

Robust IV Inference with Clustering Dependence*

Jianfei Cao

The University of Chicago

Booth School of Business

5807 S. Woodlawn, Chicago, IL 60637, USA

e-mail: jcao0@chicagobooth.edu

November 3, 2020

Abstract: Linear IV models with clustering dependence are widely used in empirical studies, while the common solution, the *cluster covariance estimator*, often produces undesirable inferential results, especially with weak instruments. In this paper I propose a method that is robust to both weak IV and (potentially heterogeneous) clustering dependence. The proposed method is based on the idea of Fama-MacBeth estimation, with group-level estimators being a truncated version of the unbiased IV estimator. Asymptotic validity is shown under both strong and weak IV sequences, as well as under general requirements. Simulation results indicate the method has good finite sample performance in both size and power. The proposed method is applied to study the effect of city compactness on population density.

Key Words: Weak dependence; Weak instruments; Fama-MacBeth method; *t*-test

1. Introduction

In linear IV models, accounting for clustering dependence has been a standard procedure in conducting statistical inference in empirical research. A common solution is to use the cluster covariance estimator (CCE), which is often referred to as the “clustered standard error” method. It has been shown that in linear models CCE methods deliver valid inference under either strong homogeneity across groups (the *large-homogeneous-group* approach, e.g.,

*I am grateful to Christian Hansen, Max Farrell, Tetsuya Kaji, Panos Toulis, Azeem Shaikh, Stéphane Bonhomme, and Alexander Torgovitsky for helpful comments and suggestions.

Bester et al., 2011), or lots of small groups (the *many-small-group* approach, e.g., Hansen and Lee, 2019). Those results provide theoretical justification for the usage of CCE methods in the linear IV model under strong IV.¹ This paper concerns statistically inference in the linear IV model with clustering dependence.

Practical issues exist in CCE. For many common settings, it is not clear that either *large-homogeneous-group* or *many-small-group* approach can justify the usage of standard clustering methods. Specifically, Bester et al. (2011) assumes all groups have the common sizes and design matrix limit which does not hold in many settings. Hansen and Lee (2019) requires that $\max_g n_g^2/n \rightarrow 0$, where n_g is the number of observations in group g and n is the sample size. In the case of equal-sized groups, this implies $n/G^2 \rightarrow 0$, where G is the number of groups. MacKinnon and Webb (2017) conduct simulation studies and show the empirical rejection can be as high as 0.1073 at a level-0.05 test, when $(n, G) = (2000, 50)$, thus with $n/G^2 = 0.8$, and group sizes are proportional to population of the 50 states in the US. A non-exhaustive search in recent empirical research suggests n/G^2 is often large, e.g., Coibion et al. (2017) with $n/G^2 = 0.46$ or 0.34 in Table 3, Dell (2012) with n/G^2 ranges from 1.21 to 16.65 in Table 7, and Deryugina et al. (2019) with $n/G^2 = 2.43$ in Table 2. For discussion on poor asymptotic approximation of inference methods based on asymptotic theory, see Ferman and Pinto (2019), Ferman (2019), MacKinnon and Webb (2017), and Young (2019).

Moreover, standard methods suffer from size distortion when weak IV is a concern. While robust inference methods such as the Anderson-Rubin test (AR, Anderson and Rubin, 1949) work under standard assumptions described in the previous paragraph, it is not well understood whether those methods have good inferential properties when standard assumptions break. In the simulation section we show the extension of AR with standard error calculated by CCE methods can result in size distortion under imbalanced group sizes, with sizes being as high as 0.116 at a level-0.05 test.

Alternatively, Fama-MacBeth methods (Fama and MacBeth, 1973; Ibragimov and Müller, 2010), sometimes referred to as mean group estimation (Pesaran and Smith, 1995; Pesaran et al., 1999), provide another inferential approach that exploits the clustering dependence structure. Those methods first perform group-level estimation for each group, and consider a weighted average of all group-level estimators. Under a wide variety of circumstances, the resulting average has well-understood properties, and simple procedure such as t -test can be

¹Hansen and Lee (2019) covers both OLS and IV, while Hansen (2007) only considers the OLS case but the results can be extended to the strong IV model.

used to attain valid inference. In this paper I introduce a group-based inference method that is built on Fama-MacBeth methods, in order to solve clustering dependence and potentially weak IV simultaneously.

In this paper I study robust inferential methods to overcome the practical issues mentioned above, based on the idea of Fama-MacBeth estimation. Since the Fama-MacBeth approach calculates the group-level estimator using only the data in a certain group, potential finite sample problem may rise in the IV estimation. In order to account for that, I propose a truncated version of the unbiased IV estimator introduced by [Andrews and Armstrong \(2017\)](#) in calculating the group-level estimator. I show that this estimator is nearly unbiased, and that using this estimator in the Fama-MacBeth approach produces valid inference. The proposed method allows for moderate number of moderate-sized groups (e.g., 30 groups of around 30 observations as in the simulation section), and is robust to both weak IV and heterogeneous clustering dependence. Table 1 summarizes whether a certain aforementioned method is robust to a non-conventional set-up.

TABLE 1
Robustness of Inferential Methods in Linear IV Models with Clustering Dependence

	$n/G^2 \gg 0$	Heterogeneous Groups	Weak IV
CCE (large- G)	NO	YES	NO
CCE (small- G)	YES	NO	NO
AR-CCE (large- G)	NO	YES	YES
AR-CCE (small- G)	YES	NO	YES
Fama-MacBeth	YES	YES	NO
Proposed method	YES	YES	YES

Notes: This table roughly summarizes whether a candidate inferential method is robust to a certain non-conventional set-up. “YES” means it generally delivers correct size and “NO” means it does not. “Large- G ” stands for the *many-small-group* approach and “small- G ” stands for the *large-homogeneous-group* one. “AR-CCE” is the natural extension of the Anderson-Rubin method to the case with clustering dependence (described in Section 4.2). “Proposed method” is the Fama-MacBeth approach with truncated unbiased estimators proposed in this paper.

Both an unbiased group-level estimator and the truncation are important in implementing Fama-MacBeth approach in this setting. Without the former, the group-level IV estimator may lead to substantial finite sample bias and cause size distortion under the null. The latter guarantees the group-level estimators have finite second moments such that the test has power. Simulation studies show that direct usage of [Andrews and Armstrong \(2017\)](#) produces far less power, and the proposed method is robust to many settings and has good power properties.

Throughout, I assume there is one endogenous variable and focus on the case of one

instrument. Cases with multiple instruments can be dealt with using the averaging method introduced by [Andrews and Armstrong \(2017\)](#). Similar to [Andrews and Armstrong \(2017\)](#), the sign of the first stage parameter is assumed to be known in order to implement the proposed method. This is often a weak assumption in empirical studies, because the sign of IV is typically embedded in the reasoning of instrument validity, and comes in before the discussion of strength of the instrument. [Mills \(2019\)](#) shows that the first-stage sign is claimed to be known in 82.35% of the papers published in the American Economic Review from 2014 to 2018 and with “instrument” in the abstract. Besides, [Mills \(2019\)](#) shows exploiting information of the first-stage sign may help improve test power. Another underlying assumption I assume throughout is the group-level normal model (see [Section 3.1](#)), for which a sufficient assumption would be weak dependence as in *large-homogeneous-group* approach ([Bester et al., 2011](#)).

The paper contributes to two streams of literature. First, the proposed method fills a gap in the literature on cluster-based inferential methods. Although those methods are extensively studied under standard assumptions such as the *large-homogeneous-group* case and the *many-small-group* case ([Bertrand et al., 2004](#); [Hansen, 2007](#); [Bester et al., 2011](#); [Cameron and Miller, 2015](#); [Hansen and Lee, 2019](#)), the properties of those methods outside the standard assumptions are largely unknown. I show through simulation that existing methods can break under many circumstances. I advocate the usage of the proposed Fama-MacBeth approach with truncated unbiased IV estimation and show its validity.

Secondly, this paper complements the recent literature on the Fama-MacBeth approach and shows its usefulness. This approach was introduced in [Fama and MacBeth \(1973\)](#) but only recently theoretically justified by [Ibragimov and Müller \(2010\)](#). The robustness and good power properties of this approach have been documented by [Ibragimov and Müller \(2010\)](#), [Canay et al. \(2017\)](#), [Cao et al. \(2019\)](#), and [Hagemann \(2019a,b\)](#). Many of their results can be either applied or extended to the strong IV case, but extension to allowing for weak IV is non-trivial.

The remainder of the paper is organized as follows. [Section 2](#) introduces a truncated version of the unbiased IV estimator with known first stage sign. [Section 3](#) proposes the inferential method that applies the truncated unbiased IV estimator. In addition, the primitive conditions for both strong and weak IV asymptotics are listed. Simulation studies are presented in [Section 4](#). In [Section 5](#), I apply the proposed method to study the effect of city compactness on population density. [Section 6](#) concludes. Proofs are relegated to the appendix.

2. Truncated Unbiased IV with Known First-stage Sign

We first consider a simple linear IV model. Let X, Y, Z be $n \times 1$ data vectors for three scalar variables. The reduced-form formulation of the linear IV model is

$$\begin{cases} Y = Z\pi\beta + U, \\ X = Z\pi + V, \end{cases} \quad (2.1)$$

where π and β are both scalars. We are interested in the structural equation parameter β . Assume the sign of π is known, and, without loss of generality, let $\pi > 0$. Assume the vector of reduced-form and first-stage estimators follows

$$\hat{\psi} = \begin{pmatrix} \hat{\gamma} \\ \hat{\pi} \end{pmatrix} = \begin{pmatrix} (Z'Z)^{-1}Z'Y \\ (Z'Z)^{-1}Z'X \end{pmatrix} \sim N(\mu, \Sigma), \quad (2.2)$$

where

$$\mu = \begin{pmatrix} \pi\beta \\ \pi \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

The usual IV estimator is $\hat{\beta}_{IV} = \hat{\gamma}/\hat{\pi}$. We assume through out that Σ is known and positive definite. Our analysis rely heavily on (2.2), which applies to cases where normality is a good approximation of the reduced-form and first-stage coefficients $\hat{\psi}$.

