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The Veto Power of The Grand Coalition in Economies With Consumption Externalities

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Abstract

The goal of this paper is to provide some new cooperative characterizations of competitive equilibria in pure exchange economies with consumption externalities. In this framework, several notions of equilibrium supported by prices are possible. The central notion analyzed in the paper is the one of A-equilibrium in which individual preferences are affected by consumption externalities in a very broad sense. Indeed, each individual i takes into account in her preferences the consumption of a group of agents A_i exogenously given. Following Hervés-Beloso and Moreno-García (2008), Hervés-Beloso et al. (2005), Hervés-Beloso and Moreno-García (2009), we show that the veto power of the grand coalition is enough to characterize A-equilibria despite the presence of externalities and provide applications in connection with strategic market games. Our results, for suitable choices of the sets Ai, imply characterizations of Walras-Nash, Berge, total and family equilibria of pure exchange economies with externalities.

JEL Classification: C71, D51, D62.

Keywords: Exchange economy, consumption externalities, Aubin core, Edgeworth equilibria, robust efficiency, two-player games.

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1 Introduction

We consider pure exchange economies with a finite number of commodities and individuals in which preferences may be affected by consumption externalities. In this framework, following Hervés-Beloso and Moreno-García (2008), Hervés-Beloso et al. (2005), Hervés-Beloso and Moreno-García (2009), we provide some new cooperative characterizations of competitive equilibria in terms of the blocking power of just one coalition, called the *grand coalition* or the *society*, formed by all the individuals in the economy.

The importance of consumption externalities has been widely recognized in the literature on other-regarding preferences, fairness and altruism (Levine (1998), Fehr and Schmidt (1999) and Dufwenberg et al. (2011)), in collective models of household consumption (Haller (2000) and Gersbach and Haller (2001)), in recent models of general equilibrium with cap-and-trade mechanisms (Hervés-Beloso and Moreno-García (2021)). In this paper, we fix our attention on the A-*economies* and A-*equilibria* introduced by Vasil'ev (2016) and studied in Graziano et al. (2023). In the model under consideration, externalities in consumption are introduced in a very broad sense assuming that each individual i takes into account the consumption of a group of agents A_i exogenously associated to her. Due to this representation of externalities, the A-equilibrium represents an adaptation of the Berge equilibrium to general equilibrium models with externalities (see Berge (1957)) as well as a generalization of the classical competitive equilibrium à la Nash given in Arrow and Hahn (1971) and Laffont (1988). Under such equilibrium, each individual chooses the best consumption bundle for each member of the associated group A_i, taking as given the commodity prices, the aggregate wealth of the members of the group, and the choices of every other agent in the economy.

The A-economy and the related A-equilibrium notion are general enough to cover important cases presented in the literature. Indeed, if for any agent i the reference group A_i reduces to the singleton represented by the agent herself, we recover the classical Walrasian economy with externalities, and the notion of competitive equilibrium à la Nash. If the group is made up of all the agents in the economy other than the individual it is associated with, we end-up with a Berge economy and the natural transposition of the Berge-Zhukovskii equilibrium for non-cooperative games [2]. When A_i coincides with the grand coalition for each agent i, the corresponding competitive equilibrium, called the *total equilibrium* in Vasil'ev (2016), is the one in which each individual is influenced by the consumption of each member of the society. Finally, a further relevant example covered by our results is represented by the *family*

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[2] See Berge (1957) and Zhukovskii (1994) for more details.

economy defined in Haller (2000) and Gersbach and Haller (2001), in which the reference sets A_i form a partition of the set of agents, and each A_i is interpreted as a family [3], provided that all the families have the same number of individuals.

In the framework of A-economies, the core is still defined as the set of allocations which cannot be blocked by a coalition of agents. But, unlike the core of selfish models, each member i of the blocking coalition S in order to be better off proposes an alternative distribution of consumption bundles for the agents in her reference set A_i. This core notion has the feature that agents in S are myopic in the sense that they ignore the choices of the other coalition members. Furthermore, each member i of the blocking coalition is optimistic with respect to the reaction of agents she is not concerned about.Indeed, she assumes that agents outside A_i stick to their status quo consumption bundles [4]. Finally, a feasibility constraint is imposed on the resources available for the blocking coalition S. This is done assuming that each agent i in S uses in the blocking process the aggregate wealth of the members of her reference group A_i.

The previous core is not equivalent to A-equilibrium allocations. However, by considering the Aubin veto mechanism to allow participation of agents in a coalition with a fraction of their endowments, a core equivalence result can be established [5]. Under the Aubin veto, the set of coalitions is extended defining a coalition α as a non-zero vector of real numbers α_i in the interval $[0, 1]$, where for each agent i in the economy, α_i represents the share of participation of i in the coalition. In particular, in the Aubin blocking mechanism defined for A-economies, each agent i in the Aubin coalition α, proposes a new consumption plan for her reference set A_i using the share α_i of the endowment ω_h of each member h of A_i.

In this note, we follow the line of investigation of Hervés-Beloso and Moreno-García (2008) [6], in which the authors notice that checking whether a given allocation belongs to the core might require to look at the whole set of possible coalitions in order to test whether any coalition can improve upon such allocation. This remark motivates an analysis of the core under restrictions imposed on coalition formation. In our framework, to check whether an al-

[3] Under this specification, any individual must belong to her reference set (her family), and all the members of the same family should be associated with the same set.

[4] See Graziano et al. (2017), Di Pietro et al. (2022) and Hervés-Beloso and Moreno-García (2021) for the difference between optimistic and pessimistic attitude of blocking coalition agents.

[5] See Vasil'ev (2016), and Graziano et al. (2023) for the Aubin core equivalence with no convexity assumption on preferences.

[6] See also Hervés-Beloso and Moreno-García (2001).

location belongs to the core implies the additional cost related to the fact that each member i in each potential blocking coalition α, may propose a new consumption bundle not only (and not necessarily) for herself, but for each of the members of the set A_i. Hence we investigate the characterizations of Aubin-core and, consequently, A-equilibrium allocations which are based on restrictions imposed on the set of coalitions.

We prove a first characterization of A-equilibrium allocations which is based on the blocking power of Aubin coalitions with rational shares of participation. This result, which is new for A-equilibria and in general for pure exchange economies with externalities, generalizes the analogous characterization given for Walras-Nash equilibria by Florenzano (1990) and relies on the construction of replica economies associated to the original A-economy. Then we go further and show that one can restrict the attention to the blocking power of just one coalition, namely the grand coalition formed by all traders in the economy. Precisely, our second characterization of A-equilibria provides a refinement of the Aubin-core equivalence theorem which is achieved by exploiting the veto power of the grand coalition. Specifically, we prove that for a given family A and the corresponding A-economy, the A-equilibrium allocations are precisely those allocations that cannot be blocked by a coalition α in which each agent i participates with a non zero fraction α_i of her initial endowment, i.e. by a coalition α with full support. Following the contributions by Hervés-Beloso and Moreno-García (2008), a blocking coalition with full support is interpreted as the society and the result is proved using a direct approach[7].

The characterization of A-equilibria in terms of the society blocking power is key for the proof of the third characterization of the paper in which A-equilibrium allocations are connected to *robustly efficient allocations* (compare Hervés-Beloso and Moreno-García (2008)). In this result, the blocking power is exercised by the grand coalition in a family of new economies associated to the original one, once the relevant allocation x has been fixed. In each of these economies, in order to block, every agent i uses as initial endowment of agent $h \in A_i$, a path from the initial endowment ω_h of the original economy to x_h. This perturbation is defined using the share α_i of participation in the coalition α as weight. We prove that under mild assumptions on the A-economy, A-equilibrium allocations are exactly robustly efficient allocations, that is allocations that cannot be blocked by the grand coalition in any of the auxiliary economies. As in Hervés-Beloso and Moreno-García (2008), as consequence of this characterization we obtain a special form of the two welfare theorems formulated in terms of the dominance relation adopted in A-economies.

[7] In Hervés-Beloso and Moreno-García (2008) and Hervés-Beloso et al. (2005), the analogous result for finite economy is a consequence of the Vind's Theorem in Vind (1972) applied in a suitable continuum associated economy.

In conclusion, the veto power of the grand coalition is enough to obtain A-equilibria. Applications of these characterizations are discussed in the last part of the paper. In Vasil'ev (2016) the notion of A-equilibrium is introduced as a natural generalization of the Berge and Nash equilibrium, building on an optimality principle which holds an intermediate position between altruism and selfishness. In particular, it is observed that in the special case of two players, each Berge equilibrium is a Nash equilibrium of an appropriate two-player game and conversely. In the last part of the paper, following Hervés-Beloso and Moreno-García (2009) and relying on the veto power of the society, we extend this remark to the more general A-equilibrium allocations. In particular, we provide a strategic interpretation of A-equilibria, as Nash equilibria of a game with only two players and within a strategic game approach. The game we consider is a two-player game played by the society without money and prices. Moreover, outcomes are given by the strategies and the characterization is proved under the assumption that preferences of agent i are separable with respect to the consumption of A_i.

Our characterization results, as well as the Aubin core equivalence theorem proved in Vasil'ev (2016), are based on the crucial assumption that the exogenous family of sets A_i associated to agents must be balanced in the sense of Bondareva with full support. This condition is not too demanding, and it is trivially satisfied by all particular examples analyzed in Section 3.

The paper is organized as follows. Section 2 presents the model and the basic assumptions; Section 3 introduces the notions of A-equilibrium, and some examples; in Section 4 we define the dominance relation, the corresponding core, the Aubin core notions and provide a first characterization of A-equilibrium allocations in terms of Aubin coalitions with rational shares of participation and Edgeworth equilibria (Theorem 15); in Section 5 we prove a characterization of A-equilibria in terms of the veto power of the grand coalition (Theorem 19) and in terms of *robustly efficient* allocations (Theorem 23). An application of previous results formulated via Nash equilibria of *society games* is proved in Section 5.3 (Theorem 28).

