Competitive Credit and Present Bias: A Stochastic Discounting Approach*

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Abstract

A prominent theme in behavioural contract theory is the study of present-biased agents represented through quasi-hyperbolic discounting. In a model of competitive credit provision, we study an alternative to this framework in which the agent has a private stochastic discount factor and may overestimate the likelihood of more patient values. Agent preferences, however, are time-consistent. While a limiting case of our model corresponds to a "fully naive" agent in work on quasi-hyperbolic discounting, another case is where the agent has correct beliefs about future discounting. In equilibrium, the agent selects options with earlier consumption in case of less patient discount factor realisations, but is penalised by receiving worse terms. Our model thus accounts for an important feature of equilibrium contracts identified in Heidhues and Kőszegi (2010). Unlike Heidhues and Kőszegi, our framework often predicts excessively backloaded consumption, including when the agent holds correct beliefs about future discounting.

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1 Introduction

A growing literature in behavioural contract theory studies present-biased agents, with agents often either naive or partially naive about their future behaviour.[1] Present bias is modeled using quasi-hyperbolic discounting following Laibson (1997) (who draws in turn on Phelps and Pollak, 1968). Agents then have time-inconsistent preferences: an agent is a collection of distinct "selves" which favour present over future utility. Naivety means that agents mispredict the preferences of future selves. In effect, they overestimate future patience.

In this paper, we suggest an alternative though complementary approach. We propose agents with time-consistent preferences, but where discounting is stochastic. While agents anticipate stochastic discount factors, they may hold misspecified beliefs and overestimate future patience. Behaviour is dynamically consistent in the sense that agents plan contingent on future discounting realisations and follow through on these plans. Nonetheless, a fully naive present-biased agent as studied in existing work on quasi-hyperbolic discounting arises as a particular limiting case in which beliefs over discounting are degenerate. Another important case is an agent with correct, non-degenerate beliefs. We apply this framework to competitive credit markets, building on Heidhues and Kőszegi (2010, henceforth HK) and Gottlieb and Zhang (2021, henceforth GZ).

We follow closely the original motivation of HK, who sought to understand how present bias of the agent interacts with competitive credit markets, although our formal model is closest to GZ. HK pursues two related sets of results: positive predictions about features of credit contracts we might expect to see, and normative findings about welfare and contractual distortions. A central positive prediction is that equilibrium credit contracts offer the option of early repayment on good terms, while present-biased agents ultimately repay later and are penalised for doing so. A central normative finding is that agents borrow too much and repay too late, as evaluated under the preferences of a long-run self that does not exhibit present bias.

Our analysis addresses both positive and normative dimensions. On the positive side, our model can capture similar contractual features, and in particular penalties for early consumption. On the normative side, time-consistent preferences seem to call for a different welfare benchmark. As explained below, we then find that consumption often occurs too late rather than too early. These implications arise not only when agent beliefs about future discounting are misspecified, but also when they are correct.

Model. Our model features an agent and at least two firms that provide credit. The agent first learns a discount factor that determines how consumption in the initial period is valued relative to the future. At this point, firms compete by offering dynamic mechanisms that determine agent consumption over time. The agent chooses a mechanism to accept and then both parties are

[1]Papers that include the study of such naive or partially naive agents include O'Donoghue and Rabin (1999), DellaVigna and Malmendier (2004), Gottlieb (2008), Heidhues and Kőszegi (2010), Murooka and Schwarz (2018), Bubb and Warren (2020), Gottlieb and Zhang (2021), Citanna and Siconolfi (2022), Heidhues et al. (2024), Sulka (2024), and Englmaier et al. (2026).

fully committed. The agent learns a new discount factor in each period, taken to be i.i.d., again determining how future consumption is valued relative to the present. After learning the discount factor in a period, the agent sends a message to the mechanism and consumption is determined. In equilibrium, the agent's strategy results in a process for consumption that satisfies an optimality condition. Namely, it maximises the agent's discounted expected payoff at the initial date given (possibly misspecified) agent beliefs. This is subject to incentive compatibility constraints and a non-negative profit condition for the firm, where expected profits are evaluated according to firm beliefs which we view as correct and which therefore potentially differ from the agent's.

Cases closer to the existing literature. We begin in Section 3 by studying the setting that is closest both in terms of the environment and the analysis to the earlier work. The agent learns a discount factor in each period that he believes to be drawn from two possible values ("patient" and "impatient"). Firms, however, correctly believe that the agent always draws the impatient value. Our view that firms hold correct beliefs is consistent with much of the behavioural industrial organisation literature (e.g., Eliaz and Spiegler, 2006, Grubb, 2009). The agent therefore persistently overestimates the likelihood of being patient. In the limiting case where the agent believes he will be patient for sure, our agent has the same preferences and beliefs as in the quasi-hyperbolic discounting model of GZ where the agent is "fully naive".