Comment 2.1. The model (2.1) has an equivalent structural formulation. Generalization of (2.1) to models with multiple instruments and/or other control variables can be done in standard methods. For multiple instruments, we can use a weighted average of the proposed estimators of single instrument, since a weighted average of (nearly) unbiased estimators is still (nearly) unbiased. Including other control variables can be done through projection on the null space by the Frisch-Waugh-Lovell theorem.

Comment 2.2. The assumption that π has known sign is often weak in empirical studies. According to a survey by Mills (2019) on papers published in the American Economic Review from 2014 to 2018, 14 out of 17 papers with “instrument” in the abstract claim to have known first-stage sign.

Comment 2.3. The normal model (2.2) is common in the literature on IV inference that is robust to weak instruments (see for example Andrews et al., 2006; Andrews and Mikusheva, 2016; Kleibergen, 2002; Moreira, 2003; Moreira and Moreira, 2019; Staiger and Stock, 1997). One motivation is that the vector $(\pi\beta, \beta)$ can be considered as a regular parameter and well-estimated under mild regularity conditions, while β itself is only weakly regular in the case

of weak instruments (Kaji, 2020). As a result, the least square estimator for $(\pi\beta, \beta)$ can often be approximated by a normal distribution. One simple example for the model (2.2) to hold is the case where Z is fixed and the rows of $[U, V]$ are i.i.d. or stationary. Then in this case the covariance matrix of $\widehat{\psi}$ is

$$\Sigma = (I_2 \otimes (Z'Z)^{-1}Z')\text{Var}[(U', V')'](I_2 \otimes (Z'Z)^{-1}Z')' \quad (2.3)$$

and can be consistently estimated. See Andrews et al. (2019) for a review on the normal approximation to the distribution of $(\widehat{\gamma}, \widehat{\pi})$.

We follow Andrews and Armstrong (2017) and define the unbiased IV estimator. Let

$$\widehat{\delta} = \widehat{\delta}(\widehat{\psi}, \Sigma) = \widehat{\gamma} - \frac{\sigma_{12}}{\sigma_2^2} \widehat{\pi}$$

and

$$\widehat{\tau} = \widehat{\tau}(\widehat{\psi}, \Sigma) = \frac{1}{\sigma_2} \frac{1 - \Phi(\widehat{\pi}/\sigma_2)}{\phi(\widehat{\pi}/\sigma_2)} = \frac{1}{\sigma_2} \Psi(\widehat{\pi}/\sigma_2),$$

where $\Psi(x) = (1 - \Phi(x))/\phi(x)$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are cdf and pdf for the standard normal distribution, respectively. The unbiased IV estimator is

$$\widehat{\beta}_U = \widehat{\beta}_U(\widehat{\psi}, \Sigma) = \widehat{\delta}\widehat{\tau} + \frac{\sigma_{12}}{\sigma_2^2}.$$

It is shown that $E[\widehat{\beta}_U] = \beta$ when $\pi > 0$.

Comment 2.4. The main idea of $\widehat{\beta}_U$ is to use the fact that $\widehat{\tau}$ is an unbiased estimator for $1/\pi$ (Voinov and Nikulin, 1993). Since $\widehat{\delta}$ can be considered as the projection of $\widehat{\gamma}$ on the null space of $\widehat{\pi}$, $\widehat{\delta}$ is independent of $\widehat{\pi}$, and thus of $\widehat{\tau}$ as a function of $\widehat{\pi}$. Those facts lead to $E[\widehat{\beta}_U] = \beta$ (Andrews and Armstrong, 2017).

Define the truncated version of the unbiased IV estimator by

$$\widetilde{\beta} = \widehat{\delta}\widetilde{\tau} + \frac{\sigma_{12}}{\sigma_2^2},$$

where

$$\widetilde{\tau} = \frac{1}{\sigma_2} \Psi\left(\frac{\max\{\widehat{\pi}, \pi^*\}}{\sigma_2}\right),$$

and π^* is some truncation parameter. That is, we “winsorize” the unbiased IV estimator according to $\widehat{\pi}$ by the threshold π , when $\widehat{\pi}$ is too small. We do this because $\Psi(\cdot)$ is positive and strictly decreasing on \mathbb{R} , and $\Psi(x) \rightarrow \infty$ as $x \rightarrow -\infty$, which causes $\widehat{\beta}_U$ to have unbounded second moment. By truncation, we eliminate extreme values of $\widehat{\beta}_U$, which is important in conducting inference.

Example 1. We visualize the truncation in Figure 1. Consider a simple case where $\widehat{\psi} = (\widehat{\gamma}, \widehat{\pi})' \sim N(\psi, I_2)$. Then the unbiased IV estimator for β is $\widehat{\beta}_U = \widehat{\delta}\widehat{\tau}$, where $\widehat{\delta} = \widehat{\gamma}$ and $\widehat{\tau} = (1 - \Phi(\widehat{\pi}))/\phi(\widehat{\pi})$. Define $\widehat{\pi}_U = 1/\widehat{\tau}$, then $\widehat{\beta}_U = \widehat{\gamma}/\widehat{\pi}_U$, i.e., $\widehat{\beta}_U$ is the slope of the line through $(\widehat{\pi}_U, \widehat{\delta})$ and the origin. Then, the proposed truncated estimator $\widetilde{\beta}$ is the slope of the line through $(\widetilde{\pi}, \widehat{\delta}) = (\max\{\widehat{\pi}_U, \pi^*\}, \widehat{\delta})$ and the origin.

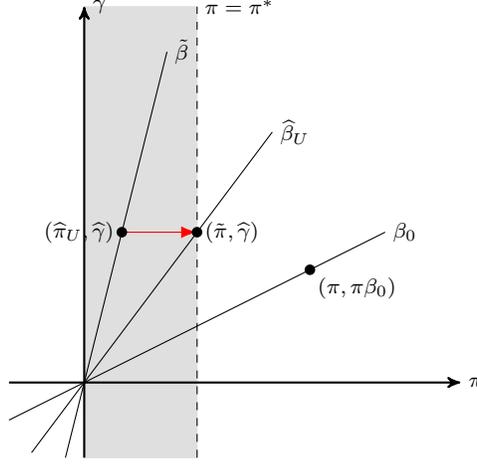


Fig 1: $\widetilde{\beta}$ is obtained through winsorizing $\widehat{\pi}_U$ in the gray area.

The following result shows that the truncated estimator is nearly unbiased when the truncation is appropriate.

Proposition 1. *Assume β is fixed. Suppose (i) $|\sigma_{12}/\sigma_2^2| < \infty$, (ii) $\pi^*/\sigma_2 \rightarrow -\infty$, and (iii) $\pi\pi^*/\sigma_2^2 \rightarrow -\infty$. Then, $E[\widetilde{\beta}] - \beta \rightarrow 0$.*

Comment 2.5. Proposition 1 gives guidance on when the proposed estimator $\widetilde{\beta}$ is approximately unbiased. A trivial example is where $\pi^* \rightarrow -\infty$ and everything else is constant, in which case $\widetilde{\beta}$ is approaching the unbiased estimator $\widehat{\beta}_U$. Under either the common strong IV asymptotics where $\sigma_2 = O(1/\sqrt{n})$ and π is constant, or the common weak IV asymptotics where $\sigma_2 = O(1/\sqrt{n})$ and $\pi = O(1/\sqrt{n})$, (ii) and (iii) require π^* to be negative and not to shrink as fast as $1/\sqrt{n}$.

3. Fama-MacBeth Inference with Truncated Unbiased IV

Consider a triangular array $\{(X_{n,i}, Y_{n,i}, Z_{n,i})\}_{i=1}^n\}_{n \geq 1}$ that follows the linear IV model (2.1),

$$\begin{cases} Y_{n,i} = Z_{n,i}\pi_n\beta + U_{n,i}, \\ X_{n,i} = Z_{n,i}\pi_n + V_{n,i}, \end{cases} \quad (3.1)$$

and a sequence of clustering dependence structures $\{\mathcal{C}_n\}_{n \geq 1}$ with $\mathcal{C}_n = \{I_{n,g}\}_{g=1}^{G_n}$ such that $G_n \rightarrow \infty$ as $n \rightarrow \infty$. That is, for any fixed n , observations are independent across groups but may be dependent within a group. As in Section 2, (X, Y, Z) is considered fixed and (U, V) is considered random. The parameter of interest is β , which does not vary with the sample size n . Our goal is to make inferential statement on the hypothesis $H_0 : \beta = \beta_0$. The first stage coefficient π_n is allowed to change with n , but stays the same across groups for each fixed n .² In the following presentation, we suppress n for simplicity. All variables and parameters (except β) should be considered as a function of n .