2 The model and the basic assumptions

There is a finite number l of commodities. The commodity space is $\mathbb{R}^l$. There is a finite number of individuals (agents or traders) denoted by the subscript $i \in N := \{1, \ldots, n\}$. The consumption set associated to each agent is the standard positive cone $X_i := \mathbb{R}^l_+$, $x_i := (x_i^1, \ldots, x_i^l)$ denotes the consumption of individual i, and $x := (x_i)_{i \in N}$ is a vector of consumption bundles for all the agents. The individual preferences $\succsim_i$ of an agent i are affected by the consumption of all the agents: $P_i(x) := \left\{x' \in \mathbb{R}^{l \cdot n}_+ \colon x' \succ_i x\right\}$ denotes the set of consumption bundles which are strictly preferred by agent i to x [8]. A price vector p is an element of $\mathbb{R}^l$, where p^c is the price of one unit of the commodity c.

The initial endowment of individual i is $\omega_i \coloneqq (\omega_i^1, \ldots, \omega_i^l) > 0$, and $\omega \coloneqq (\omega_i)_{i\in N} \in \mathbb{R}_+^{l\cdot n}$. A vector $x = (x_i)_{i\in N} \in \mathbb{R}^{l\cdot n}$ is an *allocation* if it satisfies the physical feasibility condition

$$\sum_{i\in N} x_i \leq \sum_{i\in N} \omega_i$$

and $\mathscr{F}$ denotes the set of allocations.

To each agent is exogenously associated a nonempty set $A_i \subseteq N$ describing the individuals agent i cares about or, equivalently, the set of agents influencing agent i; $A \coloneqq (A_i)_{i\in N}$ [9]. The set $D_i \coloneqq \{h \in N \colon i \in A_h\}$ denotes the agents that are influenced by i. Each agent chooses the consumption of all the individuals in her reference set A_i. In this respect, we use the symbols $X_{A_i} \coloneqq \mathbb{R}_+^{l\cdot|A_i|}$ and $x_{A_i} \coloneqq (x_h)_{h\in A_i}$, to denote the consumption set of A_i and a consumption bundle allocated to members of A_i [10]. $X_{N\setminus A_i} \coloneqq \mathbb{R}_+^{l\cdot|N\setminus A_i|}$ and $x_{N\setminus A_i} \coloneqq (x_h)_{h\in N\setminus A_i}$ have the same meaning in the case of the complementary coalition $N \setminus A_i$. Given x_{A_i} and $x_{N\setminus A_i}$, without loss of generality, we denote x by $(x_{A_i}, x_{N\setminus A_i})$. We also denote x by (x_i, x_{-i}), where $x_{-i} \coloneqq (x_k)_{k\neq i}$, a notation which turns out to be useful when we compare our economy with classical models treated in the literature. Furthermore, given an agent i, we adopt the notation $x_{(i)} \coloneqq (x_{ih})_{h\in A_i} \in X_{A_i}$, where x_{ih} represents the commodity bundle chosen (or proposed) by agent i for the consumption of an agent $h \in A_i$. Such notation will be useful to introduce the blocking mechanism behind the core notions used in the paper. Finally, we use $P_{A_i}(x) \coloneqq \left\{x'_{A_i} \in X_{A_i} \colon (x'_{A_i}, x_{N\setminus A_i}) \in P_i(x)\right\}$ to denote the set of bundles strictly preferred by i to x, when the consumption of any agent $h \in N \setminus A_i$ is fixed at x_h.

The pure exchange economy with externalities under consideration is thus formalized by the list of elements summarized below:

$$E \coloneqq \langle N, \mathbb{R}_+^l, (\succsim_i, \omega_i, A_i,)_{i\in N}\rangle.$$

A coalition in the pure exchange economy E is a non-empty subset S of the set of agents. Once we notice that each coalition S can be identified with its characteristic function, we can introduce the following generalization of ordinary coalitions called *Aubin coalition* [11] due to Aubin (1979).

[8] The strict preference relation $\succ_i \subseteq \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_+^{l\cdot n}$ is defined in the usual way, i.e., $x \succ_i y$ if and only if $x \succsim_i y$ and not $y \succsim_i x$.

[9] We are not necessarily requiring that agent i belongs to A_i.

[10] Given a set B, we denote by $|B|$ or card B its cardinality.

[11] Same as *fuzzy coalition.*

Definition 1 *An Aubin coalition is a vector* $\alpha = (\alpha_i)_{i\in N} \in [0,1]^n$ *with* $\alpha \neq 0$.

In the Aubin coalition α, an agent i may participate by employing the share α_i of her resources. If α_i takes only $\{0,1\}$-values for any i, we have usual ordinary (crisp) coalitions. For a given coalition α, we denote by $\operatorname{supp}\alpha$ its support, i.e., $\operatorname{supp}\alpha := \{i \in N\colon \alpha_i > 0\}$.

For a coalition $S \subseteq N$, we denote by $D_i(S)$ the set of all agents that influence agent i and simultaneously participate to the coalition, i.e., $D_i(S) := D_i \bigcap S$. More generally, for a fuzzy coalition α, $D_i(\alpha)$ will denote the set $D_i(\operatorname{supp}\alpha)$ formed by agents that influence agent i and simultaneously take part in α with a non-zero share α_i of resources. We say that a fuzzy coalition α has *full support* when $\alpha \gg 0$ or, equivalently, when $\operatorname{supp}\alpha = N$.

We make the following survival assumption for the aggregate endowments.

Assumption 2 *The initial endowment* ω_i *belongs to* $\mathbb{R}^l_{++}$ *for each* $i \in N$.

The basic assumptions on preference relations $\succsim_i \subseteq \mathbb{R}^{l\cdot n}_+ \times \mathbb{R}^{l\cdot n}_+$ are listed below.

Assumption 3 *For any agent* i*:*

(1) $\succsim_i$ *is complete, transitive and continuous over* $\mathbb{R}^{l\cdot n}_+$;
(2) for any vector $x_{N\setminus A_i} \in X_{N\setminus A_i}$, $\succsim_i$ *is strongly monotone over* $X_{A_i} \times \{x_{N\setminus A_i}\}$;
(3) for any vector $x_{N\setminus A_i} \in X_{N\setminus A_i}$, $\succsim_i$ *is convex over* $X_{A_i} \times \{x_{N\setminus A_i}\}$;
(4) $\succsim_i$ *is separable over* A_i*, i.e.* $(x_{A_i}, x_{N\setminus A_i}) \succsim_i (y_{A_i}, x_{N\setminus A_i})$ *for* $x_{N\setminus A_i} \in X_{N\setminus A_i}$ *implies that* $(x_{A_i}, z_{N\setminus A_i}) \succsim_i (y_{A_i}, z_{N\setminus A_i})$ *for each* $z_{N\setminus A_i} \in X_{N\setminus A_i}$.

The reader should notice that, although in this paper we do not make use of utilities, the assumptions stated for preferences ensure that the preference relation $\succsim_i$ can be represented by a continuous utility function u_i defined over the space $\mathbb{R}^{l\cdot n}_+$. Condition 4. in Assumption 3 is the separability of the utility function of each agent i on the consumption of her reference set A_i. When A_i reduces to the agent i alone, then we have the usual separability assumption on her own consumption which has been frequently adopted in models with externalities (see, Bergstrom 1970; Borglin 1973; Dufwenberg et al. 2011). When the preference $\succsim_i$ is separable on A_i, an individual preference relation $\succsim_i^{A_i}$ of agent i over consumption of members of A_i can be defined as follows: $x_{A_i} \succsim_i^{A_i} y_{A_i}$ if and only if $(x_{A_i}, x_{N\setminus A_i}) \succsim_i (y_{A_i}, x_{N\setminus A_i})$ for some $x_{N\setminus A_i}$.

In the rest of the paper, the following assumptions will be required on the exogenous family $A = (A_i)_{i\in N}$.

Assumption 4 *The family* $A = (A_i)_{i\in N}$ *satisfies the conditions:*

(1) $N \subseteq \bigcup_{i\in N} A_i$;

(2) $A = (A_i)_{i \in N}$ *is balanced in the sense of Bondareva with full support, i.e., there exists weights* $\beta = (\beta_i)_{i \in N} \in \mathbb{R}^n_{++}$*, such that* $\sum_{h \in D_i} \beta_h = 1$*, for any* $i \in N$*;*

(3) $A = (A_i)_{i \in N}$ *is symmetric in the sense that* $|D_i| = |D_j|$*, for* $i, j \in N$*.*

Point 1. of Assumption 4 assures that each agent is taken into consideration by at least one agent in the economy, i.e. $D_i \neq \emptyset$ for any i, and so, at the equilibrium solution, any agent may potentially consume. Point 2. of Assumption 4 is necessary to assure equivalence Theorems for the core f the economy E [12] Point 2 of Assumption 4 is equivalent to assume that the following set $\mathscr{B}_A$ is nonempty:

$$\mathscr{B}_A := \left\{ \beta \in \mathbb{R}^n_{++} \colon \sum_{i \in N} \beta_i \chi_{A_i} = \chi_N \right\} \neq \emptyset$$

where, for a given set $S \subseteq N$, we denote by $\chi_S := (\chi^h_S)_{h \in N} \in \mathbb{R}^n$ the characteristic function of $S \subseteq N$, i.e.,

$$\chi^h_S := \begin{cases} 1 & \text{if } h \in S \\ 0 & \text{if } h \notin S \end{cases}$$

Finally, point 3. of Assumption 4 assures that each agent is influenced by the same number of individuals.

Remark 5 In the rest of the paper, three relevant examples of an exogenous family $A = (A_i)_{i \in N}$ satisfying Assumption 4 will be used. They are described by the following cases:

$$\begin{cases} A^{WN} \equiv \{A_i = \{i\} \colon i \in N\} \\ A^B \equiv \{A_i = N \setminus \{i\} \colon i \in N\} \\ A^T \equiv \{A_i = N \colon i \in N\} \end{cases} .$$

The structure A^{WN} will describe *Walras-Nash* models of equilibria in which the only agent influencing the stability of i is the agent i herself. In this particular case, $|D_i| = 1$, for each $i \in N$. The structure A^B refers to *Berge* models, where each agent is influenced by all other agents in the economy and $|D_i| = n - 1$, for each $i \in N$. Finally, A^T will be used to define *total equilibria* in which each agent is influenced by the society. Clearly, in this situation, $|D_i| = n$, for each $i \in N$.