Our two-type model predicts contracts similar to those in GZ and HK: in equilibrium, firms offer the agent the option of highly backloaded consumption, or more front-loaded consumption but on worse terms. The agent evaluates firm offers on the basis that backloaded consumption will be chosen with positive probability, but ultimately always chooses the more front-loaded consumption. The proximity of our findings to the earlier literature (HK and GZ) suggests some initial observations about what is required in order for models to obtain the earlier predictions on contracts and behaviour. Although our model generates the same basic behaviour, preferences are time consistent because, at each date, the agent agrees on the preference ordering conditional on the realisation of the state (i.e., discount factors). Behaviour is also dynamically consistent: the agent makes a plan of action for each state and follows through on the plan, although beliefs about the likelihood of each state are incorrect. Given that the agent assigns positive probability to the low discount factor in each period, the realisation of this discount factor of itself does not suggest the possibility that the discounting process is misspecified.[2] This compares with the existing literature which views contractual form as driven by quasi-hyperbolic discounting and (possibly partial) naivety. There, agents are a sequence of "multiple selves" who would like to maintain more patient behaviour in the future, but are tempted to make decisions favouring the present. Naivety (including partial naivety) generally means that the agent has a point belief that the agent will discount the future less strongly

[2]Here, we have in mind the perspective of Galperti (2019), in which observations that a decision maker deems impossible leads to abandonment of the maintained model. In our setting, since the agent assigns positive probability to low discount factors, observing them would not lead the agent to abandon his understanding of the stochastic process governing discounting. This contrasts with models of quasi-hyperbolic discounting by a naive agent, where the agent repeatedly encounters discounting preferences that differ from his prior understanding.

than he actually does.[3]

As noted, our alternative conceptualisation suggests a different approach to welfare, which we use throughout the paper. Given time-consistent preferences, there is a single decision-making self in our model. While there is then no decision about which self to prioritise in welfare analysis, there remains the question (when the agent holds misspecified beliefs) of the appropriate choice of beliefs. We follow other industrial organisation studies with misspecified consumer beliefs by measuring welfare against firm beliefs which are assumed correct. As Grubb (2009) argues (footnote 10), this approach seems easy to rationalise on the basis of a firm that has the benefit of encountering many similar consumers and thus understanding their preferences, while individual consumers lack this advantage. Then equilibrum consumption is excessively backloaded (Proposition 2), and in particular it is inefficient as the number of periods under examination becomes large (Proposition 3). The latter claim stands in contrast to GZ (Theorem 1) and Citanna et al. (2023, Theorem 1), which find approximately efficient timing of consumption for long horizons. The reason relates to the choice of efficient benchmark: efficiency is judged against discount factor realisations equal to the low value, which are viewed as the true realisations. Our finding is perhaps intuitive: since firms' contracts cater to an agent that believes he will be more patient in future than he actually is, equilibrium consumption turns out to be excessively backloaded (we find that this intuition also carries over to richer settings – see below).

Richer settings. Section 4 then extends the model to settings with multiple values of the discount factor, and where firm and agent beliefs have full support over these values. We assume that agents weakly overestimate their degree of patience, now nesting the case of correct beliefs. We explain that it is useful to view equilibrium mechanisms as shifting utility towards more patient agent types – indeed incentive constraints pointing from lower to higher discount factors bind (Lemma 4). As a consequence, we find quite generally that more impatient type realisations receive more front-loaded consumption but on worse terms, where the terms of the contract are judged against the firms' rate of time preference (Corollary 2). This includes the case where the agent has correct beliefs. The result contrasts with HK's (p 2300) discussion of their alternative "neoclassical" model, which they argue would yield "essentially the opposite qualitative contract features" to the ones predicted by the model with quasi-hyperbolic discounting.

The similarity in the form of contracts for the setting with correct agent beliefs is a key motivation for studying our framework. While a limiting case of our model corresponds to a fully-naive agent in existing work on quasi-hyperbolic discounting (see GZ), some of the key forces in the model are preserved as we move towards a model where agent beliefs are correct. We are therefore better able to evaluate which behaviours are truly driven by departures from neoclassical assumptions in the existing literature. For instance, we discuss a possible difference of our model with correct beliefs

[3] See, however, Citanna and Siconolfi (2022), where agent preferences are time-inconsistent, but where beliefs about future self-control problems are non-degenerate. We expand on the comparison to this paper in the literature review below.

and many discounting values (i.e., many types): penalties for early consumption can be smaller as the agent smoothly adjusts contractual terms in response to lower discount factor realisations.