3.1. General results

We consider a Fama-MacBeth-type procedure. Namely, we estimate a truncated unbiased IV estimator $\tilde{\beta}_g$ for each group $g \in \{1, \dots, G\}$, using only $\{(X_i, Y_i, Z_i)\}_{i \in I_g}$. Thus, we obtain a set $\{\tilde{\beta}_g\}_{g=1}^G$ of nearly-unbiased IV estimators with bounded second moments. Define group-level quantities $\{n_g, \hat{\psi}_g, \hat{\delta}_g, \hat{\tau}_g, \pi_g^*\}_{g=1}^G$ accordingly. As in Section 2, we assume the group-level reduced-form and first-stage coefficients follow a normal distribution with known covariance Σ_g such that

$$\hat{\psi}_g = \begin{pmatrix} \hat{\gamma}_g \\ \hat{\pi}_g \end{pmatrix} \sim N(\mu_g, \Sigma_g),$$

from which the group-level truncated unbiased IV estimator $\tilde{\beta}_g$ is constructed.³ This means either that the errors (U, V) follow normal distribution or that there are at least moderate number of observations in each group. Also, define $\{\sigma_{1,g}, \sigma_{2,g}, \sigma_{12,g}, \mu_{\delta,g}, \sigma_{\delta,g}\}$ such that

$$\begin{aligned} \Sigma_g &= \begin{pmatrix} \sigma_{1,g}^2 & \sigma_{12,g} \\ \sigma_{12,g} & \sigma_{2,g}^2 \end{pmatrix}, \\ \mu_{\delta,g} &= \pi(\beta - \sigma_{12,g}/\sigma_{2,g}^2), \\ \sigma_{\delta,g}^2 &= \sigma_{1,g}^2 - \sigma_{12,g}^2/\sigma_{2,g}^2. \end{aligned}$$

²This assumption is made here for simplicity. In principle, we do not need to assume π is the same across different groups, because of the nature of group-level estimation.

³In practice, $\{\Sigma_g\}_{g=1}^G$ can be estimated by model-based or HAC-type estimators.

For the set of group-level estimates $\{\tilde{\beta}_g\}_{g=1}^G$, define the Fama-MacBeth estimator

$$\bar{\beta} = \frac{1}{G} \sum_{g=1}^G \tilde{\beta}_g$$

and the standard error

$$\text{se} = \sqrt{\frac{1}{G(G-1)} \sum_{g=1}^G (\tilde{\beta}_g - \bar{\beta})^2}.$$

The corresponding t -statistic is

$$t = \frac{\bar{\beta} - \beta_0}{\text{se}}.$$

We show t is asymptotically normal when the estimator is properly truncated.

Assumption 1. (i) $\limsup_n \sup_g |\sigma_{12,g}/\sigma_{2,g}^2| < \infty$;

(ii) $\sup_g \pi_g^*/\sigma_{2,g} \rightarrow -\infty$, as $n \rightarrow \infty$;

(iii) $\sup_g \pi \pi_g^*/\sigma_{2,g}^2 \rightarrow -\infty$, as $n \rightarrow \infty$.

Define $M = \sup_g \Psi(\pi_g^*/\sigma_{2,g})/\sigma_{2,g}$. Conceptually, M guides the overall level of truncation across groups with respect to $\hat{\tau}_g$. This is because $\Psi(\cdot)$ is a strictly decreasing one-to-one map such that $\hat{\tau}_g \geq \pi_g^*$ if and only if $\hat{\tau}_g \leq \Psi(\pi_g^*/\sigma_{2,g})/\sigma_{2,g}$.

Assumption 2. The truncation parameter M satisfies

$$M = o\left(\frac{B}{\bar{\sigma}_\delta(\kappa G)^{1/3}}\right),$$

where

$$B^2 = \sum_{g=1}^G E[(\tilde{\beta}_g - E[\tilde{\beta}_g])^2],$$

$$\bar{\sigma}_\delta = \max_g \sigma_{\delta,g},$$

$$\kappa = \max_g K\left(-\frac{3}{2}, \frac{1}{2}; -\frac{\mu_{\delta,g}^2}{2\sigma_{\delta,g}^2}\right)$$

and $K(a, b; z)$ is Kummer's confluent hypergeometric function.

Comment 3.1. Assumption 1 and 2 are high-level conditions that allow for many IV configurations. Both a fixed π (strong IV) or a local drifting sequence that shrinks at the rate of $n^{-1/2}$ (weak IV) are discussed below. Assumption 1 is generally weak. Assumption 1(i) implies σ_{12} and $\sigma_{2,g}^2$ are approximately of the same scale. This is reasonable, since they are typically $O(1/n_g)$ with weak dependence. Assumption 1(ii) & (iii) requires both $\pi_g^*/\sigma_{2,g}$ and $\pi \pi_g^*/\sigma_{2,g}^2$ to go to $-\infty$, uniformly. This is weak under strong IV as long as π_g^* is negative and bounded away from zero. Under weak IV where $\pi = O(1/\sqrt{n})$, 1(ii) is weak, and 1(iii)

holds when $\inf_g n_g/\sqrt{n}$ does not go to zero too fast, i.e., number of groups increases too fast. Assumption 2 puts restrictions on the truncation parameter. Practical suggestions of how the truncation parameters are chosen are given in Appendix A.

Theorem 1. *Under Assumption 1 and 2, $t \xrightarrow{d} N(0, 1)$.*

Comment 3.2. This result implies that the test $\psi = \mathbb{1}\{|t| > z_{\alpha/2}\}$ delivers asymptotically correct size at level α , where $z_{\alpha/2}$ is the $(1 - \alpha/2)$ -quantile of the standard normal distribution. In practice, some quantities in constructing the t -statistic need to be estimated. The implementation details are in Appendix A.

3.2. Strong IV asymptotics

In this subsection I give the primitive assumptions under which the proposed method delivers valid inference under strong IV.

Define

$$\begin{cases} \bar{\sigma}_2 = \max_g \sigma_{2,g} \\ \underline{\sigma}_2 = \min_g \sigma_{2,g} \end{cases}. \quad (3.2)$$

Assumption S1. (i) $\liminf_n \pi > 0$;

(ii) $\limsup_n \sup_g |\sigma_{12,g}/\sigma_{2,g}^2| < \infty$;

(iii) $\underline{\sigma}_2 M \rightarrow \infty$ and $\bar{\sigma}_2 = O(1)$.

Assumption S2. (i) $\bar{\sigma}_2/\underline{\sigma}_2 = O(1)$;

(ii) $M = o(BG^{-1/3})$.

Comment 3.3. S1(i) implies a strong IV sequence, and includes the case where π is fixed as $n \rightarrow \infty$. S1(ii) is the same as Assumption 1(i). The first half of S1(iii) together with S2(ii) provides guidance on choice of M . The second half of S1(iii) is weak as long as groups are not diminishing. S2(i) requires there is no severe size imbalance across groups. Together with the assumptions under the weak IV asymptotics in Section 3.3, these assumptions have implications on the truncation parameter selection. Practical suggestions are given in Appendix A.

Proposition 2. *Under Assumption S1 and S2 (strong IV sequence), Assumption 1 and 2 hold.*

3.3. Weak IV asymptotics

In this subsection I give the preliminary assumptions under which the proposed method delivers valid inference under weak IV where the first stage strength parameter π follows a drifting sequence towards 0 at the rate of $n^{-1/2}$.

Let $\bar{\sigma}_2$ and $\underline{\sigma}_2$ be defined in Equation (3.2). Similarly, define $\bar{\sigma}_\delta = \max_g \sigma_{\delta,g}$ and $\underline{\sigma}_\delta = \min_g \sigma_{\delta,g}$.

Assumption W1. (i) $\pi = \pi_0/\sqrt{n}$;

(ii) $\sup_n \sup_g |\sigma_{12,g}/\sigma_{2,g}^2| < \infty$;

(iii) $n^{-1/2}\Psi^{-1}(\underline{\sigma}_2 M)/\bar{\sigma}_2 \rightarrow -\infty$.

Assumption W2. (i) $\pi^2/\underline{\sigma}_\delta^2 \rightarrow 0$;

(ii) $M = o(B\bar{\sigma}_\delta^{-1}G^{-1/3})$.

Comment 3.4. W1(i) is standard in the weak IV literature (e.g., [Staiger and Stock, 1997](#)).

In the case of weak dependence with approximately balanced groups, $\sigma_{2,g} = O(n_g^{-1/2})$, so W1(iii) implies $\Psi^{-1}(\underline{\sigma}_2 M)/\sqrt{G} \rightarrow -\infty$; $\underline{\sigma}_\delta = O(1/\min_g n_g)$, so W2(i) implies $\max_g n_g/n \rightarrow 0$ (cf. $\max_g n_g^2/n \rightarrow 0$ in [Hansen and Lee, 2019](#)).

Proposition 3. *Under Assumption W1 and W2 (weak IV sequence), Assumption 1 and 2 hold.*

4. Simulation

In this section we study the finite sample performance of the proposed estimator. In all the following settings, the data generating process follows the linear IV model (3.1) where $n = 900$, $G = 30$, such that $n/G^2 = 1$, which deviates from the usual asymptotics. The null hypothesis is $H_0 : \beta = 0$. For each setting, 1000 replications are conducted to calculate the empirical rejection rate.