Remark 6 Assume that $\mathscr{H} := \{H_j \colon j = 1, \ldots, k\}$, with $k \leq n$ is a partition of the set of agents, and interpret each $H_j \in \mathscr{H}$ as a family. For every $i \in N$, define $A^F \equiv \{A_i\}_{i \in N}$ where A_i is the element of $\mathscr{H}$ containing i. Notice that, in this case, the sets A_i are not necessarily different, and $i \in A_i$ for each $i \in N$. In particular, observe that members of the same family are associated with

[12] Compare Theorem 15.

the same A_i and $D_i = A_i$, for each agent i. The structure A^F will describe the *family economies* in which the agents influencing the stability of i are the members of her family[13]. In this setting, if all the families have the same cardinality, then all points of Assumption 4 are satisfied.

Finally, we introduce the notion of r-fold replica of the pure exchange economy E necessary to define in the next section, the Edgeworth equilibrium.

Definition 7 *If r is any positive integer, then the r-fold replica economy of E, denoted by E_r, is an economy with rn agents indexed by i_q with $i = 1, \dots, n$, $q = 1, \dots, r$ such that each agent i_q has the following characteristics:*

1. $X_{i_q} \coloneqq \mathbb{R}^l_+$;
2. $\omega_{i_q} \coloneqq \omega_i$;
3. *for any* $x = (x_{h_s})_{\substack{h \in N \\ s=1,\cdots,r}} \in \mathbb{R}^{l \cdot n_r}_+$ *and* $y = (y_{h_s})_{\substack{h \in N \\ s=1,\cdots,r}} \in \mathbb{R}^{l \cdot n_r}_+$, *the preference relation* $\succsim^r_{i_q} \subseteq \mathbb{R}^{l \cdot n_r}_+ \times \mathbb{R}^{l \cdot n_r}_+$ *is defined by*

$$x \succsim^r_{i_q} y \iff (x_{h_q})_{h \in N} \succsim_i (y_{h_q})_{h \in N}$$

   where $N(r) \coloneqq \{i_q = (i, q) : i \in N, q = 1, \dots, r\}$ *is the set of the agents in the r-fold replica, and n_r denotes its cardinality;*
4. $A^r_{i_q} \coloneqq \{h_q = (h, q) \in N \times \{q\} : h \in A_i\}$.

As for the economy E, also in the replica economy we use the notation $X_{A^r_{i_q}} \coloneqq \mathbb{R}^{l \cdot |A^r_{i_q}|}_+ = \mathbb{R}^{l \cdot |A_i|}_+$ to denote the consumption possibilities of the reference set $A^r_{i_q}$. For a given $i \in N$, agents i_q with $q = 1, \dots, r$ can be interpreted as the agents of type i in the replica E^r. The r-fold replica economy of E is formalized by the list of elements summarized below:

$$E^r \coloneqq \langle N(r), \mathbb{R}^l_+, (A_{i_q}, \succsim^r_{i_q}, \omega^r_{i_q})_{i_q \in N(r)} \rangle$$

If $r = 1$ then the 1-fold replica economy E^1 coincides with E. Given a vector $x \in \mathbb{R}^{l \cdot n_r}_+$, in a natural way, $P_{i_q}(x) \coloneqq \left\{ x' \in \mathbb{R}^{l \cdot n_r}_+ : x' \succ^r_{i_q} x \right\}$ denotes the set of consumption bundles which are strictly preferred by agent i_q to x, and $P_{A^r_{i_q}}(x) \coloneqq \left\{ x'_{A^r_{i_q}} \in X_{A^r_{i_q}} : (x'_{A^r_{i_q}}, x_{N(r) \setminus A^r_{i_q}}) \in P_{i_q}(x) \right\}$ is the set of bundles strictly preferred by i_q to x, when the consumption of any agent $h_q \in N \setminus A^r_{i_q}$ is fixed at x_{h_q}[14]. Notice that, for $x \in \mathbb{R}^{l \cdot n_r}_+$

$$x'_{(i_q)} \in P_{A^r_{i_q}}(x) \iff x'_{(i_q)} \in P_{A_i}((x_{h_q})_{h \in N}).$$

[13] See for instance Haller (2000) and Gersbach and Haller (2001).

[14] The strict preference relation $\succ^r_{i_q} \subseteq \mathbb{R}^{l \cdot n_r}_+ \times \mathbb{R}^{l \cdot n_r}_+$ is defined by $x \succ^r_{i_q} y$ if and only if $(x_{h_q})_{h \in N} \succsim_i (y_{h_q})_{h \in N}$ and not $(y_{h_q})_{h \in N} \succsim_i (x_{h_q})_{h \in N}$.

We denote by $\mathscr{F}^r$ the set of allocation for the r-fold replica economy E^r, i.e.,

$$\mathscr{F}^r := \left\{ x \in \mathbb{R}_+^{l \cdot n_r} \colon \sum_{i_q \in N(r)} x_{i_q} \leq \sum_{i_q \in N(r)} \omega_{i_q} \right\}.$$

Remark 8 If $x \in \mathbb{R}_+^{l \cdot n}$ is an allocation for the economy E, then the vector $x^{(r)} \in \mathbb{R}_+^{l \cdot n_r}$ defined by $x_{i_q}^{(r)} = x_i$ for any $q = 1, \cdots, r$ and for any $i \in N$ is an allocation for the r-fold replica economy E^r. Indeed,

$$\sum_{i_q \in N(r)} x_{i_q} = \sum_{i \in N} r x_i \leq \sum_{i \in N} r \omega_i = \sum_{i_q \in N(r)} \omega_{i_q}$$

where the last inequality follows from the feasibility of x. So, an allocation of E gives rise in a natural way to an allocation of E^r.

In line with the notations introduced in Section 2, we denote by $D_{i_q}^r := \{h_s \in N(r) \colon i_q \in A_{h_s}^r\}$ the set of agents influencing i_q. Given a coalition $S \subseteq N(r)$, $D_{i_q}^r(S) = D_{i_q}^r \cap S$. Notice also that for each agent i_q, the cardinalities of the sets $D_{i_q}^r$ and $D_{i_q}^r(S)$ are respectively the same of the cardinalities of D_i and $D_i(S_q)$, where $S_q = \{h \in N \colon h_q = (h, q) \in S\}$.

3 A-Equilibrium

In this section, we introduce a competitive equilibrium for the A-economy. The A-equilibrium is a natural generalization of the classical Walras-Nash competitive equilibrium in the presence of externalities, and it is in the spirit of the Berge equilibrium for non-cooperative games (see Berge (1957)).

Definition 9 (A-equilibrium) $(x, p) \in \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_{++}^l$ *is an A-equilibrium for the economy E if*

1. $x_{A_i} \in B_{A_i}(p, \omega)$ *for all* $i \in N$;
2. $P_{A_i}(x) \bigcap B_{A_i}(p, \omega) = \emptyset$ *for all* $i \in N$;
3. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$

where $B_{A_i}(p, \omega) := \left\{ x_{A_i} \in X_{A_i} \colon p \cdot \left(\sum_{h \in A_i} x_h \right) \leq p \cdot \left(\sum_{h \in A_i} \omega_h \right) \right\}$ *denotes the budget set of agent i.*

Notice that agent i uses the endowments of all the agents belonging to A_i in its budget constraint. By the first two conditions, the affordable bundle

x_{A_i} maximizes the preference of agent i over the budget set. Condition 3. is the classical market clearing condition. We denote by $\Omega(E)$ the set of A-equilibria, and by $\mathscr{W}(E)$ the set of A-equilibrium allocations, i.e., $\mathscr{W}(E) \coloneqq \Big\{x \in \mathbb{R}_+^{l \cdot n} \mid \exists\, p \gg 0\colon (x,p) \in \Omega(E)\Big\}$.

It is easy to show that the notion of competitive equilibrium introduced with Definition 9 generalizes well know competitive equilibrium notions depending on the specification of the exogenous family A. Consider the three structures presented in Remark 5 [15] and, for simplicity, suppose that preferences are represented by utility functions [16] . When we consider the family A^{WN} corresponding to the partition of the set of agents $\{A_i = \{i\} : i \in N\}$, i.e. the situation in which the only agent contributing to i's wealth is i, (x,p) is an A-equilibrium if

1. $x_i \in \underset{x_i' \in \mathbb{R}_+^l}{\arg\max}\{u_i(x_i', x_{-i}) \mid p \cdot x_i' \leq p \cdot \omega_i\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

Hence, the A-economy coincides with the standard pure exchange economy with externalities, and the definition of A-equilibrium coincides with the one of competitive equilibrium à la Nash [17] . The existence of competitive equilibria à la Nash is proved, among other authors, by Dufwenberg et al. (2011) and Florenzano (1990).

Consider now the structure A^B in which $A_i = N \setminus \{i\}$ for every $i \in N\}$, i.e. the situation in which each agent is affected by all the other agents. Thus, (x,p) is an A-equilibrium if

1. $x_{-i} \in \underset{x_{-i}' \in \mathbb{R}_+^{l \cdot (n-1)}}{\arg\max}\ \{u_i(x_{-i}', x_i) \mid p \cdot \sum_{h \in N \setminus \{i\}} x_h' \leq p \cdot \sum_{N \setminus \{i\}} \omega_h\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

This definition can be considered as an adaptation of the classical Berge equilibrium in the sense of Zhukovskii (1994) for non-cooperative games [18] . The issue related to the existence of a Berge equilibrium is a difficult task that have been addressed for example in Abalo and Kostreva (2005) and Nessah et al. (2007). However, some special cases of A-equilibria when $A = A^B$ are presented in Graziano et al. (2023).

