^{n-1} = \frac{\mathrm{H}(y)}{\mathrm{H}(x)} \tag{48}$$

which is independent of n. As the distribution of $\boldsymbol{x}$ is FOSD decreasing across competitive equilibria and the distribution of $\boldsymbol{y}$ given $\boldsymbol{x}$ is invariant across equilibria, it follows the *joint* of $\boldsymbol{x}, \boldsymbol{y}$ is FOSD decreasing across equilibria, completing the argument.

Point ii)- Monopoly and Duopoly Comparison.

Suppose $(y \leq x \leq) q^{\mathrm{M}} \leq q^{\mathrm{FB}}$. As both market statistics realize (w.p. 1) in the full bunching region, the competitive allocation (22) will assign every type the undamaged quality x at price $g(x) - g(y)$. Per (29), type dependent surplus is

$$\mathrm{W}^{x,y}(\theta) = g(y) + \theta x$$

Using the law of iterated expectation, we can write total surplus as

$$\mathrm{W}_2 = \mathbb{E}_2[\mathrm{W}^{\boldsymbol{x},\boldsymbol{y}}] = \mathbb{E}_x\left[\mathbb{E}_{y|x}[g(\boldsymbol{y})] + \frac{1}{2}\boldsymbol{x}\right] \tag{49}$$

Since

$$\mathrm{H}_x(y) = \frac{g'(x)}{c'(x)}\frac{c'(y)}{g'(y)}, \quad y \in [0, x]$$

is the conditional CDF of $\boldsymbol{y}$ given $\boldsymbol{x} = x$ it follows

$$\begin{aligned}
\mathbb{E}_{y|x}\left[g\left(\boldsymbol{y}\right)\right] &= \int_0^x g\left(y\right)\mathrm{d}\mathrm{H}_x\left(y\right) &&= \frac{1}{\mathrm{H}(x)}\left(\mathrm{H}\left(y\right)g\left(y\right)\Big|_0^x - \int_0^x g'\left(y\right)\mathrm{H}\left(y\right)\mathrm{d}y\right) \\
&= \frac{1}{\mathrm{H}(x)}\left[\left(\mathrm{H}\left(x\right)g\left(x\right)\right) - \int_0^x g'\left(y\right)\frac{c'(y)}{g'(y)}\mathrm{d}y\right] &&= g\left(x\right) - \frac{c(x)}{\mathrm{H}(x)} = g\left(x\right) - c\left(x\right)\frac{g'(x)}{c'(x)}
\end{aligned} \tag{50}$$

Notice that function

$$s\left(x\right) = g\left(x\right) - c\left(x\right)\frac{g'\left(x\right)}{c'\left(x\right)}$$

is positive and monotonically increasing in $\left[0, q^{\mathrm{M}}\right]$ since $s\left(0\right) = 0$ and

$$\begin{aligned}
s'\left(x\right) &= g'\left(x\right) - c'\left(x\right)\frac{g'(x)}{c'(x)} - c\left(x\right)\frac{g''(x)c'(x) - g'(x)c''(x)}{[c'](x)^2} \\
&= -c\left(x\right)\frac{g''(x)c'(x) - g'(x)c''(x)}{[c'](x)^2} > 0
\end{aligned}$$

Substituting (50) into (49) we get

$$\begin{aligned}
\mathrm{W}_2 &= \mathbb{E}_2\left[g\left(\boldsymbol{x}\right) - c\left(\boldsymbol{x}\right)\frac{g'(\boldsymbol{x})}{c'(\boldsymbol{x})} + \frac{1}{2}\boldsymbol{x}\right] \\
&= \int_0^{q^{\mathrm{M}}}\left[g\left(x\right) - c\left(x\right)\frac{g'(x)}{c'(x)} + \frac{1}{2}x\right]\mathrm{d}\mathrm{H}^2\left(x\right)
\end{aligned}$$

Integrating by parts the last expression we finally obtain

$$\underbrace{g\left(q^{\mathrm{M}}\right) - c\left(q^{\mathrm{M}}\right) + \frac{1}{2}q^{\mathrm{M}}}_{\mathrm{W}^{\mathrm{M}}} - \int_0^{q^{\mathrm{M}}}\left[\underbrace{g'\left(x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(c\left(x\right)\frac{g'\left(x\right)}{c'\left(x\right)}\right)}_{s'(x)>0} + \frac{1}{2}\right]\mathrm{H}^2\left(x\right)\mathrm{d}x$$

that quantifies the competitive losses from a full-bunching monopolist.