For each setting, we observe $\{(X_i, Y_i, Z_i)\}_{i=1}^n$ and a partition $\{I_g\}_{g=1}^G$ of $\{i\}_{i=1}^n$. Let consecutive observations belong to the same group, i.e., $I_1 = \{1, 2, \dots, |I_1|\}$, $I_2 = \{|I_1| + 1, \dots, |I_1| + |I_2|\}$, etc., where $|\cdot|$ is cardinality. The data are drawn according to the following

process.

$$Y_i = Z_i\pi\beta + U_i$$

$$X_i = Z_i\pi + V_i$$

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} \sim N\left(0, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\right), \text{ if } i = 1 + \sum_{h=1}^g |I_h| \text{ for some } g = 0, 1, \dots, G-1$$

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} = 0.5 \begin{pmatrix} U_{i-1} \\ V_{i-1} \end{pmatrix} + \sqrt{1 - 0.5^2} \begin{pmatrix} \varepsilon_i^U \\ \varepsilon_i^V \end{pmatrix}, \text{ if } i \neq 1 + \sum_{h=1}^g |I_h| \text{ for any } g = 0, 1, \dots, G-1$$

$$\begin{pmatrix} \varepsilon_i^U \\ \varepsilon_i^V \end{pmatrix} \sim N\left(0, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\right) \text{ and is i.i.d. across } i$$

Also, each dimension of the k -dimensional instruments Z_i takes one draw from the distribution of $\{U_i\}_{i=1}^n$ and is fixed across replications. Thus, (U_i, V_i) within each group follows an AR(1) process and is independent across different groups. The parameters $(\beta, \pi, \{I_g\}_{g=1}^G, k)$ vary accordingly across settings.

4.1. Debiasing and truncation

We first investigate three Fama-MacBeth-type inferential procedures and show necessity of debiasing and truncation. We consider t -test on group-level 2SLS estimators (FM), t -test on group-level unbiased IV estimators (FMU), and the proposed t -test on group-level truncated unbiased IV estimators (FMUT), with truncation parameter selected as suggested in Appendix A. The full-sample 2SLS with CCE estimates of standard errors is also reported for comparison.

In this experiment, we have 5 instrument ($k = 5$) and one endogenous variable. Groups are imbalanced in sizes, with 5 groups of 90 observations and 25 groups of 18 observations. For each group, the observations follow an AR(1) process as described before. The first stage coefficient is $\pi = (0.1, 0.1, 0.1, 0.1, 0.1)' / \sqrt{5}$ such that $\|\pi\|_2 = 0.1$.

The power curves are reported in Figure 1. Estimators used in CCE and FM are both biased. FM has large bias between the two, since it uses group-level 2SLS estimators with much larger finite sample bias compared with the full-sample estimator. Compared with FMUT, FMU is less powerful. The reason is that the unbiased IV estimator does not have bounded second moment, such that the resulting t -statistic has a tail that is too large.

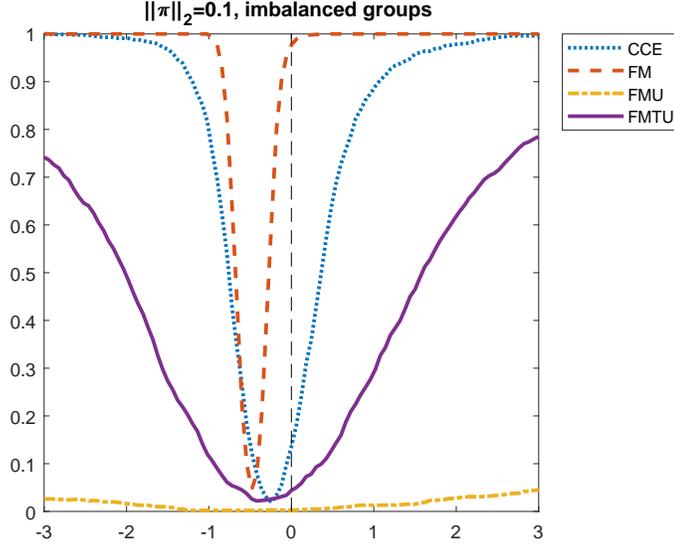


Fig 2: Power comparison among Fama-MacBeth procedures ($\alpha = 0.05$)

4.2. Comparison with other methods

Here we compare the proposed method with the existing inferential procedure. We consider the “clustered standard error” approach (CCE), and the natural extension of Anderson-Rubin test to the our settings (AR-CCE). To implement the AR-CCE method, we apply CCE to the regression of $Y - X\beta_0$ on Z , where β_0 is the hypothesized value as in $H_0 : \beta = \beta_0$. In our case, we test $H_0 : \beta = 0$, so AR-CCE is equivalent to perform CCE to test the hypothesis $H_0 : \gamma = 0$ in the regression $Y = Z\gamma + U$.

We look at several configurations. The number of instruments k varies in the set $\{1, 5, 10\}$. The first-stage strength is chosen such that $\|\pi\|_2 \in \{0.1, 0.5\}$, with $\pi = \|\pi\|_2 \iota_k / \sqrt{k}$ and ι_k being a k -vector of ones. For example, in the case of $\|\pi\|_2 = 0.1$ and $k = 5$, we have $\pi = (0.1, 0.1, 0.1, 0.1, 0.1)' / \sqrt{5}$. We also consider both balanced and imbalanced groups. In the *Balanced Groups* case, there are 30 groups of 30 observations; in the *Imbalanced Groups* case, there are 5 groups of 90 observations and 25 groups of 18 observations.

The sizes are reported in Table 2 and the power curves are in Figure 3, 4, and 5. Among all methods, only the proposed method FMUT is able to deliver robust inference result at the null across all settings. CCE displays a noticeable bias under $\|\pi\|_2$ and over-identification. AR-CCE is robust to weak instruments, but not under group imbalance.

TABLE 2
Summary ($\alpha = 0.05$)

			Balanced Groups			Imbalanced Groups		
			Median	MAD	Size	Median	MAD	Size
$k = 1$	$\pi = 0.5$	CCE	0.002	0.050	0.040	0.001	0.049	0.062
		AR-CCE	-	-	0.040	-	-	0.060
		FMTU	-0.010	0.057	0.039	-0.035	0.087	0.052
	$\pi = 0.1$	CCE	0.010	0.254	0.043	0.007	0.251	0.048
		AR-CCE	-	-	0.040	-	-	0.060
		FMTU	0.032	0.303	0.066	0.002	0.354	0.048
$k = 5$	$\pi = 0.5$	CCE	0.011	0.051	0.051	0.010	0.052	0.063
		AR-CCE	-	-	0.037	-	-	0.092
		FMTU	-0.068	0.110	0.033	-0.078	0.141	0.034
	$\pi = 0.1$	CCE	0.194	0.256	0.119	0.202	0.256	0.136
		AR-CCE	-	-	0.037	-	-	0.092
		FMTU	0.073	0.260	0.047	0.029	0.312	0.044
$k = 10$	$\pi = 0.5$	CCE	0.022	0.051	0.076	0.021	0.050	0.082
		AR-CCE	-	-	0.063	-	-	0.116
		FMTU	-0.055	0.105	0.046	-0.074	0.131	0.037
	$\pi = 0.1$	CCE	0.277	0.284	0.259	0.279	0.285	0.267
		AR-CCE	-	-	0.063	-	-	0.116
		FMTU	0.067	0.247	0.075	0.026	0.294	0.046

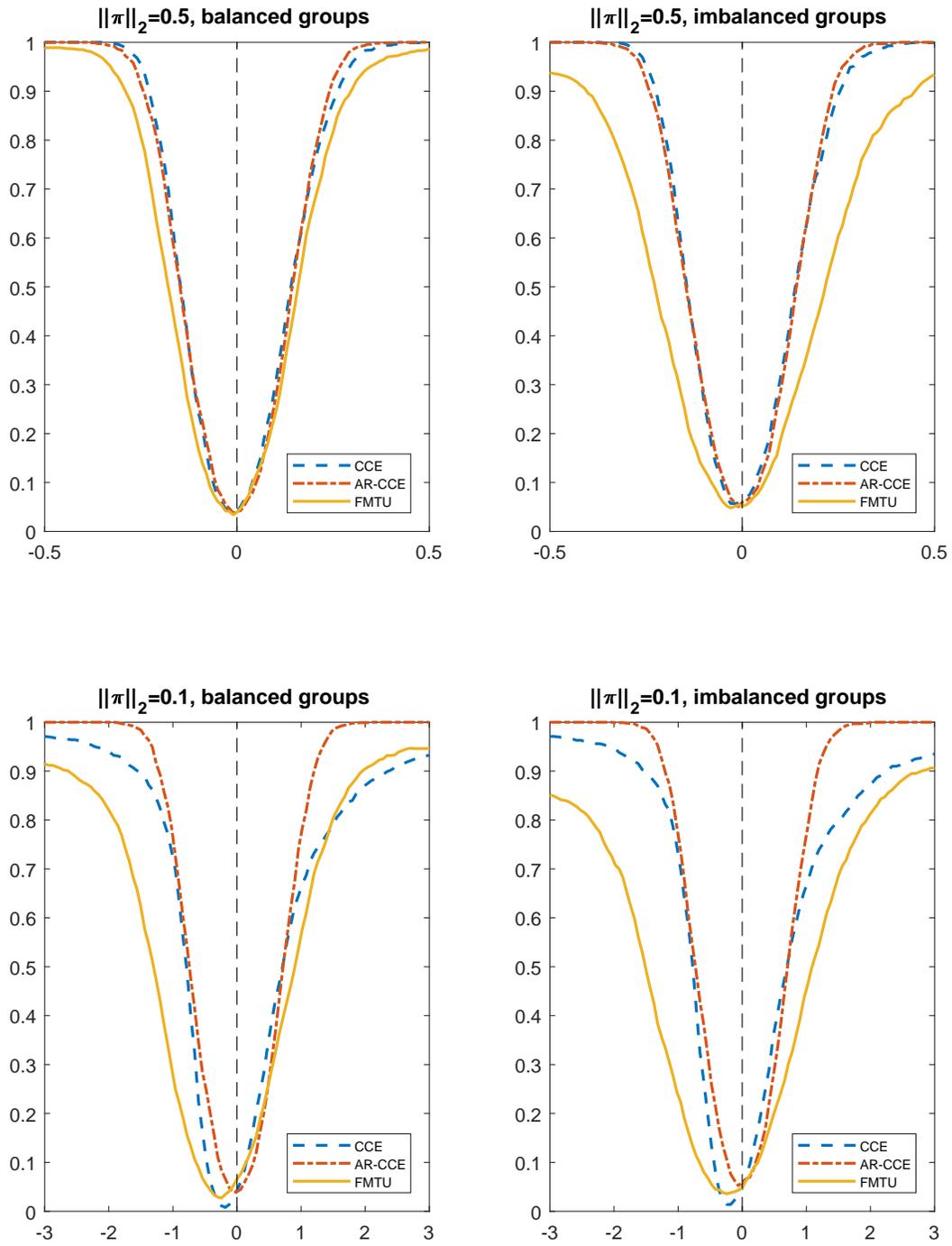


Fig 3: Power curves with nominal size $\alpha = 0.05$ and $k = 1$.