We also show (in Proposition 6) that excessive backloading of consumption can arise, including when agent beliefs are correct. The reason for this finding is that backloading of consumption relaxes upward incentive constraints and therefore allows higher payoffs to be delivered in case of higher discount factor realisations. Delivering higher payoffs for higher discount factors is in fact something called for by efficiency (see Proposition 4). In this sense, equilibrium mechanisms can feature distortions in two directions: too little consumption after high discount factors, but also consumption that is too backloaded. The use of backloading of consumption to relax upward incentive constraints is similar to the backloading of the option that the agent ultimately does not choose in the baseline models of GZ and HK. The difference when the agent has correct beliefs is that the agent is correct in anticipating the probability of selecting options with backloaded consumption.

Although the version of the model with correct agent beliefs also features distortions, the implications for the role of regulation are quite different from existing work.[4] Here, the appropriate welfare criterion is unambiguous. Moreover, the agent's expected payoffs are maximal, subject to firms breaking even and to incentive compatibility. Since regulatory intervention would be subject to the same incentive and firm participation constraints, a regulator cannot improve agent welfare relative to the laissez-faire outcome.

Finally, we note that our model with stochastic discount factors can help to understand the role of agent misprediction of future discounting quite generally. We derive (in Proposition 7) inverse Euler equations that characterise equlibrium and efficient consumption, using a perturbation analysis related to Rogerson (1985). In the case of logarithmic utility (see Corollary 3), this has an interpretation in terms of expected consumption. When agent beliefs are correct, the rate of growth in expected consumption is the same in equilibrium as for the efficient benchmark. Otherwise, when the agent overestimates patience, expected consumption grows more quickly in equilibrium. The intuition is that equilibrium mechanisms pander to more patient discount factor realisations, because the agent mistakenly places excessive weight on the probability of these realisations.

1.1 Further literature

This paper is related to the literature on dynamic mechanism design. While many papers (see Pavan, Segal, and Toikka, 2014) consider persistent types, we specialise here to the case of i.i.d. types. This is common, for instance, in early literature on incentive-compatible intertemporal lending and insurance, including Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992). Our work differs from these earlier papers especially due to private information on discount factors rather than income/endowments, and because we allow agent beliefs to be misspecified.

[4] See HK and Citanna et al. (2025) who consider regulatory interventions in competitive credit markets with present-biased agents.

Work involving stochastic discount factors is quite common in macroeconomics and finance – see Stachurski and Zhang (2021) for a review. In our model of the agent, discount factors can be interpreted as subjective states, in the sense of Kreps (1979) and Dekel et al. (2001). Gul and Pesendorfer (2004) and Higashi et al. (2009) provide applications of this framework to dynamic decision problems. Notably, Higashi et al. provide a decision theoretic foundation for modeling an agent with an i.i.d. stochastic discount factor, as in our paper.

Our application to dynamic credit markets warrants further discussion of Citanna and Siconolfi (2022). This paper, like ours, allows the agent to have non-degenerate but possibly misspecified beliefs about future discounting. However, key differences are that the agent has time-inconsistent preferences and (as the authors remark, p 11) there is disagreement between the preferences of each self and those anticipated for future selves (ruling out the case often descibed in the literature as "fully naive", see for instance GZ, p 797). Citanna and Siconolfi show that, unlike in the model of GZ, inefficiencies need not vanish asymptotically. They show that there may be too little or too much consumption at the initial date ("underborrowing" or "overborrowing"). Our findings on asymptotic inefficiency in Section 3 are driven by different considerations, especially related to the choice of welfare benchmark. Our results on excessive backloading of consumption go beyond statements about consumption at the initial date.

2 General Model

Environment and payoffs. A number $J \geq 2$ of firms, $j = 1, 2, \dots, J$, offer long-term lending contracts to a single representative agent over a finite number $T \geq 3$ periods, $t = 1, 2, \dots, T$. The firm whose contract is accepted can be viewed as making payments to and receiving payments from the agent, with the agent able to make payments out of available income that he receives as an endowment in each period. This arrangement can equally be represented as the contract determining a non-negative agent consumption c_t in period t, and the lender being claimant on all remaining income. Let $R \geq 1$ be the gross interest rate of firms (all firms face the same interest rate). Then we let $I_T > 0$ denote a firm's date-1 value of the agent's deterministic income stream when the horizon length is T. It is natural to view I_T as increasing in T and converging to some $I_\infty < \infty$ with T. A firm's profit in a setting with horizon length T is

$$I_T - \sum_{t=1}^{T} \frac{c_t}{R^{t-1}}.$$

We sometimes refer to the date-t NPV of a consumption stream $(c_\tau)_{\tau=t}^{T}$ as the firm's cost of the agent's consumption.