We now prove the general welfare decomposition (30). As for Proposition 2, we consider the integral representation $\mathrm{W}^{x,y} = \int_0^x \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{W}^{x,y}\left(q\right)\mathrm{d}q$ and study how competitive welfare *grows* relative to monopolist. In the relevant case $y > q^{\mathrm{FB}}$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{W}^{x,y}\left(q\right) = \begin{cases} w^{\star}\left(q\right) - c'\left(q\right) & q < y \\ w^{\mathrm{M}}\left(q\right) & q \in [y, x] \end{cases}$$

for the following reason: quality increments below y are distributed undamaged (as in first-best) but production costs are incurred twice; for versions $[y, x]$ instead competitive and monopolist marginal welfare coincide since those increments are distributed to the same types (and cost incurred only once). Therefore, the net welfare gains in a (x, y) competitive market,

$W^{x,y} - W^{M}$ can be written as

$$\begin{aligned}
& \left(\int_0^y w^\star(q) - c'(q)\,\mathrm{d}q + \int_y^x w^M(q)\,\mathrm{d}q\right) - \left(\int_0^{q^{FB}} w^\star(q)\,\mathrm{d}q + \int_{q^{FB}}^{q^M} w^M(q)\,\mathrm{d}q\right) \\
= & -c(y) + \underbrace{\int_0^{q^{FB}} w^\star(q) - w^\star(q)\,\mathrm{d}q}_{=0} + \int_{q^{FB}}^{y} \underbrace{w^\star(q) - w^M(q)}_{=d(q)}\mathrm{d}q + \underbrace{\int_y^x w^M(q) - w^M(q)\,\mathrm{d}q}_{=0} - \int_x^{q^M} w^M(q)\,\mathrm{d}q \\
= & -c(y) + \int_{q^{FB}}^{y} d(q)\,\mathrm{d}q - \int_x^{q^M} w^M(q)\,\mathrm{d}q
\end{aligned}$$

Integrating out the market statistics $(\boldsymbol{x},\boldsymbol{y})$ – maintaining the convention $\int_{q^{FB}}^{y} d(q)\,\mathrm{d}q = 0$ whenever $y < q^{FB}$ – we obtain (30).

We are now left to prove competitive domination under extremely convex acquisition costs. WLOG, assume $q^{FB} < 1$ and consider the class of convex functions $c_\alpha(q) = q^\alpha$. Irrespectively of the revenue function V, it holds

$$q_\infty^M = \lim_{\alpha\to\infty} q^M(\alpha) = 1, \quad c_\infty^M = \lim_{\alpha\to\infty}\left(q^M(\alpha)\right)^\alpha = 0 \tag{51}$$

Moreover,

$$H_\alpha^{EQ(2)}(q) = \begin{cases} \frac{c'(q)}{V'(q)} & q \le q^M(\alpha) \\ 1 & q > q^M(\alpha) \end{cases} \longrightarrow \begin{cases} 0 & q < q_\infty^M \\ 1 & q \ge q_\infty^M \end{cases} \tag{52}$$

namely firms' strategies converge in probability to q_∞^M. Finally, since $\boldsymbol{y} \le q^M(\alpha)$ then

$$\mathbb{E}_\alpha[c(\boldsymbol{y})] \le c\left(q_\alpha^M\right) \to 0 \tag{53}$$

Plugging (51)-(52)-(53) in (30) we get limit welfare difference of

$$W_2 - W^M \to \int_{q^{FB}}^{q_\infty^M} d(q)\,\mathrm{d}q - c_\infty^M - \int_{q_\infty^M}^{q_\infty^M} w^M(q)\,\mathrm{d}q = \int_{q^{FB}}^{1} d(q)\,\mathrm{d}q > 0$$

completing the argument.