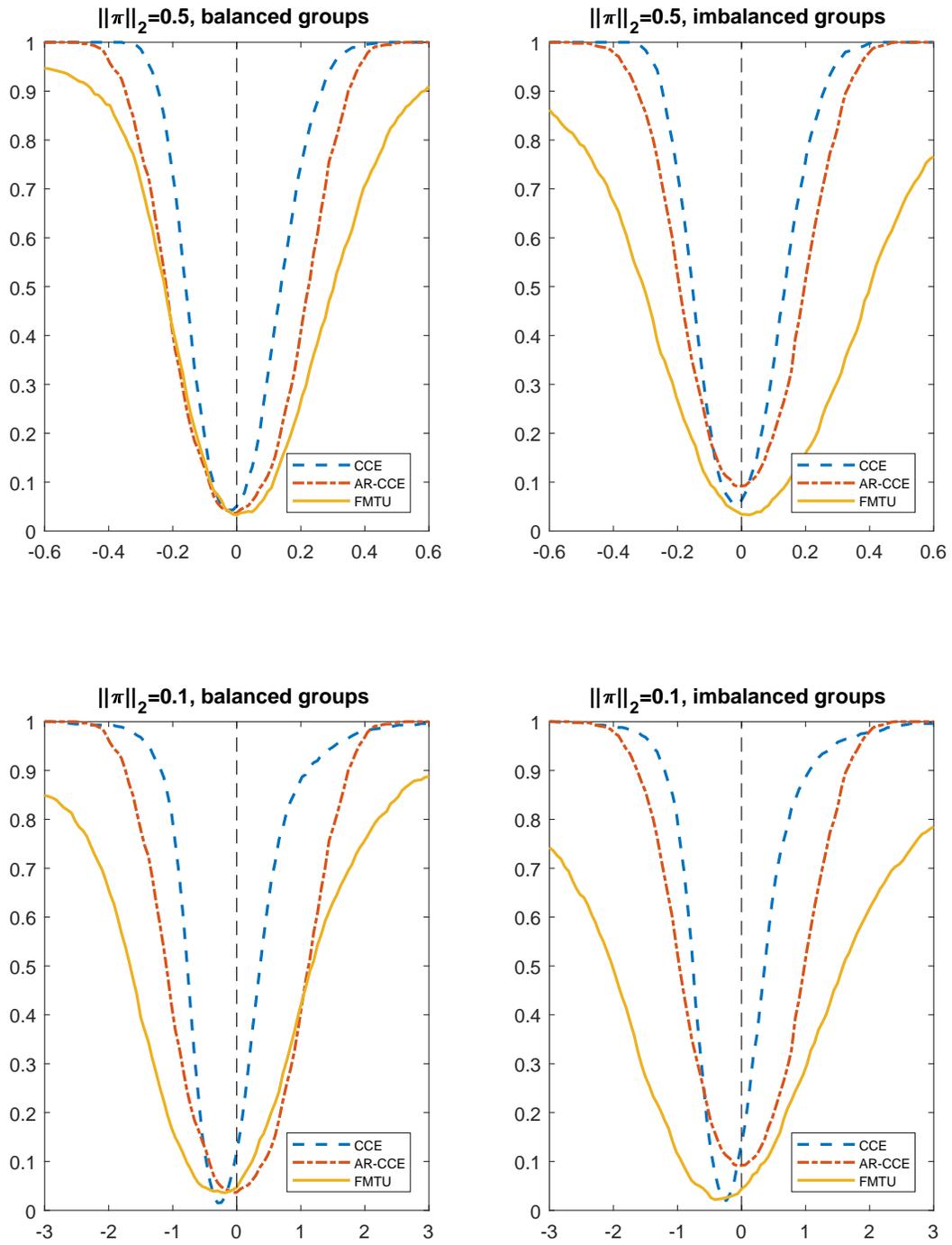


Fig 4: Power curves with nominal size $\alpha = 0.05$ and $k = 5$.

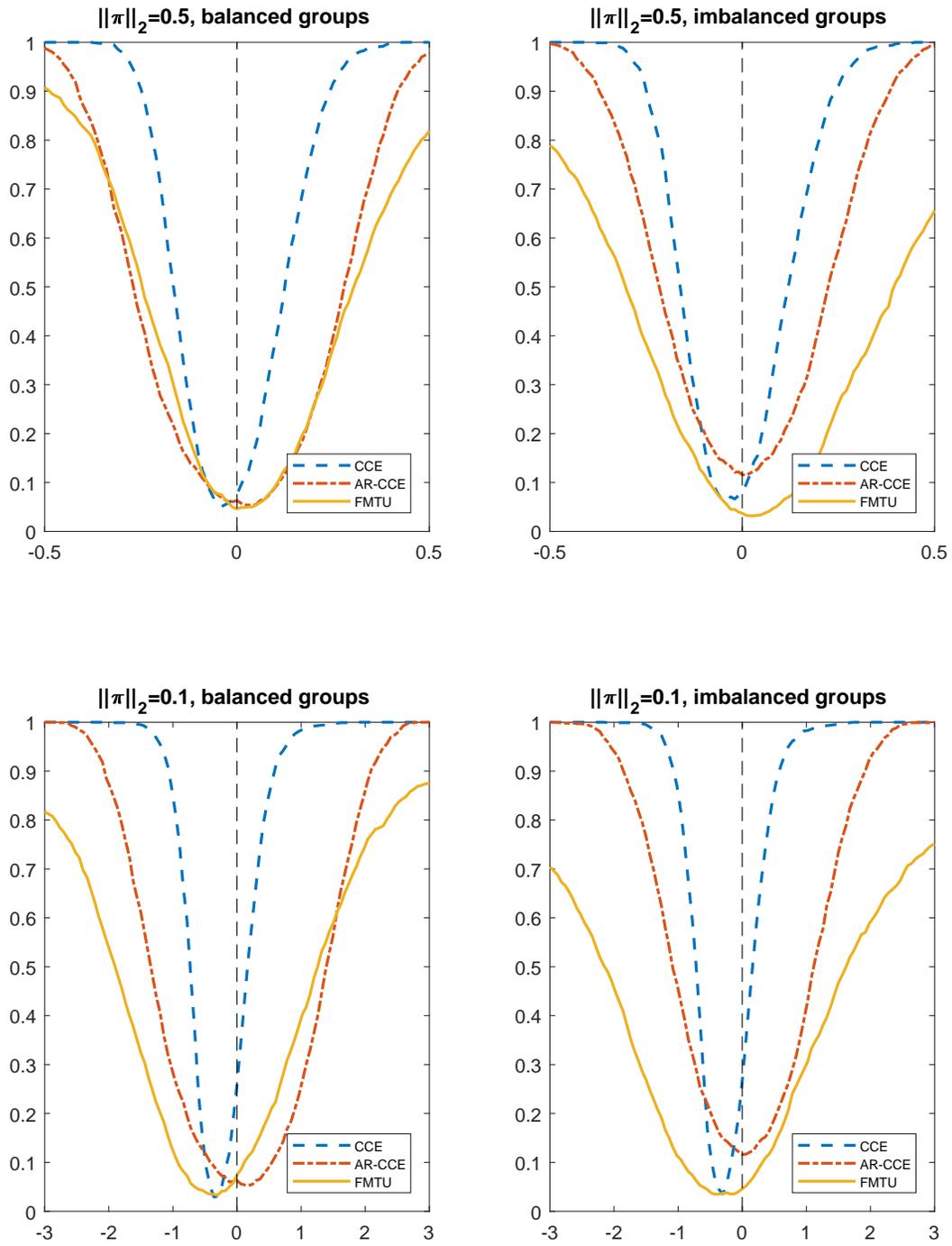


Fig 5: Power curves with nominal size $\alpha = 0.05$ and $k = 10$.

5. Empirical Application: Urban Geometry in India

In this section I use the proposed inferential method to study the effect of city shape on population density. The data used in this section was originally collected and analyzed in [Harari \(2020\)](#). The shape of a city affects its compactness, where compactness is measured by how convenient its residents travel for daily activities. Ideally, a compact city should look like a circle, while cities develop into various shapes for many reasons, including geographic constraints. Compact cities are attractive to residents since their daily activities operate more efficiently than those in cities that are less compact. This argument suggests the hypothesis that more compact cities should have higher population density. However, city shape is highly endogenous, because itself is the outcome of economic activities. [Harari \(2020\)](#) proposes a solution to this by utilizing geographic obstacles such as mountains and lakes as an instrument. I apply the method proposed in this paper, FMTU, in order to obtain a more robust set of empirical results.