[15] Further examples are presented in Graziano et al. (2023).
[16] As already mentioned, under Assumption 3 the preference relations can be represented by continuous utility functions.
[17] See for example Borglin (1973).
[18] See for instance,Vasil'ev (2016).

Finally, in the case of the structure A^T in which $A_i = N$, for each $i \in N\}$, i.e. the situation in which each agent is affected by all the agents including herself, (x, p) is an A-equilibrium if

1. $x \in \arg\max_{x' \in \mathbb{R}_+^{l \cdot n}} \{u_i(x') \mid p \cdot \sum_{h \in N} x'_h \leq p \cdot \sum_{h \in N} \omega_h\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

and the notion of A-equilibrium introduced with Definition 9 requires that agents maximize their utility functions on the same budget set. The common budget set is formed by allocations whose value does not exceed the value at the fixed price system of the total initial endowment and agents propose a consumption bundle for each of the traders in the society. We call this equilibrium a *total equilibrium*[19]. Examples in which a total equilibrium exists are given in Graziano et al. (2023).

In very general terms, for the pure exchange economy studied in this paper, the fact that different agents may share the same A-relevant arguments in their utility functions, makes standard approaches to the existence problem not applicable (see the discussion in Vasil'ev (2016) and Graziano et al. (2023)). On the other hand, the variety of equilibrium notions included in the notion of A-equilibrium makes the general problem of existence of A-equilibria an open questions that deserve to be analyzed. For each benchmark case analyzed above, the equivalence of A-equilibria with cooperative solutions like the core is relevant since may provide a way to overcome issues related to the existence of the equilibria by studying the non-emptiness of the core. In the following sections, we shall present equivalence results for A-equilibria focusing in particular on the blocking power of the grand coalition.

4 Core allocations and Edgeworth equilibria

We introduce now some properties that a feasible allocation may satisfy. The following notions do not depend on prices, but on the blocking mechanism implemented by coalitions. The blocking mechanism in pure exchange economies with externalities may be defined in several ways, depending on the assumptions made on the behavior of agents outside the blocking coalition. This behavior may affect the utility of coalition's members due to consumption externalities (see Graziano et al. (2017) for an overview of cooperative solutions in models with externalities). In the blocking procedure introduced below, a member i of a coalition blocking a status quo allocation x, proposes a new

[19] A total equilibrium can be considered also as a special case of *family equilibrium* studied by Gersbach and Haller (2001) when all agents belong to the same family.

commodity bundle for each of the agents in the set A_i. Moreover, agents in a blocking coalition are myopic with respect the decision taken by the others coalition members. Furthermore, this notions is based on an *optimistic* behavior of the blocking coalition with respect to the reactions of the outsiders meaning that outsiders' consumption is fixed at the status quo.

Definition 10 (Core) *Given an allocation $x \in \mathscr{F}$ and a coalition $S \subseteq N$, we say that S improves upon x whenever, for every agent $i \in N$, there exists a vector $x'_{(i)} \in X_{A_i}$ such that*

1. $x'_{(i)} \in P_{A_i}(x)$ *for any* $i \in S$;

2. $$\sum_{i \in N} \sum_{h \in D_i(S)} x'_{hi} \leq \sum_{i \in N} \sum_{h \in D_i(S)} \omega_i.$$

The set of allocations which cannot be improved upon by any coalition is called the Core, and it is denoted by $\mathscr{C}(E)$. The set of allocations which cannot be improved upon by the grand coalition N is denoted by $\mathscr{N}(E)$.

It is clear that the inclusion $\mathscr{C}(E) \subseteq \mathscr{N}(E)$ holds true. Moreover, since the actions of coalition members are independent from the outsiders, the previous notions reduce respectively to the standard core and the set of Pareto optimal allocations in a pure exchange economy with selfish agents. The blocking mechanism introduced above and the corresponding core can be extended to Aubin coalitions as follows.

Definition 11 (Aubin Core) *Given an allocation $x \in \mathscr{F}$ and a coalition $\alpha \in [0,1]^n$ with $\alpha \neq 0$, we say that α improves upon x whenever, for every agent $i \in N$, there exists a vector $x'_{(i)} \in X_{A_i}$ such that*

1. $x'_{(i)} \in P_{A_i}(x)$ *for any* $i \in \mathrm{supp}(\alpha)$;

2. $$\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i \in N} \sum_{h \in D_i(\alpha)} \omega_i.$$

The set of allocations which cannot be improved upon by any Aubin coalition is called the Aubin Core, and it is denoted by $\mathscr{C}^A(E)$. The set of feasible allocations which cannot be improved upon by any Aubin coalition with rational shares of participation for each agent, i.e. by any $\alpha \in ([0,1] \cap Q)^n$, $\alpha \neq 0$, is denoted by $\mathscr{C}^A_{rat}(E)$. The set of allocations which cannot be improved by Aubin coalitions with full support is denoted by $\mathscr{N}^A(E)$.

Notice that this more general definition also ensures that usual notions of Aubin core can be recovered in pure exchange economies with selfish agents. Obviously, since an enlargement of the blocking coalitions reduces the corresponding core, the inclusions $\mathscr{C}^A(E) \subseteq \mathscr{C}(E)$, $\mathscr{C}^A(E) \subseteq \mathscr{C}^A_{rat}(E)$ and $\mathscr{C}^A(E) \subseteq$

$\mathscr{N}^A(E)$ hold true.

We introduce now the notion of Edgeworth equilibrium. Following the Edgeworth conjecture, Debreu and Scarf (1963) have showed that, by enlarging in an appropriate way the number of agents in the economy, all non-equilibrium allocations are ruled out from the limit core. The first result of the paper shows the validity of the Edgeworth conjecture in our setting. We proceed by showing the equivalence between the Aubin core and the limit core i.e. the Edgeworth equilibrium and then apply the fuzzy core equivalence Theorem proved in Vasil'ev (2016).

Definition 12 (Edgeworth Equilibrium) *Given a positive integer r, an allocation $x \in \mathscr{F}$ and a coalition $S \subseteq N(r)$ with $S \neq \emptyset$, we say that S improves upon x in the r-fold replica economy E^r whenever S improves upon the equal treatment allocation $x^{(r)}$ associated with x, i.e. for every agent $i_q = (i,q) \in N(r)$, there exists a vector $x'_{(i_q)} \in X_{A^r_{i_q}}$ such that*

1. $x'_{(i_q)} \in P_{A^r_{i_q}}(x^{(r)})$ *for any* $i_q \in S$;

2. $\displaystyle\sum_{i_q \in N(r)} \sum_{h_s \in D^r_{i_q}(S)} x'_{h_s i_q} \leq \sum_{i_q \in N(r)} \sum_{h_s \in D^r_{i_q}(S)} \omega_{i_q}.$

The set of allocations of E which cannot be improved upon by any coalition in the r-fold replica economy E^r is denoted by $\mathscr{C}_r(E)$. An allocation x of E is an Edgeworth equilibrium if $x \in \mathscr{C}_r(E)$ for any $r \in \mathbb{N}$. The set of Edgeworth equilibria is denoted by $\mathscr{C}^E(E)$.

Remark 13 Notice that, by definition, the equality $\mathscr{C}^E(E) \coloneqq \bigcap_{r\in\mathbb{N}} \mathscr{C}_r(E)$ holds true. Moreover, the family of sets $\{\mathscr{C}_r(E)\}_{r\geq 1}$ is decreasing. Indeed, the following easy argument shows that $x \in \mathscr{C}_{\widetilde{r}}(E)$ implies $x \in \mathscr{C}_r(E)$, when $\widetilde{r} > r$. Suppose by contradiction that there exist a coalition $S \subseteq N(r)$ and vectors $x'_{(i_q)}$, with $i_q \in N(r)$ such that conditions 1. and 2. in Definition 12 hold. Define the coalition of in $N(\widetilde{r})$ $\widetilde{S} \coloneqq S$ and $\widetilde{x}_{(i_q)} \in \mathbb{R}^{l \cdot n_{\tilde{r}}}_+$ as follows,

$$\tilde{x}_{i_q} \coloneqq \begin{cases} x'_{i_q} & \text{if } q = 1, \ldots, r; \\ 0 & \text{if } q = r+1, \ldots, \tilde{r}. \end{cases}$$

Notice that, by Definition 7 , we get $\tilde{x}_{i_q} \in P_{A^{\tilde{r}}_{i_q}}(x^{(\tilde{r})})$ for any agent $i_q \in \tilde{S}$. Furthermore,

$$\sum_{i_q \in N(\tilde{r})} \sum_{h_q \in D_{i_q}(\tilde{S})} \tilde{x}_{h_q i_q} = \sum_{i_q \in N(r)} \sum_{h_q \in D_{i_q}(S)} x'_{h_q i_q} \leq$$

$$\leq \sum_{i_q \in N(r)} \operatorname{card} D^r_{i_q}(S) \cdot \omega_{i_q} = \sum_{i_q \in N(\tilde{r})} \operatorname{card} D^{\tilde{r}}_{i_q}(\tilde{S}) \cdot \omega_{i_q}$$

where the last equality follows by

$$\sum_{i_q\in N(\tilde r)} \operatorname{card} D^{\tilde r}_{i_q}(\tilde S)\cdot \omega_{i_q} = \sum_{i_q\in N(r)} \operatorname{card} D^{r}_{i_q}(S)\cdot \omega_{i_q} + \sum_{i_q\in N(\tilde r)\setminus N(r)} \operatorname{card} D^{\tilde r}_{i_q}(\tilde S)\cdot \omega_{i_q}$$

and the fact that the second component of the right side of the previous equation is equal to zero. Thus, the conclusion follows.

The equivalence of Aubin core allocations and Edgeworth equilibria can be extended to A-economies.