C Proofs of Additional Results

C.1 Proofs of Appendix A.1 (NS Monopolist)

Proof of Proposition 6

The quality-conditional pricing problem reads

$$V^{NS}(q) = \max_\theta [g(q) + \theta q](1-\theta)$$

The first order condition $\theta q - \theta g(q) - \theta^2 q \ge 0$ gives

$$n(q) \quad = \quad \max\left\{\tfrac{q-g(q)}{2q}, 0\right\}$$

From which it follows that the no-exclusion threshold $n^{-1}(0)$ satisfies $g(q) = q$. As $q^{\mathrm{FB}} = \beta(0)$ solves instead $g'(q) = 1$, $q^{\mathrm{FB}} < n^{-1}(0)$ follows from the Lagrange's Theorem and the fact that g is concave, which also implies

$$n(q) = \frac{q - g(q)}{2q} < \frac{1 - g'(q)}{2} = b(q)$$

whenever $g'(q) < 1$.

iii) The marginal revenue function $\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{V}^{\mathrm{NS}}(q)$ is equal to $g'(q)$ when $q - g(q) < 0$ and we have full bunching (serve everyone at price $g(q)$). If $n(q) \in (0,1)$, we can use the envelope theorem to obtain

$$\begin{array}{rclcl} \mathrm{V}^{\mathrm{NS}}(q) & = & \max_{\theta}[g(q) + \theta q](1-\theta) & & \\ \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{V}^{\mathrm{NS}}(q) & = & [g'(q) + n(q)](1 - n(q)) & = & \frac{(q+g(q))(q-g(q)+2qg'(q))}{4q^2} \end{array}$$

from which we can observe that $\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{V}^{\mathrm{NS}}(q)$ is continuously but not smoothly pasted at $n^{-1}(0)$. To prove that $q^{\mathrm{NS}} \leq q^{\mathrm{M}}$ it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}q}\Pi^{\mathrm{NS}}(q) \leq \mathrm{V}'(q)$$

strictly when $g'(q) \leq 1$. If $q \in \left[q^{\mathrm{FB}}, n^{-1}(0)\right]$, then

$$\mathrm{V}'(q) = \left(\frac{1 + g'(q)}{2}\right)^2 > g'(q) = \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{V}^{\mathrm{NS}}(q)$$

If $q > n^{-1}(0)$, then the derivative of the profit function is given by

$$\begin{array}{rclcl} \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{V}^{\mathrm{NS}}(q) & = & \frac{(q+g(q))(q-g(q)+2qg'(q))}{4q^2} & < & \frac{(q+q)(g(q)-g(q)+2qg'(q))}{4q^2} \\ & = & g'(q) & \leq & \left(\frac{1+g'(q)}{2}\right)^2 \\ & = & \mathrm{V}'(q) & & \end{array}$$

where the first inequality uses $g(q) < q$ (twice) and the second uses $g'(q) \leq 1$.

Proof of Proposition 7

The welfare of type θ under the NS allocation is

$$\mathrm{W}^{\mathrm{NS}}(\theta) = \begin{cases} \left(\theta - n\left(q^{\mathrm{NS}}\right)\right)q^{\mathrm{NS}} & \theta > n\left(q^{\mathrm{NS}}\right) \\ 0 & \text{else} \end{cases}$$

which is zero at the exclusion threshold and grows linearly in θ with slope q^{NS}. If q^{M} is in region B, then $n\left(q^{\mathrm{NS}}\right) = 0$ and

$$q^{\mathrm{M}} > q^{\mathrm{NS}} > q^{\mathrm{FB}}$$

By continuity of β, there exists some $\overline{\theta} > 0$ for which

$$q^{\mathrm{NS}} > \beta(\theta) = \rho^{\mathrm{M}}(\theta) \qquad \forall \theta \in \left[0, \overline{\theta}\right]$$

Now pick $\theta \in \left(0, \overline{\theta}\right)$ and notice

$$\mathrm{W}(\theta) \;=\; \int_0^{\theta} \beta(\theta')\mathrm{d}\theta' \;<\; \int_0^{\theta} q^{\mathrm{NS}}\mathrm{d}\theta' \;=\; \theta q^{\mathrm{NS}} \;=\; \mathrm{W}^{\mathrm{NS}}(\theta)$$

proving that θ is better-off under the NS policy.