5.1. Methodology

To facilitate quantitative analysis, [Harari \(2020\)](#) proposes a shape metric that is based on the average distance between any two points in a polygon, in order to measure the compactness of a city. Namely, the Shape index is defined by

$$Shape = \frac{1}{B(B-1)} \sum_{i=1}^B \sum_{j \neq i} d_{ij},$$

where i and j stand for two points sampled from interior points of the city, d_{ij} is the Euclidean distance between i and j , and B is the number of sampled points. We consider the *Normalized Shape* obtained by dividing *Shape* by the radius of the Equivalent Area Circle (EAC), where EAC is the circle with the same area as the city. That is, *Normalized Shape* measures how much the city shape is different from a circle.

The instrument is the *Normalized Shape* index for the projected city. Constructing the projected city is a two-step procedure. First, predict the area that a city should occupy in a given year, based on its projected historical population growth. Second, predict the shape of the city given projected area and geographic constraints. We then instrument the shape of the actual city with shape of the potential one.

I consider the same setup as [Harari \(2020\)](#) does. The regression of interest is

$$\Delta Population\ density = \alpha + \beta \Delta Normalized\ shape + U,$$

where the dependent variable is the change of population density from 1951 to 2011, the endogenous variable is the change of city shape index from 1950 to 2010, and the instrument is the change of city shape index for the projected city expansion from 1950 to 2010. The instrument is the difference of *Normalized Shape* for projected cities. This model can be interpreted as a *difference-in-difference* design with continuous treatment and endogeneity.

I consider the potential dependence among observations by applying the framework suggested by Cao et al. (2019). Namely, I first applied k -medoids to generate a partition of cities using their geographic locations, and then use the given clustering structure to perform the group-based inference method proposed in this paper. I use this method to obtain inference results robust to spatial correlation. Factors that affect population density in a city may include climate, culture, economy, personal preferences, etc. Those factors are multidimensional and the natural administrative division⁴ does not necessarily capture the underlying dependence structure. That is, cities in neighboring states may be highly correlated in factors that contribute to population density.

The idea of Cao et al. (2019) is to use k -medoids, a clustering algorithm, to generate a partition of observations that helps obtain robust results in group-based inferential methods. Cao et al. (2019) show the clustering generated by k -medoids satisfies *group-balance* and *diminishing-boundary*. The former, *group-balance*, requires there is no diminishingly small group, and the latter, *diminishing-boundary*, requires across-group dependence is approximately ignorable. The algorithmic details are represented in Appendix E. I apply k -medoids to generate a clustering of 10 group. The resulting structure is visualized in Figure 6.

5.2. Results

Table 3 compares the original results in Table 8 of Harari (2020) with those obtained from the proposed method. Note that the first-stage t -statistic being 5.311 does not imply we can ignore the instrument strength. Lee et al. (2020) show that in order to have a level-0.05 second-stage test in a single IV model, the first-stage F -statistic needs to exceed 104.7, which translates into a t -statistic of 10.23. Comparing 2SLS and the proposed method of this paper, the estimates are qualitatively similar (-171.79 versus -199.26), while the standard errors are quite different. While the new p -value still suggests rejecting the null at some levels such as 0.1, the implication is vastly different from the original p -value, suggesting that including spatial correlation in analysis is crucial.

⁴India is a federal union comprising 28 states and 8 union territories.

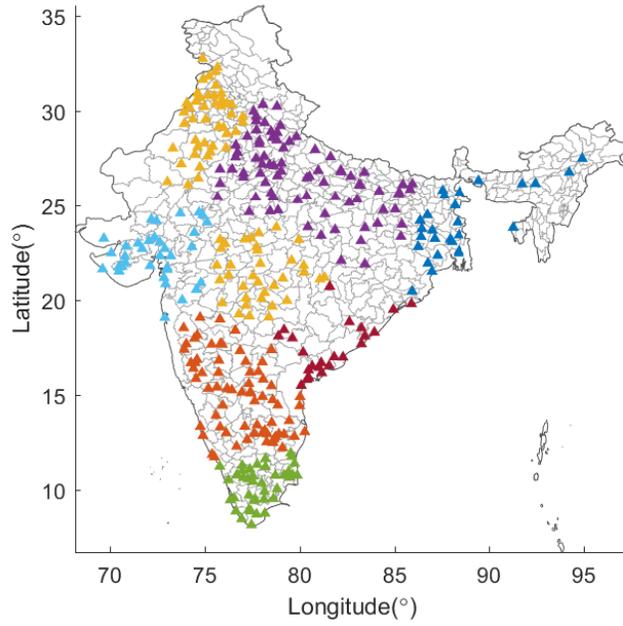


Fig 6: Partition of cities in Indian by k -medoids using 10 clusters. Distances are Euclidean distances based on latitude and longitude coordinates recorded at cities centroids. Different colors correspond to different clusters in the partition. Marks are plotted at city centroids.

TABLE 3

	Δ Normalized shape	Δ Population density	
	First stage (1)	2SLS (2)	FMUT (3)
Δ Potential normalized shape	0.0996 (0.0188)		
Δ Normalized shape		-171.8 (37.32)	-199.3 (88.35)
t -stat	5.311	-4.603	-2.255
p -value		0.000	0.051
Observations	351	351	351

Notes: This table reports estimates of the impacts of city shape on population. Column 1 reports the first-stage results. Column 2 reports 2SLS results with White robust standard error. Column 3 reports results from the truncated unbiased Fama-MacBeth method. The p -value for FMUT is calculated using a Student's t -distribution of 9 degrees of freedom.

6. Conclusion

In the setting of IV regression, this paper proposes an inferential method that is based on the idea of Fama-MacBeth estimation, in order to deal with weak IV and heterogeneous clustering dependence. To overcome the finite sample bias of IV regression, the group-level estimator is a truncated version of the unbiased IV estimator proposed by [Andrews and Armstrong \(2017\)](#). I give high-level conditions under which the proposed method is asymptotically valid. Asymptotic validity is also shown under both strong- and weak-IV sequences. Finite sample performance is shown by simulation. The proposed method is applied to study the effect of city compactness on population density.

Appendix A: Implementation

In this section I describe the details in implementing the proposed procedure. The idea is simply to replace each quantity by its sample analog. Section [A.1](#) discuss the case with only one instrument. Section [A.2](#) is for the case with more than one instruments.

A.1. One single instrument

The algorithm consists of three steps: group-level estimation, debiasing and truncation, and t -test.

Step 1 We fix some group g and only use observations in this group. Let the corresponding group-level estimators be $\hat{\psi}_g = (\hat{\psi}_{1,g}, \hat{\psi}_{2,g})'$ as in [\(2.2\)](#), and the residuals be $\{\hat{U}_i, \hat{V}_i\}_{i \in I_g}$. Let $\hat{\Lambda}_g$ be a Heteroskedasticity and Auto-correlation Correction estimator (HAC) of

$$\text{Var} \left[\frac{1}{\sqrt{n_g}} \sum_{i \in I_g} \begin{pmatrix} Z_i U_i \\ Z_i V_i \end{pmatrix} \right].$$

In the simulation section, we use the Newey-West estimator with $\lfloor 4(T/100)^{1/4} \rfloor$ lags ([Newey and West, 1987](#)). The estimator for $\text{Var}[\hat{\psi}]$ is thus

$$\hat{\Sigma}_g = \begin{pmatrix} Q_{ZZ,g}^{-1} & 0 \\ 0 & Q_{ZZ,g}^{-1} \end{pmatrix} \hat{\Lambda}_g \begin{pmatrix} Q_{ZZ,g}^{-1} & 0 \\ 0 & Q_{ZZ,g}^{-1} \end{pmatrix},$$

where $Q_{ZZ,g} = n_g^{-1} \sum_{i \in I_g} Z_i Z_i'$. The group-level $(\hat{\delta}, \hat{\tau})$ is given by

$$\hat{\delta}_g = \hat{\delta}(\hat{\psi}_g, \hat{\Sigma}_g), \quad \hat{\tau}_g = \hat{\tau}(\hat{\psi}_g, \hat{\Sigma}_g),$$

and the unbiased IV is $\hat{\beta}_g = \hat{\delta}\hat{\tau} + \hat{\sigma}_{12,g}/\hat{\sigma}_{2,g}^2$.

Step 2 Consider a uniform truncation parameter where $\pi_g^* = \pi^*$ for each g . Let

$$\pi_{SIV}^* = \min_g \frac{1}{\sqrt{n_g}} \Psi^{-1} \left(c \sqrt{\frac{\bar{n}}{n_g}} \right)$$

and

$$\pi_{WIV}^* = \min_g \frac{1}{\sqrt{n_g}} \Psi^{-1} \left(\sqrt{\frac{\bar{n}}{n_g}} \Psi \left(-c \sqrt{\frac{n}{\bar{n}}} \right) \right),$$

where $\bar{n} = \max_g n_g$ and $\underline{n} = \min_g n_g$. The former is suggested by assumptions under the strong IV asymptotics in Section 3.2 and the latter by the weak IV asymptotics in Section 3.3. The truncation parameter is chosen to be $\pi^* = \min\{\pi_{SIV}^*, \pi_{WIV}^*\}$. In practice, I recommend to use $c = 10$. Using the selected threshold π^* , we can obtain a set of group-level truncated unbiased IV estimators $\{\tilde{\beta}_g\}_{g=1}^G$.