Theorem 14 *Let E be an A-economy. Assume that conditions (1)-(3) of Assumption 3 and condition (1) of Assumption 4 are satisfied. Then*

$$\mathscr{C}^A(E) = \mathscr{C}^A_{rat}(E) = \mathscr{C}^E(E)$$

Proof. By definition, it is true that $\mathscr{C}^A(E) \subseteq \mathscr{C}^A_{rat}(E)$. We claim that $\mathscr{C}^A_{rat}(E) = \mathscr{C}^E(E)$.

Let $x \in \mathscr{C}^A_{rat}(E)$ and assume by contradiction that $x \notin \mathscr{C}^E(E)$. Then there exists a positive integer r such that $x \notin \mathscr{C}_r(E)$. Consequently, there exists a coalition $S \subseteq N(r)$ which improves upon $x^{(r)}$ via vectors x'_{i_q}, with $i_q = (i,q) \in N(r)$. This means that

1. $x'_{(i_q)} \in P_{A^r_{i_q}}(x^{(r)})$ for any $i_q \in S$;
2. $\displaystyle\sum_{i_q\in N(r)} \sum_{h_q\in D^r_{i_q}(S)} x'_{h_q i_q} \le r \sum_{i\in N} \sum_{h_q\in D^r_{i_q}(S)} \omega_{i_q}.$

For each type h, denote by t_h the number of agents h_q which belong to S, that is,

$$t_h \coloneqq \operatorname{card} S_h \quad \text{and} \quad S_h \coloneqq \{q\in\{1,\dots,r\} \colon h_q \in S\}$$

For any type h, define the non-null coalition γ of E with rational shares by $\gamma_h \coloneqq \frac{t_h}{r} \in [0,1]$, and vectors $z_{(h)} = (z_{hi})_{i\in N} \in \mathbb{R}^{l\cdot n}_+$ with $z_{hi} \coloneqq \frac{1}{t_h}\left(\sum_{q\in S_h} x'_{h_q i_q}\right)_{i\in N} \in \mathbb{R}^l_+$ if $h \in \operatorname{supp}\gamma$ and $z_{hi} = 0$ otherwise. Notice that,

$$\begin{aligned}
\sum_{i\in N}\sum_{h\in D_i(\gamma)} \gamma_h z_{hi} &= \frac{1}{r}\sum_{i\in N}\sum_{h\in D_i(\gamma)}\sum_{q\in S_h} x'_{h_q i_q} = \frac{1}{r}\sum_{i\in N}\sum_{q=1}^{r}\sum_{h_q\in D^r_{i_q}(\gamma)} \hat{x}_{h_q i_q} = \\
&= \frac{1}{r}\sum_{i_q\in N(r)}\sum_{h_q\in D^r_{i_q}(S)} x'_{h_q i_q} \le \frac{1}{r}\sum_{i_q\in N(r)} \operatorname{card} D^r_{i_q}(S)\cdot\omega_{i_q} = \\
&= \frac{1}{r}\sum_{i\in N}\sum_{q=1}^{r} \operatorname{card} D^r_{i_q}(S)\cdot\omega_i = \frac{1}{r}\sum_{i\in N}\sum_{h\in N} t_h\cdot\hat{\omega}_{hi} = \\
&= \sum_{i\in N}\sum_{h\in N}\gamma_h\hat{\omega}_{hi} = \sum_{i\in N}\sum_{h\in D_i(\gamma)}\gamma_h\hat{\omega}_{hi} = \sum_{i\in N}\gamma^A_i\omega_i
\end{aligned}$$

where,

$$\hat{x}_{h_q i_q} := \begin{cases} x'_{i_q} & \text{if } q \in S_h \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{\omega}_{h_q i_q} := \begin{cases} \omega_i & \text{if } h \in D_i(S) \\ 0 & \text{otherwise} \end{cases}.$$

Consider now a type $i \in \operatorname{supp}\gamma$. By $x'_{(i_q)} \in P_{A^r_{i_q}}(x^{(r)})$ for any $q \in S_i$, one gets $z_{(h)} \in P_{A_i}(x)$ by the convexity of the preferences and the fact that $z_{(h)}$ is a linear convex combination of $(x'_{(iq)})_{q\in S_h}$. Thus a contradiction is obtained since $x \in \mathscr{C}^A_{rat}(E)$.

Conversely, let $x \in \mathscr{C}^E(E)$ and suppose by contradiction that $x \notin \mathscr{C}^A_{rat}(E)$. Then there exist an Aubin coalition $\gamma \in ([0,1] \cap Q)^n$, and vectors $x'_{(i)}$, with $i \in N$ such that

1. $x'_{(i)} \in P_{A_i}(x)$ for any $i \in \operatorname{supp}\gamma$;
2. $\sum_{i\in N} \sum_{h\in D_i(\gamma)} \gamma_h x'_{hi} \leq \sum_{i\in N} \sum_{h\in D_i(\gamma)} \gamma_h \omega_i.$

Let $\nu \in \mathbb{N}$ and for any $i \in N$, denote by ν_i the image of the ceiling function evaluated at $\gamma_i\nu$, i.e., $\nu_i := \lceil \gamma_i \nu \rceil$, which is a natural number since $\nu \in \mathbb{N}$ and $\gamma_i > 0$ [20].

For any $i \in \operatorname{supp}\gamma$, consider the sequence $(\varepsilon^n)_{n\in\mathbb{N}} := (\frac{\gamma_i n}{n_i})_{n\in\mathbb{N}}$. Notice that: (1) for each $n \in \mathbb{N}$ one has $0 < \varepsilon^n \leq 1$; (2) up to a subsequence, ε^n converges to 1.

Then, by continuity of preferences, there exists $r \in \mathbb{N}$ such $(z_{(h)}) := \frac{\gamma_i r}{n_i} x'_{(i)} \in P_{A_i}(x)$ for each $i \in \operatorname{supp}\gamma$ and

$$\sum_{i\in N} \sum_{h\in D_i(\gamma)} r_h \cdot z_{hi} \leq \sum_{i\in N} \sum_{h\in D_i(\gamma)} r_h \cdot \omega_i.$$

Define the coalition S of E^r by $S := \cup_{h\in \operatorname{supp}(\gamma)} \{(h,q) : q = 1,\ldots,r_h\}$ and notice that

$$\sum_{i_q\in N(r)} \sum_{h_s\in D^r_{i_q}(S)} z_{h_s i_q} = \sum_{i\in N} \sum_{q\in\{1,\ldots r\}} \sum_{h_q\in D_{i_q}(S)} z_{h_q i_q} =$$
$$= \sum_{i\in N} \sum_{h\in D_i(S)} \sum_{q\in S_h} z_{h_q i_q} = \sum_{i\in N} \sum_{h\in D_i(\gamma)} r_h \cdot z_{hi} \leq$$

[20] We remind that the ceiling function $\lceil\ \rceil : y \in \mathbb{R} \mapsto \lceil y \rceil \in \mathbb{R}$ is define by $\lceil y \rceil := \min\{n' \in \mathbb{Z} \colon n' \geq y\}$.

$$\leq \sum_{i\in N}\sum_{h\in D_i(\gamma)} r_h \cdot \omega_i \leq \sum_{i_q\in N(r)} \operatorname{card} D^r_{i_q}(S)\cdot \omega_{i_q}$$

which contradicts $x \in \mathscr{C}^E(E)$. The contradiction proves our claim.

Since $\mathscr{C}^A(E) \subseteq \mathscr{C}^A_{rat}(E) = \mathscr{C}^E(E)$, to conclude the proof of the Theorem it is enough to show that $\mathscr{C}^A_{rat}(E) \subseteq \mathscr{C}^A(E)$. Let x be an allocation of $\mathscr{C}^A_{rat}(E)$ and assume that $x \notin \mathscr{C}^A(E)$. Then there exists a coalition $\alpha \in [0,1]^n$ with $\alpha \neq 0$, and for every agent $i \in N$, there exists a vector $x'_{(i)} \in X_{A_i}$ such that

1. $x'_{(i)} \in P_{A_i}(x)$ for any $i \in \operatorname{supp}\alpha$;
2. $\sum_{i\in N}\sum_{h\in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i(\alpha)} \alpha_h \omega_i$.

By continuity, we can find a positive ε such that for each $\lambda \in (1-\varepsilon, 1)$

$$\lambda x'_{(i)} + (1-\lambda)\omega \in P_{A_i}(x) \text{ for any } i \in \operatorname{supp}\alpha.$$

Let $\gamma_i = 0$ for each $i \notin \operatorname{supp}\alpha$ and γ_i be a rational number such that $\frac{\alpha_i}{\gamma_i} \in (1-\varepsilon, 1)$ for $i \in \operatorname{supp}\alpha$. Then, for any agent $i \in N$, $D_i(\alpha) = D_i(\gamma)$ and moreover, from

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)} \gamma_h \frac{\alpha_h}{\gamma_h} x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i(\alpha)} \gamma_h \frac{\alpha_h}{\gamma_h}\omega_i$$

we derive

$$\sum_{i\in N}\sum_{h\in D_i(\gamma)} \gamma_h \left[\frac{\alpha_h}{\gamma_h} x'_{hi} + (1-\frac{\alpha_h}{\gamma_h})\omega_i\right] - \sum_{i\in N}\sum_{h\in D_i(\gamma)} \gamma_h \omega_i \leq 0$$

which implies a contradiction. ■

The first characterization of A-equilibria follows now from Theorem 14 and (Vasil'ev, 2016, Thm 3.1).

Theorem 15 (Equivalence Theorem) *Let E be an A-economy. Assume that Assumption 2, conditions (1)-(3) of Assumption 3 and conditions (1)-(2) of Assumption 4 are satisfied. Then*

$$\mathscr{W}(E) = \mathscr{C}^A(E) = \mathscr{C}^A_{rat}(E) = \mathscr{C}^E(E).$$

Remark 16 As showed by Graziano et al. (2023), the convexity requirement given in Point 3 of Assumption 3 is not necessary for the validity of the equivalence theorem, if one follows the idea of Husseinov (1994), and allows agents to participate in more than one coalition simultaneously. However, we can not dispense with convexity to show that A-equilibria are Edgeworth equilibria. We also notice that for the validity of Theorem 15, Assumption 2 can be weakened requiring that the aggregate initial endowment is strictly positive.