ii) Since in Region B no one is excluded by the NS monopolist, total surplus is

$$\begin{aligned}
\mathrm{W}^{\mathrm{NS}} &= g\left(q^{\mathrm{NS}}\right) - c\left(q^{\mathrm{NS}}\right) + \tfrac{1}{2}q^{\mathrm{NS}} \\
&= \int_0^{q^{\mathrm{NS}}} g'\left(q^{\mathrm{NS}}\right) - c'\left(q^{\mathrm{NS}}\right) + \tfrac{1}{2}\mathrm{d}q
\end{aligned}$$

Subtracting this from monopolist surplus 47 and breaking down the integral in regions $q^{\mathrm{FB}} < q^{\mathrm{NS}} < q^{\mathrm{M}}$ we get

$$\begin{aligned}
\mathrm{W}^{\mathrm{NS}} - \mathrm{W}^{\mathrm{M}} &= \int_{(g')^{-1}(1)}^{q^{\mathrm{NS}}} \left(\tfrac{1}{2} + g'(q) - c'(q)\right)\mathrm{d}q - \int_{(g')^{-1}(1)}^{q^{\mathrm{M}}} \tfrac{3}{2}\left(\tfrac{1+g'(q)}{2}\right)^2 - c'(q)\,\mathrm{d}q \\
&+ \quad 0 - \int_{q^{\mathrm{NS}}}^{q^{\mathrm{M}}} \tfrac{3}{2}\left(\tfrac{1+g'(q)}{2}\right)^2 - c'(q)\,\mathrm{d}q \\
&= \int_{q^{\mathrm{FB}}}^{q^{\mathrm{NS}}} d(q)\,\mathrm{d}q - \int_{q^{\mathrm{NS}}}^{q^{\mathrm{M}}} \tfrac{3}{2}\left(\tfrac{1+g'(q)}{2}\right)^2 - c'(q)\,\mathrm{d}q
\end{aligned}$$

first term are welfare gains from undoing damaging, second are losses from underacquisition (compared to monopolist).

iii) We keep fixed q^{M}. As $c''\left(q^{\mathrm{M}}\right) \to \infty$, then $c'\left(q^{\mathrm{M}}\right) - c'(q) \to \infty$ for each $q > q^{\mathrm{M}}$. As $\mathrm{V}'\left(q^{\mathrm{M}}\right) - \left(\mathrm{V}^{\mathrm{NS}}\right)'\left(q^{\mathrm{M}}\right)$ is positive but finite this means $q^{\mathrm{NS}} \to q^{\mathrm{M}}$ and by *ii*) above

$$\mathrm{W}^{\mathrm{NS}} - \mathrm{W}^{\mathrm{M}} \to \int_{q^{\mathrm{FB}}}^{q^{\mathrm{M}}} d(q)\,\mathrm{d}q$$

when convex costs shut down underacquisition, NS policy in Region B has the only (welfare increasing) effect of undoing screening inefficiencies.

C.2 Proof of Proposition 8 (Stability, Appendix XX)

One firm offering the monopolist pricing function $p^{\mathrm{M}}(\cdot)$, everyone else abstaining, is again the only candidate equilibrium. In this case, however, an inactive firm can offer pricing function $p^{\mathrm{M}}(\cdot) - \epsilon$ (so that allocations would be unchanged), pay $c\left(q^{\mathrm{M}}\right)$ and make revenues that are ϵ–close to $\mathrm{V}\left(q^{\mathrm{M}}\right)$, a profitable deviation. So, there is no ex-post equilibrium and the monopolist equilibrium is fully ex-post unstable (but fully interim stability per Proposition 4 *i*)).