Step 3 We apply t -test to the set of group-level estimators $\{\tilde{\beta}_g\}_{g=1}^G$. Namely, let

$$t = \frac{\bar{\beta} - \beta_0}{\text{se}},$$

where

$$\bar{\beta} = \frac{1}{G} \sum_{g=1}^G \tilde{\beta}_g$$

and

$$\text{se} = \sqrt{\frac{1}{G(G-1)} \sum_{g=1}^G (\tilde{\beta}_g - \bar{\beta})^2}.$$

We reject the null hypothesis $H_0 : \beta = \beta_0$ if $|t| > cv$, where cv is the $(1 - \alpha)$ -quantile of the t -distribution of $G - 1$ degrees of freedom.

A.2. Multiple instruments

When multiple instruments are available, we follow [Andrews and Armstrong \(2017\)](#) and use a weighted average of unbiased IV estimators with respect to all instruments. Namely, for the j -th instrument, we perform Step 1 and 2 as in Section A.1 and obtain the j -th unbiased IV estimator $\tilde{\beta}_{g,j}$. The group level estimator is then given by

$$\tilde{\beta}_g = \sum_j w_j \tilde{\beta}_{g,j},$$

where $\{w_j\}_{j=1}^k$ is a set of weights that sum up to one. See [Andrews and Armstrong \(2017\)](#) for a discussion on optimal weight selection. In the simulation section, $\{w_j\}_{j=1}^k$ are simply chosen to be equal weights. Finally, we follow Step 3 in Section A.1 using $\{\tilde{\beta}_g\}_{g=1}^G$.

Appendix B: Truncation Parameter Choices

In this section we investigate the impact of the truncation parameter choice. As in Appendix A, we recommend to use $\pi^* = \min\{\pi_{SIV}^*, \pi_{WIV}^*\}$ with $c = 10$ as the truncation parameter. We look into different choices of c in this experiment.

The data regenerating process is the same as in Section 4.1. FMUT methods with 3 different values of c are reported. The CCE method is also reported for comparison. The power curves are shown in Figure 7. Generally, the proposed method is quite robust to the choice of the truncation parameter in terms of null rejection rate. Moreover, Figure 7 exhibits a “bias-variance” tradeoff. That is, a smaller c corresponds to high power but causes more bias.

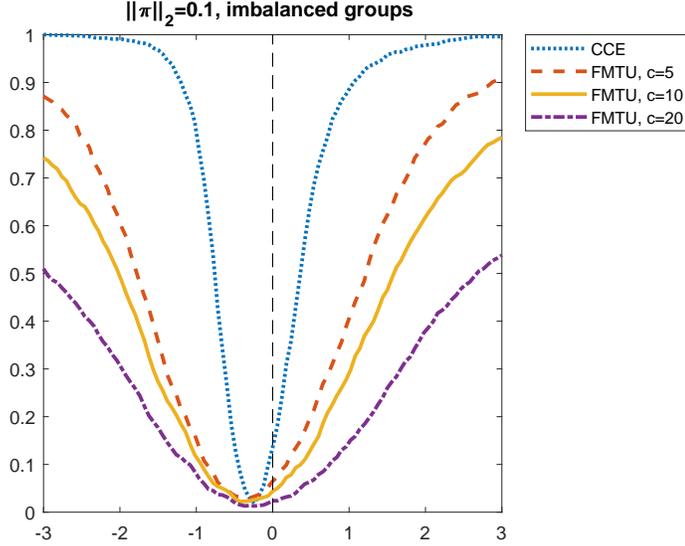


Fig 7: Power comparison among truncation parameter choices ($\alpha = 0.05$)

Appendix C: Useful Results

In this section I present some results that are useful for proofs in Section D.

Lemma 1. $E[\widehat{\tau} \mathbb{1}\{\widehat{\pi} \geq \pi^*\}] = \eta \pi^{-1}$, where

$$\eta = (1 - \Phi((\pi^* - \pi)/\sigma_2)) - (1 - \Phi(\pi^*/\sigma_2)) \exp(\pi\pi^*/\sigma_2^2 - \pi^2/(2\sigma_2^2)).$$

Proof.

$$\begin{aligned}
& E[\widehat{\tau} \mathbb{1}\{\widehat{\pi} \geq \pi^*\}] \\
&= E \left[\frac{1}{\sigma_2} \cdot \frac{1 - \Phi(\widehat{\pi}/\sigma_2)}{\phi(\widehat{\pi}/\sigma_2)} \mathbb{1}\{\widehat{\pi} \geq \pi^*\} \right] \\
&= \int_{\pi^*/\sigma_2}^{\infty} \frac{1}{\sigma_2} \cdot \frac{1 - \Phi(x)}{\phi(x)} \phi(x - \pi/\sigma_2) dx \\
&= \frac{1}{\sigma_2} \int_{\pi^*/\sigma_2}^{\infty} (1 - \Phi(x)) \exp(x\pi/\sigma_2 - \pi^2/(2\sigma_2^2)) dx \\
&= \frac{1}{\pi} \exp(-\pi^2/(2\sigma_2^2)) \left((1 - \Phi(x)) \exp\left(\frac{\pi}{\sigma_2} x\right) \Big|_{\pi^*/\sigma_2}^{\infty} - \int_{\pi^*/\sigma_2}^{\infty} \exp(x\pi/\sigma_2) d(1 - \Phi(x)) \right) \\
&= \frac{\eta}{\pi}.
\end{aligned}$$

The fourth equality is integration by parts. \square

Kummer's confluent hypergeometric functions

$$K(a, b, z) = \sum_{k=0}^{\infty} \frac{a^{\bar{k}} z^k}{b^{\bar{k}} k!},$$

where the rising factorial is defined by

$$x^{\bar{k}} = x(x+1) \dots (x+n-1).$$

For $z < 0$,

$$K(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} [1 + O(|z|^{-1})],$$

where the Gamma function is

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du.$$

See [Abramowitz and Stegun \(1965\)](#) for reference.

Appendix D: Proofs

Proof of Proposition 1. Let η be defined as in Lemma 1. Note that

$$\begin{aligned}
|E[\widehat{\beta}] - E[\widehat{\beta}_U]| &= |E[\widehat{\delta}(\widehat{\tau} - \widehat{\tau})]| \\
&= |E[\widehat{\delta}]| \cdot |E[\widehat{\tau} - \widehat{\tau}]| \\
&= |E[\widehat{\delta}]| \cdot E[(\widehat{\tau} - \tau^*) \mathbb{1}\{\widehat{\pi} < \pi^*\}] \\
&\leq |E[\widehat{\delta}]| \cdot E[\widehat{\tau} \mathbb{1}\{\widehat{\pi} < \pi^*\}] \\
&= |\pi(\beta - \sigma_{12}/\sigma_2^2)| \cdot (\pi^{-1} - \eta\pi^{-1}) \\
&= |\beta - \sigma_{12}/\sigma_2^2| \cdot (1 - \eta) \\
&\rightarrow 0.
\end{aligned}$$

The second equality uses the independence between $\widehat{\delta}$ and $\widehat{\pi}$, which implies the independence between $\widehat{\delta}$ and functions of $\widehat{\pi}$. The inequality is because Ψ is strictly decreasing and thus $0 \leq \widehat{\tau} - \tau^* \leq \widehat{\tau}$ under the event $\widehat{\pi} < \pi^*$. The fourth equation is by Lemma 1 and the fact that $E[\widehat{\tau}] = 1/\pi$. The convergence is because $\eta \rightarrow 1$ under (i), (ii), and (iii), and σ_{12}/σ_2^2 is bounded. \square

Proof of Theorem 1. By Assumption 1 and follow the proof of Proposition 1,

$$\sup_g |E[\widetilde{\beta}_g] - \beta| \rightarrow 0.$$

So

$$t = t^* + \frac{G^{-1} \sum_{g=1}^G (E[\widetilde{\beta}_g] - \beta)}{\text{se}} = t^* + o_p(1),$$

where

$$t^* = \frac{G^{-1} \sum_{g=1}^G (\widetilde{\beta}_g - E[\widetilde{\beta}_g])}{\text{se}}.$$

Under Assumption 2, for some absolute constant C ,

$$\begin{aligned} \sup_x |\mathbb{P}(t^* < x) - \Phi(x)| &\leq CB^{-3} \sum_{g=1}^G E[|\widetilde{\beta}_g - \beta|^3] \\ &\lesssim CB^{-3} G \max_g E[|\widehat{\delta}_g|^3] E[|\widetilde{\tau}_g|^3] \\ &\lesssim CB^{-3} G \bar{\sigma}_\delta^3 \kappa M^3 \\ &= o(1). \end{aligned}$$

The inequality is by a Berry-Esseen bound for Student's statistic in [Bentkus et al. \(1996\)](#). The third line uses a representation of the third raw absolute moment of normal distribution (for example, see [Winkelbauer, 2012](#)). Combining this with $t = t^* + o_p(1)$, we obtain $t \xrightarrow{d} N(0, 1)$. \square

Proof of Proposition 2. Assumption 1(i) holds automatically by Assumption S1(ii). For 1(ii), note

$$\max_g \frac{\pi_g^* - \pi}{\sigma_{2,g}} = \max_g \Psi^{-1}(\sigma_{2,g}M) - \frac{\pi}{\sigma_{2,g}} \leq \Psi^{-1}(\underline{\sigma}_2 M) \rightarrow -\infty$$

by S1(iii) and the fact that $\Psi^{-1}(x) \rightarrow -\infty$ as $x \rightarrow \infty$. For 1(iii), note that

$$\max_g \frac{\pi \pi_g^*}{\sigma_{2,g}^2} = \max_g \frac{\pi \Psi^{-1}(\sigma_{2,g}M)}{\sigma_{2,g}} \leq \frac{\pi \Psi^{-1}(\underline{\sigma}_2 M)}{\bar{\sigma}_2} \rightarrow -\infty$$

by S1(iii).