5 Further Characterizations of A-equilibria

We are going to show in this section that A-equilibria can be characterized in terms of the veto power of the grand coalition. This will be done under two different approaches. In Section 5.1, we will see that A-equilibria are exactly those allocations that cannot be blocked by Aubin coalitions with full support. Then, in Section 5.2, it will be proved that A-equilibrium allocations coincide with robustly efficient allocations, that is they are robust with respect to the veto power exercised in a family of auxiliary economies in which agent i proposes for the members of her reference set A_i an initial endowment modified in a precise direction. The characterizations in terms of the blocking power of the grand coalition is key to show in Section 5.3 that A-equilibrium allocations are Nash equilibria of a suitable game.

5.1 A-equilibria and coalitions with full support

In this section we focus on the veto power of the grand coalition. It will be exercised allowing all agents to take part in Aubin coalitions with a (non-zero) share of their endowments. This means that we shall concentrate on the set $\mathscr{N}^A(E)$. We have already noticed that $\mathscr{C}^A(E) \subseteq \mathscr{N}^A(E)$. Our aim is to find conditions under which the equality holds true. Then, we will derive a characterization of A-equilibria in terms of the blocking power of the grand coalition.

Lemma 17 *Let E be an A-economy satisfying Assumption 2 and conditions (1) and (2) of Assumption 3. Let $x \in \mathscr{F}$ be an allocation such that $x \notin \mathscr{C}^A(E)$. Then there exists a coalition $\alpha \in [0,1]^n$ with $\alpha \neq 0$ and there exists a vector $z_{(i)} \in X_{A_i}$ for every agent $i \in N$ such that*

1. $z'_{(i)} \in P_{A_i}(x)$ *for any* $i \in \operatorname{supp} \alpha$;

2. $\displaystyle\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h z'_{hi} \ll \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h \omega_i.$

Proof. From the fact that $x \notin \mathscr{C}^A(E)$, it follows that there exist a coalition $\alpha \in [0,1]^n$ with $\alpha \neq 0$ and a vector $x'_{(i)} \in X_{A_i}$ for every agent $i \in N$, such that

1. $x'_{(i)} \in P_{A_i}(x)$ for any $i \in \operatorname{supp}(\alpha)$;

2. $\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h \omega_i$.

By continuity of preferences, there exists $\varepsilon > 0$ such that $z_{(i)} \coloneqq \left[\varepsilon x'_{(ih)} + (1-\varepsilon)\omega_h\right]_{h \in A_i} \in P_{A_i}(x)$, for each $i \in \operatorname{supp}\gamma$. Moreover,

$$\sum_{i \in N} \sum_{h \in D_i(\alpha)} \frac{\alpha_h}{\varepsilon} z_{hi} = \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h x'_{hi} + \sum_{i \in N} \sum_{h \in D_i(\alpha)} \frac{\alpha_h}{\varepsilon}\omega_i - \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h \omega_i \leq \sum_{i \in N} \sum_{h \in D_i(\alpha)} \frac{\alpha_h}{\varepsilon}\omega_i.$$

By survival assumption, $z_{(i)} \gg 0$. Then, given a positive δ such that $(\delta \cdot z_{(i)}, x_{N \setminus A_i}) \in P_{A_i}(x)$, for each $i \in \operatorname{supp}\gamma$, we obtain that

$$\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h \delta z_{hi} \ll \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h z_{hi} \leq \sum_{h \in D_i(\alpha)} \alpha_h \omega_i$$

and the conclusion follows with $z'_{(i)} = \delta z_{(i)}$. ■

Theorem 18 *Let E be an A-economy. Assume that Assumption 2 and conditions (1) and (2) of Assumption 3 are satisfied. Then*

$$\mathscr{C}^A(E) = \mathscr{N}^A(E).$$

Proof. By definition, $\mathscr{C}^A(E) \subseteq \mathscr{N}^A(E)$. Let $x \in \mathscr{N}^A(E)$ and assume by contradiction that $x \notin \mathscr{C}^A(E)$. Then by Lemma 17, there exists a coalition $\alpha \in [0,1]^n$ with $\alpha \neq 0$ and there exists a vector $z_{(i)} \in X_{A_i}$ for every agent $i \in N$ such that

1. $z'_{(i)} \in P_{A_i}(x)$ for any $i \in \operatorname{supp}\alpha$;

2. $\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h z'_{hi} \ll \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h \omega_i$.

Let $w \coloneqq \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h z'_{hi} - \sum_{i \in N} \alpha_i^A \omega_i \ll 0$ and λ a positive number such that $w + \sum_{i \in N} \sum_{h \in D_i^c(\alpha)} \lambda z'_{hi} \leq 0$, where $D_i^c(\alpha) = D_i \cap (N \setminus \operatorname{supp}\alpha)$. Define now $z_{(i)} \coloneqq z'_{(i)}$ for $i \in \operatorname{supp}\alpha$ and $z_{(i)} \coloneqq [x_j + \omega_j]_{j \in A_i}$, for each agent $i \in N \setminus \operatorname{supp}\alpha$. By monotonicity, $z_{(i)} \in P_{A_i}(x)$ for any $i \in N$. Moreover, the Aubin coalition β defined by $\beta_i \coloneqq \alpha_i$, for each $i \in \operatorname{supp}\alpha$ and $\beta_i \coloneqq \lambda$, for each $i \in N \setminus \operatorname{supp}\alpha$ has full support and it is true that

$$\sum_{i\in N}\sum_{h\in D_i}\beta_h z_{hi} = \sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h z'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)}\lambda(x_{hi}+\omega_i) =$$

$$= \sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h z'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)}\lambda x_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)}\lambda\omega_i =$$

$$= w + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)}\lambda x_{hi} + \sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h\omega_i + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)}\lambda\omega_i \leq \sum_{i\in N}\sum_{h\in D_i}\beta_h\omega_i$$

and a contradiction follows. ■

As direct consequence of Theorem 18 and (Vasil'ev, 2016, Thm 3.1), we obtain a second characterization of A-equilibria in terms of allocations which cannot be dominated by the society using the Aubin veto, i.e. by Aubin coalitions with full support.

Theorem 19 *Let E be an A-economy. Assume that Assumption 2, conditions (1) and (2) of Assumption 3 and conditions (1) and (2) of Assumption 4 are satisfied. Then*

$$\mathscr{W}(E) = \mathscr{C}^A(E) = \mathscr{N}^A(E).$$

5.2 A-equilibria and robust efficiency

The main aim in this Section is to show that A-equilibria coincide with *robustly efficient allocations*. This characterization has been introduced for competitive equilibria in Hervés-Beloso and Moreno-García (2008). There the proof relies on the use of Vind theorem about the measure of blocking coalitions in suitable atomless economies (see Vind (1972)), that in its turn applies the Lyapunov convexity theorem. To extend the result to economies with consumption externalities, we proceed first proving that Aubin core allocations are robustly efficient. Then we use the characterization of A-equilibria in terms of Aubin core allocations. Notice that we provide a direct proof of the equivalence in which we can dispense with the use of Lyapunov convexity theorem and Vind result. In particular, our proof does not use atomless economies, whose definition in the presence of consumption externalities is not immediate.

To define robustly efficient allocations, we observe that in the Aubin blocking procedure introduced with Definition 11, each agent $h \in N$ enters the market using $\alpha_h\omega_i$, i.e. by aggregating the share α_h of agent i initial endowment ω_i, for each $i \in A_h$. Given an allocation $x \in \mathscr{F}$ and the Aubin coalition α, consider the blocking mechanism in which each agent $h \in N$ proposes instead a slight perturbation of ω_i, for each member $i \in A_h$. This perturbation is defined for each $i \in A_h$ as the convex combination with share α_h of agent i initial endowment ω_i and the commodity bundle x_i allocated to i under x. Precisely, it is defined as the aggregated quantity of resources given by

$$\omega_{(\alpha,h)} := [\alpha_h \omega_i + (1-\alpha_h)x_i]_{i\in A_h}.$$

The blocking procedure is formalized as follows and gives rise to a notion of *robustly efficient* allocations which is inspired by the analogous notion given by Hervés-Beloso and Moreno-García (2008) in selfish models.

Definition 20 *An allocation $x \in \mathscr{F}$ is said to be robustly efficient if there do not exist vectors $x'_{(i)} \in X_{A_i}$, $i \in N$, such that*

1. $x'_{(i)} \in P_{A_i}(x)$ *for any* $i \in N$;
2. $\sum_{i\in N}\sum_{h\in D_i} x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i} [\alpha_h\omega_i + (1-\alpha_h)x_i]$.

The set of robustly efficient allocations will be denoted by $\mathscr{RE}(E)$.

Remark 21 The set of allocations which cannot be improved upon by the grand coalition N has been denoted by $\mathscr{N}(E)$. For an allocation $x \in \mathscr{N}(E)$ there do not exist vectors $x'_{(i)} \in X_{A_i}$, $i \in N$, such that

1. $x'_{(i)} \in P_{A_i}(x)$ for any $i \in N$;
2. $\sum_{i\in N}\sum_{h\in D_i} x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i} \omega_i$

where each agent $h \in D_i$ uses the resources ω_i of agent i. Consequently, an allocation is robustly efficient if and only if it is not dominated by the society in the families of A-economies in which each agent $h \in D_i$ proposes for i the perturbation of ω_i given by $\alpha_h\omega_i + (1-\alpha_h)x_i$.

Theorem 22 *Let E be an A-economy. Assume that Assumption 2, conditions (1)-(3) of Assumption 3 and conditions (1) and (3) of Assumption 4 are satisfied. Then*

$$\mathscr{C}^A(E) = \mathscr{RE}(E).$$

Proof. Let $x \in \mathscr{C}^A(E)$ and assume by contradiction that $x \notin \mathscr{RE}(E)$. Then the exist vectors $x'_{(i)} \in X_{A_i}$, $i \in N$, such that

1. $x'_{(i)} \in P_{A_i}(x)$ for any $i \in N$;
2. $\sum_{i\in N}\sum_{h\in D_i} x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i} [\alpha_h\omega_i + (1-\alpha_h)x_i]$.