To prove *ii*) and *iii*) we ask when a competitive market (x, y) is immune to (interim and ex-post) deviations. Notice preliminarily that

$$0 = \mathrm{R}\left(q^{\mathrm{M}}, q^{\mathrm{M}}\right) < c\left(q^{\mathrm{M}}\right) < \mathrm{R}\left(q^{\mathrm{M}}, 0\right) = \mathrm{V}\left(q^{\mathrm{M}}\right)$$

and $\mathrm{R}\left(q^{\mathrm{M}},z\right)$ is monotonically decreasing, so there is a threshold $z^{\star}$ such that $\mathrm{R}\left(q^{\mathrm{M}},z^{\star}\right)=c\left(q^{\mathrm{M}}\right)$. At the interim stage, a potential entrant solves

$$\mathrm{ID}(x)=\max_{q}\mathrm{R}(q,x)-c(q)$$

the objective is always maximized at q^{M} and delivers positive value $\mathrm{ID}(x)\geq 0 \iff x\leq z^{\star}$. At the ex-post stage a deviator that enters at q^{M} can push out of the market the interim monopolist – by granting a small discount to all types in $[b(y),b(x)]$ – and earn revenues arbitrarily close to $\mathrm{R}\left(q^{\mathrm{M}},y\right)$, exceeding investment costs and therefore inducing entry whenever $y\leq z^{\star}$.

The degree of interim stability of the n–equilibrium is therefore $\mathrm{H}_{\boldsymbol{x}}^{\mathrm{EQ}(n)}\left(z^{\star}\right)$, while the degree of ex-post stability is $\mathrm{H}_{\boldsymbol{y}}^{\mathrm{EQ}(n)}\left(z^{\star}\right)$. Point ii) now follows from $\boldsymbol{x}\geq\boldsymbol{y}$, while the n–ranking in iii) follows from the FOSD ranking of market statistics proved in Proposition 5-i).

C.3 Detailed Derivations for Section 5

Value of Information

With no information, the agent randomizes and gets value $\frac{1}{2}$, independently of τ_p. As $a_1(s)=\mathrm{sign}(s)=\mathrm{sign}(\omega+\epsilon)$ for $\epsilon\sim\mathcal{N}\left(0,q^{-1}\right)$ independent of ω,

$$\mathbb{E}_{\mathrm{S}\times\Omega}\left[u_1\left(a_1(s),\omega\right)\right]-\frac{1}{2}=\mathbb{E}\left[\mathrm{sign}\left(\omega\cdot(\omega+\epsilon)\right)\right]=\mathbb{P}\left[\omega\cdot(\omega+\epsilon)\geq 0\right]-\left(\mathbb{P}\left[\omega\cdot(\omega+\epsilon)\leq 0\right]\right)-\frac{1}{2}$$

$$=2\mathbb{P}\left[\omega\cdot(\omega+\epsilon)\geq 0\right]-\frac{3}{2}=\frac{3\pi-2\arctan\left(\frac{\tau_p}{q}\right)+2\arctan\left(\frac{q}{\tau_p}\right)}{2\pi}-\frac{3}{2}$$

In the second problem agents choose the conditional expectation $a_2(s)=\frac{qs}{q+\tau_p}$ and the value is given by the reduction of the conditional variance of ω

$$\mathbb{E}_{\mathrm{S}\times\Omega}\left[u_2\left(a_2(s),\omega\right)\right]=\mathbb{V}_{prior}-\mathbb{V}_{post}=-\frac{1}{q+\tau_p}+\frac{1}{\tau_p}=\frac{q}{\tau_p\left(q+\tau_p\right)}$$

Now we need to show that there exists a concave function $g:v_1\left(\left[0,\tau_p\right]\right)\to\mathbb{R}$ such that $v_1(q)=g\left(v_2(q)\right)$. Recall that this is equivalent to requiring that

$$\mathrm{A}_{v_1}=-\frac{v_1''(q)}{v_1'(q)}>-\frac{v_2''(q)}{v_2'(q)}=\mathrm{A}_{v_2}$$