To see Assumption 2, first note for some constant C_1, C_2 ,

$$K \left(-\frac{3}{2}, \frac{1}{2}; -\frac{\mu_\delta^2}{2\sigma_\delta^2} \right) \leq C_1 \left(\frac{\mu_\delta^2}{2\sigma_\delta^2} \right)^{3/2} + C_2 \left(\frac{\mu_\delta^2}{2\sigma_\delta^2} \right)^{1/2} \lesssim C_1 \underline{\sigma}_\delta^{-3}$$

by properties of Kummer's confluent hypergeometric function (for example, 13.1.5 of [Abramowitz and Stegun, 1965](#)). Therefore,

$$M = o\left(\frac{B}{\sigma_2(\bar{\sigma}_2^{-3}G)^{1/3}}\right) = o\left(\frac{B}{\sigma_2(\kappa G)^{1/3}}\right).$$

□

Proof of Proposition 3. Assumption 1(i) and 1(ii) follow the same reasoning as in the proof of Proposition 2. For 1(iii), note that

$$\max_g \frac{\pi \pi_g^*}{\sigma_{2,g}^2} \lesssim \frac{\Psi^{-1}(\sigma_2 M)}{\sqrt{n} \sigma_2} \rightarrow -\infty.$$

To see Assumption 2, for some constant C ,

$$K\left(-\frac{3}{2}, \frac{1}{2}; -\frac{\mu_\delta^2}{2\sigma_\delta^2}\right) \leq 1 + C \left| -\frac{\mu_\delta^2}{2\sigma_\delta^2} \right| = 1 + O\left(\frac{\pi^2}{\sigma_\delta^2}\right),$$

by properties of Kummer's confluent hypergeometric function. Combining this with W2(ii) gives Assumption 2. □

Appendix E: k -Medoids Algorithm

This section states the k -medoids algorithm used in Section 5. Let (\mathbf{X}, d) be a metric space with a finite set of locations \mathbf{X} and a distant metric d . For some cluster $\mathbf{C} \subseteq \mathbf{X}$ and medoid $i \in \mathbf{X}$, define the cost to be

$$\text{cost}(\mathbf{C}, i) = \sum_{j \in \mathbf{C}} d(i, j)^2.$$

Let $\mathcal{C} = \{\mathbf{C}_g\}_{g=1}^G$ be a partition of \mathbf{X} , i.e., clustering structure. Define the total cost for \mathcal{C} with a set of medoids $\{i_{\mathbf{C}}\}_{\mathbf{C} \in \mathcal{C}}$ by summing over clusters

$$\text{total cost}(\mathcal{C}, \{i_{\mathbf{C}}\}_{\mathbf{C} \in \mathcal{C}}) = \sum_{\mathbf{C} \in \mathcal{C}} \text{cost}(\mathbf{C}, i_{\mathbf{C}}).$$

Algorithm k -medoids Clustering

Input. (\mathbf{X}, d) , G .

Procedure.

1. Initialize cluster centroids $\{i_1, \dots, i_G\} \subset \mathbf{X}_n$ arbitrarily.
2. While total cost decreases,
 - a. For each $k \leq G$, for each $j \notin \{i_1, \dots, i_G\}$ compute the cost with new medoids $\{i_1, \dots, i_{k-1}, j, i_{k+1}, \dots, i_G\}$;
 - b. Assign new medoids membership if the new set of medoids has less total cost.

Output. \mathcal{C} with $|\mathcal{C}| = G$.

References

- Abramowitz, M. and Stegun, I. A. (1965). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover Publications.
- Anderson, T. W. and Rubin, H. (1949). Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations. *The Annals of Mathematical Statistics*, 20(1):46–63.
- Andrews, D. W. K., Moreira, M. J., and Stock, J. H. (2006). Optimal Two-Sided Invariant Similar Tests for Instrumental Variables Regression. *Econometrica*, 74(3):715–752.
- Andrews, I. and Armstrong, T. B. (2017). Unbiased instrumental variables estimation under known first-stage sign. *Quantitative Economics*, 8(2):479–503.
- Andrews, I. and Mikusheva, A. (2016). Conditional Inference With a Functional Nuisance Parameter. *Econometrica*, 84(4):1571–1612.
- Andrews, I., Stock, J. H., and Sun, L. (2019). Weak Instruments in Instrumental Variables Regression: Theory and Practice. *Annual Review of Economics*, 11(1):727–753.
- Bentkus, V., Bloznelis, M., and Götze, F. (1996). A Berry-Esséen bound for student’s statistic in the non-i.i.d. case. *Journal of Theoretical Probability*, 9(3):765–796.
- Bertrand, M., Duflo, E., and Mullainathan, S. (2004). How Much Should We Trust Differences-In-Differences Estimates? *The Quarterly Journal of Economics*, 119(1):249–275.
- Bester, C. A., Conley, T. G., and Hansen, C. B. (2011). Inference with dependent data using cluster covariance estimators. *Journal of Econometrics*, 165(2):137–151.
- Cameron, A. C. and Miller, D. L. (2015). A Practitioner’s Guide to Cluster-Robust Inference. *Journal of Human Resources*, 50(2):317–372.
- Canay, I. A., Romano, J. P., and Shaikh, A. M. (2017). Randomization Tests Under an Approximate Symmetry Assumption. *Econometrica*, 85(3):1013–1030.
- Cao, J., Hansen, C., Kozbur, D., and Villacorta, L. (2019). Inference for Dependent Data with Cluster Learning. *Working paper*.
- Coibion, O., Gorodnichenko, Y., and Koustas, D. (2017). Consumption Inequality and the Frequency of Purchases. *American Economic Journal: Macroeconomics*, forthcoming.
- Dell, M. (2012). Path Dependence in Development: Evidence from the Mexican Revolution. *Working paper*.
- Deryugina, T., Heutel, G., Miller, N. H., Molitor, D., and Reif, J. (2019). The mortality and medical costs of air pollution: Evidence from changes in wind direction. *American Economic Review*, 109(12):4178–4219.

- Fama, E. F. and MacBeth, J. D. (1973). Risk, Return, and Equilibrium: Empirical Tests. *Journal of Political Economy*, 81(3):607–636.
- Ferman, B. (2019). A simple way to assess inference methods.
- Ferman, B. and Pinto, C. (2019). Inference in Differences-in-Differences with Few Treated Groups and Heteroskedasticity. *The Review of Economics and Statistics*, 101(3):452–467.
- Hagemann, A. (2019a). Permutation inference with a finite number of heterogeneous clusters. *arXiv preprint arXiv:1907.01049*.
- Hagemann, A. (2019b). Placebo inference on treatment effects when the number of clusters is small. *Journal of Econometrics*, 213(1):190–209.
- Hansen, B. E. and Lee, S. (2019). Asymptotic theory for clustered samples. *Journal of Econometrics*, 210(2):268–290.
- Hansen, C. B. (2007). Asymptotic properties of a robust variance matrix estimator for panel data when T is large. *Journal of Econometrics*, 141(2):597–620.
- Harari, M. (2020). Cities in Bad Shape: Urban Geometry in India. *American Economic Review*, 110(8):2377–2421.
- Ibragimov, R. and Müller, U. K. (2010). t-Statistic Based Correlation and Heterogeneity Robust Inference. *Journal of Business & Economic Statistics*, 28(4):453–468.
- Kaji, T. (2020). Theory of Weak Identification in Semiparametric Models.
- Kleibergen, F. (2002). Pivotal statistics for testing structural parameters in instrumental variables regression. *Econometrica*, 70(5):1781–1803.
- Lee, D. L., McCrary, J., Moreira, M. J., and Porter, J. (2020). Valid t-ratio Inference for IV.
- MacKinnon, J. G. and Webb, M. D. (2017). Wild Bootstrap Inference for Wildly Different Cluster Sizes. *Journal of Applied Econometrics*, 32(2):233–254.
- Mills, B. (2019). Inference Under First-Stage Sign Information in the Instrumental Variables Model. *Working paper*.
- Moreira, H. and Moreira, M. J. (2019). Optimal two-sided tests for instrumental variables regression with heteroskedastic and autocorrelated errors. *Journal of Econometrics*, 213(2):398–433.
- Moreira, M. J. (2003). A Conditional Likelihood Ratio Test for Structural Models. *Econometrica*, 71(4):1027–1048.
- Newey, W. K. and West, K. D. (1987). A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. *Econometrica*, 55(3):703.
- Pesaran, M. H., Pesaran, M. H., Shin, Y., and Smith, R. P. (1999). Pooled Mean Group

- Estimation of Dynamic Heterogeneous Panels. *Journal of the American Statistical Association*, 94(446):621–634.
- Pesaran, M. H. and Smith, R. (1995). Estimating long-run relationships from dynamic heterogeneous panels. *Journal of Econometrics*, 68(1):79–113.
- Staiger, D. and Stock, J. H. (1997). Instrumental Variables Regression with Weak Instruments. *Econometrica*, 65(3):557–586.
- Voinov, V. G. and Nikulin, M. S. (1993). *Unbiased Estimators and Their Applications, Vol. 1: Univariate Case*. Kluwer Academic Publishers, Dordrecht.
- Winkelbauer, A. (2012). Moments and Absolute Moments of the Normal Distribution. *arXiv preprint arXiv:1209.4340*.
- Young, A. (2019). Channeling Fisher: Randomization tests and the statistical insignificance of seemingly significant experimental results. *Quarterly Journal of Economics*, 134(2):557–598.