Denote by $D_i^c(\alpha)$ the set $D_i \setminus D_i(\alpha)$. From condition 2. it follows that

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)} x'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(\alpha)} x'_{hi} \leq$$

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h\omega_i+\sum_{i\in N}\sum_{h\in D_i}x_i-\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h x_i\leq$$
$$\leq\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h\omega_i+\sum_{i\in N}\sum_{h\in D_i}\omega_i-\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h x_i$$

and consequently

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)}x'_{hi}+\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h x_i+\sum_{i\in N}\sum_{h\in D_i^c(\alpha)}x'_{hi}\leq\sum_{i\in N}\sum_{h\in D_i(\alpha)}(1+\alpha_h)\,\omega_i+\sum_{h\in D_i^c(\alpha)}\omega_i.$$

The last inequality implies that

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)}(1+\alpha_h)\left[\frac{1}{1+\alpha_h}x'_{hi}+\frac{\alpha_h}{1+\alpha_h}x_i\right]+\sum_{i\in N}\sum_{h\in D_i^c(\alpha)}x'_{hi}\leq$$
$$\leq\sum_{i\in N}\sum_{h\in D_i(\alpha)}(1+\alpha_h)\,\omega_i+\sum_{h\in D_i^c(\alpha)}\omega_i$$

inequality which in its turn is equivalent to

$$\sum_{i\in N}\sum_{h\in D_i}\beta_h z_{hi}\leq\sum_{i\in N}\sum_{h\in D_i}\beta_h\omega_i$$

where

$$\beta_h=\begin{cases}1+\alpha_h, & h\in\operatorname{supp}\alpha\\ 1 & h\in N\setminus\operatorname{supp}\alpha\end{cases}$$

and, for any $h\in N$ and $i\in A_h$,

$$z_{hi}=\begin{cases}\frac{1}{1+\alpha_h}x'_{hi}+\frac{\alpha_h}{1+\alpha_h}x_i, & h\in\operatorname{supp}\alpha\\ x'_{hi} & h\in N\setminus\operatorname{supp}\alpha\end{cases}.$$

The convexity of preferences and the fact that β is an Aubin coalition with $D_i=D_i(\beta)$ for each $i\in N$, imply now a contradiction.

To show the other inclusion, assume that $x\in\mathscr{RE}(E)$ and, by contradiction, that $x\notin\mathscr{C}^A(E)$. Then there exists a vector $x'_{(i)}\in X_{A_i}$ such that

1. $x'_{(i)}\in P_{A_i}(x)$ for any $i\in\operatorname{supp}\alpha$;

2. $\sum_{i\in N}\sum_{h\in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i\in N}\sum_{h\in D_i(\alpha)} \alpha_h \omega_i$.

By Theorem 18 we can assume that $\operatorname{supp}\alpha = N$ and $D_i(\alpha) = D_i$. Let $\varepsilon \in (0,1)$ be such that $\varepsilon x'_{(i)} \in P_{A_i}(x)$, for any $i \in N$. Then, from condition 2. we derive that

$$\sum_{i\in N}\sum_{h\in D_i} \alpha_h \varepsilon x'_{hi} + \sum_{i\in N}\sum_{h\in D_i} \alpha_h(1-\varepsilon)x'_{hi} + \sum_{i\in N}\sum_{h\in D_i} \alpha_h(1-\alpha_h)x_i \leq$$
$$\sum_{i\in N}\sum_{h\in D_i} \alpha_h\omega_i + \sum_{i\in N}\sum_{h\in D_i} \alpha_h(1-\alpha_h)x_i.$$

Define now $S \coloneqq \{h \in N : \alpha_h \neq 1\}$, $D_i(S) = D_i \cap S$ and $D_i^c(S) = D_i \setminus D_i(S)$. From the previous inequality we derive that

$$\sum_{i\in N}\sum_{h\in D_i(S)} \alpha_h \varepsilon x'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(S)} \varepsilon x'_{hi} +$$
$$+\sum_{i\in N}\sum_{h\in D_i(S)} \alpha_h(1-\varepsilon)x'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(S)} (1-\varepsilon)x'_{hi} + \sum_{i\in N}\sum_{h\in D_i(S)} (1-\alpha_h)x_i \leq$$
$$\leq \sum_{i\in N}\sum_{h\in D_i(S)} [\alpha_h\omega_i + (1-\alpha_h)x_i] + \sum_{i\in N}\sum_{h\in D_i^c(S)} \omega_i$$

and consequently

$$\sum_{i\in N}\sum_{h\in D_i(S)} \alpha_h \varepsilon x'_{hi} + \sum_{i\in N}\sum_{h\in D_i^c(S)} x'_{hi} +$$
$$+\sum_{i\in N}\sum_{h\in D_i(S)} (1-\alpha_h)\left[x_i + \frac{\alpha_h(1-\varepsilon)}{1-\alpha_h}x'_{hi}\right] \leq$$
$$\leq \sum_{i\in N}\sum_{h\in D_i(S)} [\alpha_h\omega_i + (1-\alpha_h)x_i] + \sum_{i\in N}\sum_{h\in D_i^c(S)} \omega_i$$

which can be written as

$$\sum_{i\in N}\sum_{h\in D_i(S)} \left[\alpha_h \varepsilon x'_{hi} + (1-\alpha_h)(x_i + \frac{\alpha_h(1-\varepsilon)}{1-\alpha_h}x'_{hi})\right] + \sum_{i\in N}\sum_{h\in D_i^c(S)} x'_{hi} \leq$$
$$\leq \sum_{i\in N}\sum_{h\in D_i(S)} [\alpha_h\omega_i + (1-\alpha_h)x_i] + \sum_{i\in N}\sum_{h\in D_i^c(S)} \omega_i.$$

Consider the consumption bundles $z_{(h)} := \left[\alpha_h \varepsilon x'_{hi} + (1-\alpha_h)(x_i + \frac{\alpha_h(1-\varepsilon)}{1-\alpha_h} x'_{hi})\right]_{i \in A_h}$, for any $h \in S$ and $z_{(h)} := (x'_{(h)})$, otherwise. By monotonicity and convexity, $z_{(i)} \in P_{A_i}(x)$ for any $i \in N$ and from the previous inequalities we obtain

$$\sum_{i \in N} \sum_{h \in D_i} \alpha_h z_{hi} \leq \sum_{i \in N} \sum_{h \in D_i} [\alpha_h \omega_i + (1-\alpha_h) x_i]$$

which implies a contradiction. ∎

As direct consequence of Theorem 22 and (Vasil'ev, 2016, Thm 3.1), we obtain a third characterization of A-equilibria.

Theorem 23 *Let E be an A-economy. Assume that Assumption 2, conditions (1)-(3) of Assumption 3 and conditions (1)-(3) of Assumption 4 are satisfied. Then*

$$\mathscr{W}(E) = \mathscr{R}\mathscr{E}(E).$$

In particular, the characterization in terms of robust efficiency extends to the Walrasian-Nash, Berge and total equilibria of economies with consumption externalities the corresponding result proved by Hervés-Beloso and Moreno-García (2008) for selfish pure exchange economies.

Remark 24 Notice that from the characterization of competitive equilibria of selfish economies, Hervés-Beloso and Moreno-García (2008) derive the two welfare theorems (see Remark 3.2 in their paper). It is well known that, in the presence of externalities, the two welfare theorems do not hold. However, our previous characterization and Remark 21 imply the two welfare theorems formulated in terms of the A-dominance. Hence, each A-equilibrium allocation x is Pareto optimal in the sense that it not blocked by the society N in the following sense: it is not possible to find the vectors $x'_{(i)} \in X_{A_i}$ for $i \in N$ such that

1. $x'_{(i)} \in P_{A_i}(x)$ for each $i \in N$;
2. $\sum_{i \in N} \sum_{h \in D_i} x'_{hi} \leq \sum_{i \in N} \sum_{h \in D_i} \omega_i$,

where D_i represents the set of agents influencing i. On the other hand, under the assumptions of Theorem 23, in particular due to point 3. of Assumption 4 and the feasibility of x, if the allocation x is Pareto optimal in the previous sense, then x is also a Pareto optimal allocation in the A-economy E_x in which the initial endowment allocation is x. Note that in this case, all the auxiliary economies described in Remark 21 and associated to E_x are equal to E_x. Hence we can apply Theorem 23 to the economy E_x to obtain that an

A-equilibrium allocation is Pareto optimal in the economy E_x, i.e. the second welfare theorem formulated for the A-dominance.

5.3 A-equilibria and Nash equilibria of a society game

Our aim in this section is to show that the A-equilibria of a pure exchange economy with consumption externalities coincide with Nash equilibria of a two-player game. This result, which extends to economies with consumption externalities the analogous characterization introduced by Hervés-Beloso and Moreno-García (2009), provides a strategic interpretation of Walras-Nash, Berge and total equilibria. The characterization is a consequence of Theorem 18 and requires the introduction of a suitable game which is adapted to our framework of A-economies from Hervés-Beloso and Moreno-García (2009).

We notice that, contrary to usual strategic interpretation of Walrasian equilibria, the approach proposed by Hervés-Beloso and Moreno-García (2009) does not consider money and prices. The game associated to the market economy is played by two players regardless of the number of agents. It is referred as a *society game* because it is a game in which the society plays in two different roles. In our proof of the result, preferences of each agent i will be separable with respect to the consumption of her reference set A_i.

For the remainder of the Section, we shall assume that condition (3) of Assumption 4 is satisfied.