By direct computations,

$$\mathrm{A}_{v_1}=\frac{1}{2q}+\frac{1}{q+\tau_p}$$

$$\mathrm{A}_{v_2}=\frac{2}{q+\tau_p}$$

and therefore $A_{v_1} > A_{v_2} \iff q < \tau_p$. Then, letting $y = \frac{q}{\tau_p(q+\tau_p)} \in \left[0, \frac{1}{\tau_p^2}\right]$ we can finally write the utility as $g(y) + \theta y$ with

$$g(y) = \frac{\arctan\left(\sqrt{\frac{\tau_p y}{1-\tau_p y}}\right) - \arctan\left(\sqrt{\frac{1-\tau_p y}{\tau_p y}}\right)}{\pi},$$

$$g'(y) = \frac{\sqrt{\frac{\tau_p y}{1-\tau_p y}}}{\pi y}$$

from which we easily check the Inada condition at 0 and that

$$y^{FB} = (g')^{-1}(1) = \frac{\frac{\pi}{\tau_p} + \sqrt{\frac{\pi^2 - 4\tau_p^2}{\tau_p}}}{2\pi}$$

Inverting y^{FB} delivers the expression for q^{FB} given in the text, as well as the non-triviality upper bound

$$y^{FB} < \frac{1}{\tau_p^2} \iff \tau_p < \frac{\pi}{2}.$$

Correlation as product heterogeneity

In the text we argued that perfect correlation of experts' conjecturing effort – implied by specification (33) – drives the implausible (for the application) implications of Proposition 3. A natural extension is to suppose firms observe

$$\begin{array}{rrcl} & dX_t^j & = & \omega dt + dW_t^j \\ \text{where} & dW_t^j & = & \rho dW_t + \sqrt{1-\rho^2} dZ_t^j \end{array} \tag{54}$$

where dZ_t^j is a firm-specific process and $\rho \in [0,1]$. Consider, for expositional convenience,[49] a duopoly in which firms i, j have produced $q_i > q_j$. It holds

$$Cov\left(X_{q_j}, X_{q_i}\right) = \rho \min\left\{q_i, q_j\right\}$$

meaning that, if $\rho < 1$, X_{q_j} contains information about θ even after conditioning on X_{q_i}. Consumers, still characterized by preference (32), only care about the combined precision of their signal, so $u\left(q_i, q_j, \theta\right)$ admits the aggregator representation

$$u\left(q_i, q_j, \theta\right) = u\left(\Psi\left(q_i, q_j\right), \theta\right)$$

where $\Psi : Q^2 \to Q$, derived by the normal updating formulas is the piecewise convex function given by

$$\Psi\left(q_i, q_j\right) = \frac{q_i\left(q_j(1-2\rho) + q_i\right)}{q_i - q_j \rho^2}$$

[49]The expression for the quality aggregator Ψ and the social cost function $\bar{c}$ are immediately extended to a setting with n active firms.

Notice that if $\rho = 0$ (signals are uncorrelated), then $\Psi(q) = q_i + q_j$ and we have a model of additive social value of production as in Kreps and Scheinkman (1983), though the distribution problem is subject to the screening frictions. If $\rho = 1$ then we are back to homogenous products and maximum aggregator $\Psi(q) = q_i$ which we have analyzed throughout this paper. The following full deterioration property holds instead for any ρ as it is just a consequence of continuity of Ψ: for each q_i, q_j and $q' < \Psi(q_i, q_j)$ we can find $(q_i', q_j') \leq (q_i, q_j)$ such that $q' = \Psi(q_i', q_j')$.

Define the cost $\bar{c} : Q \to \mathbb{R}$

$$\bar{c}(q) = \min_{q_i, q_j} c(q_i) + c(q_j), \qquad \text{s.t. } \Psi(q_i, q_j) \geq q \tag{55}$$

Given the full deterioration property, the characterization of first and second best allocations follow immediately from our analysis by working in the aggregate quality space and replacing the cost function c with $\bar{c}$. Second best is equivalent to a monopolist that owns all production sources and distributes damaged qualities only subject to information frictions: he allocates packages $\{q_i(\theta), q_j(\theta)\}_{\theta \in [0,1]}$ subject to IR and IC where each type θ can only choose among *profiles* $\{q_i(\theta'), q_j(\theta')\}$ (cannot pick $q_i(\theta')$ and $q_j(\theta'')$ for $i \neq j$). Proposition (1) applies verbatim to characterize the second best distribution of *aggregate qualities* which are produced at cost (55). However, this second best contract cannot be decentralized as a pricing stage equilibrium among competitive firms.