In the remainder of the section we adopt the notation $x_A \coloneqq (x_{(i)})_{i\in N}$, where $x_{(i)} = (x_{ih})_{h\in A_i} \in X_{A_i}$, and x_{ih} can be interpreted as the commodity bundle chosen (or proposed) by agent i for the consumption of an agent $h \in A_i$. Clearly, x_A belongs to $\Pi_{i\in N} X_{A_i}$. In particular, given an allocation $x \equiv (x_1, \dots, x_n) \in \mathscr{F}$, and an agent $i \in N$, define the vector $\widetilde{x}_{(i)} \coloneqq (x_h)_{h\in A_i}$ in which i proposes to each $h \in A_i$ the same commodity bundle that h receives under x, i.e. $x_{ih} = x_h$ for each $i \in N$ and $h \in A_i$, and by $\widetilde{x} \coloneqq (\widetilde{x}_{(i)})_{i\in N}$. Notice that under the symmetric condition (3) contained in Assumption 4, and from the feasibility of x it follows that

$$\sum_{i\in N}\sum_{h\in D_i} \widetilde{x}_{hi} = \sum_{i\in N} |D_i|\, x_i \leq \sum_{i\in N} |D_i|\, \omega_i = \sum_{i\in N}\sum_{h\in D_i} \omega_i$$

In the spirit of Hervés-Beloso and Moreno-García (2009), our aim is to characterize an A-equilibrium in terms of a two player non cooperative game. Hence we assume that there are two players, $I \coloneqq \{a, b\}$ which can be identified as the society playing two different roles.

With notation introduced above, the strategy set of players a is given by

$$S_a := \left\{ x_A \in \Pi_{i\in N} X_{A_i} |\ x_{ih} \neq 0\ \forall\, i \in N,\, h \in A_i \text{ and } : \sum_{i\in N}\sum_{h\in D_i} x_{hi} \leq \sum_{i\in N}\sum_{h\in D_i} \omega_i \right\}.$$

Notice that $S_a \neq \emptyset$ since $\widetilde{\omega} = (\widetilde{\omega}_{(i)})_{i\in N} \in S_a$.

The strategy set of players b is given by

$$S_b := \left\{ (\alpha, y_A) \in (0,1]^n \times \Pi_{i\in N} X_{A_i} : \sum_{i\in N}\sum_{h\in D_i} \alpha_h y_{hi} \leq \sum_{i\in N}\sum_{h\in D_i} \alpha_h \omega_i \right\}.$$

Notice that $S_b \neq \emptyset$ since $s_b := (\chi_N, \widetilde{\omega}) \in S_b$, where the vector $\chi_N \in \mathbb{R}^n$ is the characteristic function of N.

Given a profile of strategies $s \in S$, where $s := (s_a, s_b) = (x_A, \alpha, y_A)$ and $S := S_a \times S_b$, denote the payoff functions of the two players by $\pi_a := S \longrightarrow \mathbb{R}$ and $\pi_b := S \longrightarrow \mathbb{R}$ and let (G, π_a, π_b) the associated game.

Definition 25 *(G, π_a, π_b) is a society game if the following conditions are satisfied.*

1. *$x_A = y_A$ implies that $\pi_a(s) = \pi_b(s) = 0$;*
2. *$(x_{A_i}, y_{N\setminus A_i}) \in P_i(y_{A_i}, y_{N\setminus A_i})$ for each $i \in N$ if and only if $\pi_a(s) > 0$;*
3. *$(y_{A_i}, x_{N\setminus A_i}) \in P_i(x_{A_i}, x_{N\setminus A_i})$ for each $i \in N$ if and only if $\pi_b(s) > 0$;*
4. *$s' = (z_A, \alpha, y_A) \in S$ and $(z_{A_i}, y_{N\setminus A_i}) \in P_i(x_{A_i}, y_{N\setminus A_i})$ for each $i \in N$ implies that $\pi_a(s') > \pi_a(s)$.*

Theorem 26 *Let E be an A-economy. Assume that Assumption 2, Assumption 3 and Assumption 4 are satisfied. Then a society game (G, π_a, π_b) satisfies the following properties:*

1. *For each $s \in S$, $\pi_a(s)$ and $\pi_b(s)$ cannot be both positive;*
2. *if $x \in \mathscr{F}$ and $s = (\widetilde{x}, \alpha, y_A)$ is a Nash equilibrium, then x is not dominated by the grand coalition;*
3. *if $s = (\widetilde{x}, \alpha, \widetilde{x}) \in S$ and $x \in \mathscr{F}$ is not dominated by the grand coalition, then player a cannot improve her payoff;*
4. *if $x \in \mathscr{F}$ and $\widetilde{x} \in S_a$, then $x \notin C^A(E)$ if and only if $\pi_b(\widetilde{x}, \alpha, y_A) > 0$ for a strategy $(\alpha, y_A) \in S_b$.*

Proof. The statement of property 1. follows from conditions 2. and 3. of Definition 25 and the separability assumptions on preferences.

To prove property 2., let x be an allocation such that $s = (\widetilde{x}, \alpha, y_A)$ is a Nash equilibrium and assume by contradiction that x is dominated by the grand coalition. Then there exist vectors $z_{(i)} \in X_{A_i}$, $i \in N$, such that

1. $z_{(i)} \in P_{A_i}(x)$ for any $i \in N$;

2. $\sum_{i \in N} \sum_{h \in D_i} z_{hi} \leq \sum_{i \in N} \sum_{h \in D_i} \omega_i$.

Let $z_A := (z_{(i)})_{i \in N}$ be the vector which collects consumption proposal of agent i for A_i, for each $i \in N$. It follows from 2. that $s' = (z_A, \alpha, y_A)$ belongs to S. Moreover, by 1. and separability of preferences, $\pi_a(s') > \pi_a(s)$ which is impossible.

Consider now the strategy $s = (\widetilde{x}, \alpha, \widetilde{x}) \in S$ such that x is an allocation which cannot be dominated by the grand coalition. First observe that $\pi_a(s) = 0$ by definition of society game. Assume, by contradiction, that for a given $z_A \in S_a$ it holds true that $\pi(z_A, \alpha, \widetilde{x}) > 0$. Set $z_{(i)} := z_{A_i}$ for each $i \in N$ and observe that

1. $z_{(i)} \in P_{A_i}(x)$ for any $i \in N$;

2. $\sum_{i \in N} \sum_{h \in D_i} z_{hi} \leq \sum_{i \in N} \sum_{h \in D_i} \omega_i$

which implies a contradiction.

In order to show property 4., assume first that $x \notin C^A(E)$. Then, by Theorem 19, there exists α with full support and for every agent $i \in N$, there exists a vector $y_{(i)} \in X_{A_i}$ such that

1. $y_{(i)} \in P_{A_i}(x)$ for any $i \in N$;

2. $\sum_{i \in N} \sum_{h \in D_i} \alpha_h y_{hi} \leq \sum_{i \in N} \sum_{h \in D_i} \alpha_h \omega_i$.

Set $z_{A_i} := z_{(i)}$ for each $i \in N$. Then condition 2. says that $(\alpha, y_A) \in S_b$ and condition 1. ensures that $\pi_b(\widetilde{x}, \alpha, y_A) > 0$, and a contradiction. Assume conversely that for a given allocation x it is true that $\pi_b(\widetilde{x}, \alpha, y_A) > 0$ for a strategy $(\alpha, y_A) \in S_b$. By definition of society game, it is also true that $y_{(i)} = y_{A_i} \in P_{A_i}(x)$ for any $i \in N$ and consequently $x \notin C^A(E)$. ■

Theorem 27 *Let E be an A-economy. Assume that Assumption 2, Assumption 3 and Assumption 4 are satisfied. If the economy E has an A-equilibrium, then the society game (G, π_a, π_b) a Nash equilibrium.*

Proof. It is enough to observe that if x is an A-equilibrium allocation, then $s = (\widetilde{x}, \chi_N, \widetilde{x})$ belongs to S and Theorem 19 ensures that x is not dominated by the grand coalition. This implies that $\pi_a(s)$ cannot improve. On the other hand, from $x \in C^A(E)$ it follows that $\pi_b(\widetilde{x}, \chi_N, \widetilde{x}) = 0 = \pi_b(\widetilde{x}, \alpha, y_A)$, for each (α, y_A), and consequently also player b cannot improve. ■

The next Theorem provides our last characterization of A-equilibrium allocations. This is given in terms of Nash equilibria of a society game.

Theorem 28 *Let E be an A-economy. Assume that Assumption 2, Assumption 3 and Assumption 4 are satisfied. Let $x \in \mathscr{F}$ be an allocation. The following properties hold true:*

1. *if $s = (\widetilde{x}, \alpha, y_A)$ is a Nash equilibrium with $\pi_a(s) = \pi_b(s) = 0$, then x is an A-equilibrium allocation;*
2. *if x is an A-equilibrium allocation, then $s = (\widetilde{x}, \alpha, \widetilde{x})$ is a Nash equilibrium with $\pi_a(s) = \pi_b(s) = 0$.*

Proof. To show condition 1., assume that the allocation x is not an A-equilibrium. Then by Theorem 15, $x \notin C^A(E)$ and $\pi_b(\widetilde{x}, \beta, z_A) > 0 = \pi_b(\widetilde{x}, \alpha, \widetilde{x})$, which is impossible. To show condition 2., consider an A-equilibrium x for which $s = (\widetilde{x}, \alpha, \widetilde{x})$ is not a Nash equilibrium. Then, by Theorem 26, an improvement of player a contradicts the fact that x cannot be dominated by the grand coalition while an improvement of b implies that x is not in the Aubin core. ■

Example 29 Let E be an A-economy for which the Assumptions of Theorem 28 are satisfied. Then preference relations of player i can be represented by an utility function which is separable with respect to the consumption of her reference set A_i, for each $i \in N$. Define the payoff functions of the two players a and b as

$$\pi_a(x_A, \alpha, y_A) \coloneqq \min\{u_i(x_{A_i}) - u_i(y_{A_i})\colon i \in N\}$$

$$\pi_b(x_A, \alpha, y_A) \coloneqq \min\{\alpha_i[u_i(y_{A_i}) - u_i(x_{A_i})]\colon i \in N\}$$

then it is easy to show that the payoff functions define a society game (G, π_a, π_b) for which Theorem 28 holds true.

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