The agent enjoys a payoff from a consumption stream $(c_t)_{t=1}^{T}$ which is determined through a per-period utility function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ and the realisation of a sequence of discount factors

$(\delta_t)_{t=1}^{T-1}$, with $\delta_t > 0$ for each t. In particular, the agent's realised utility is

$$u(c_1) + \sum_{t=2}^{T} \Pi_{s=1}^{t-1} \delta_s u(c_t).$$

We assume that u is continuous, strictly increasing, strictly concave, twice continuously differentiable on $(0, \infty)$, and that $\lim_{c\to\infty} u(c) = \infty$ while $\lim_{c\to\infty} u'(c) = 0$. In Section 3, we impose that $u(0) = 0$, so there is a finite lower bound on the utility that may be obtained (that the minimum utility is zero is a convenient normalisation, without loss of generality). Our motivation here is to bring our environment as close as possible to HK and especially GZ, to show that our framework captures similar economic logic and to highlight differences. In Section 4, we then allow also the case where $\lim_{c\downarrow 0} u(c) = -\infty$ (and in such cases we may as well specify $u(0) = -\infty$). This will have the advantage of guaranteeing that there are no corner solutions.

Agent discount factors (types). Our general model of agent preferences (although markedly specialised in Section 3) is as follows. In each period t, the agent privately learns his discount factor δ_t that determines how future payoffs are discounted relative to the present. There is a finite number $N \geq 2$ of possible values for δ_t in each period (the agent's date-t "type"), given by $\delta^{(n)}$ for $n = 1, \ldots, N$. Here, $\delta^{(n)}$ is strictly increasing in n. Let Δ be the collection of the N possible types. The value δ_t is the realisation of a random variable $\tilde{\delta}_t$ that is commonly known to be distributed i.i.d. across periods. Firms correctly perceive the probability of type $\delta^{(n)}$ to be $p_n \geq 0$ whereas the agent perceives the probability to be $q_n \geq 0$ (with agent beliefs known to firms). Naturally, $\sum_{n=1}^{N} p_n = \sum_{n=1}^{N} q_n = 1$. We have in mind an agent who believes all types are possible and so assume $q_n > 0$ for all n (we refer to this requirement as agent beliefs having "full support"). We assume that the agent is "more optimistic" than the firms about patience in the sense of first-order stochastic dominance: i.e., for all m, $\sum_{n=1}^{m} p_n \geq \sum_{n=1}^{m} q_n$. We say the agent is "strictly more optimistic" if he is more optimistic and the two distributions differ. The reason we focus on cases where the agent is more optimistic than the firms is consistency with the focus of the literature on present bias, the agent believes he will on average act more patiently in future than he actually does. However, because we allow that $p_n = q_n$ for all n, we nest the case with common (and hence correct) beliefs.

A history of types up to date $t \leq T - 1$ is denoted $H^t \in \Delta^t$. We identify with any history H^T the $T-1$-tuple corresponding to the first $T - 1$ realisations. In general, we write $H_t = \delta_t$ for the t^{th} element of the history and write $H_T = \delta_T = \emptyset$, understanding that $(H_1, H_2, \ldots, H_{T-1}, \emptyset) = (H_1, H_2, \ldots, H_{T-1})$. We write for $\tau \geq t$, $H_t^\tau = (H_t, H_{t+1}, \ldots, H_\tau)$ (and again $H_t^T = H_t^{T-1}$, the restriction to the first $T - t$ elements).

Feasible mechanisms.[5] Each firm j makes a one-time simultaneous initial move by offering a

[5] In defining the available mechanisms, and subsequently restricting equilibria, we ensure that equilibria are characterised through optimisation problems that have a similar structure to the ones in HK and GZ. In particular, the agent's date-1 expected payoff is maximised (for each initial type $\delta_1 = \delta^{(n)}$) subject to incentive constraints and a firm

dynamic mechanism M_j. The agent can accept only one mechanism, and if he accepts the mechanism of firm j, then both are committed to it. The agent sends a message in each period and enjoys the resulting consumption. We assume that contractual offers are private to the agent and not contractible/verifiable, so mechanisms that are not accepted play no further role.[6] We also restrict attention to deterministic mechanisms with a finite set of messages in each period. While we believe these latter restrictions can be relaxed without affecting the results, they simplify the analysis and seem justifiable on grounds of realism.

Equilibrium restrictions. We restrict attention to equilibria in which the agent plays a pure message strategy and breaks indifferences in favour of the firm: If, at any history, the agent is indifferent over a set of messages in the mechanism he has accepted from a given firm, then he selects a message which maximises the firm's continuation payoff (expected NPV of profits). We also restrict attention to equilibria in which firms choose pure strategies.

Given the restriction of feasible mechanisms and the equilibrium restrictions above, there is no loss of generality in restricting attention to equilibria involving only deterministic incentive-compatible direct mechanisms, with the set of messages in each period $t \leq T-1$ equal to Δ. By this, we mean that, for any equilibrium in which a firm or firms offer a mechanism that is not direct, there is another equilibrium with the same distribution over outcomes where all offered mechanisms are direct.[7] A deterministic direct mechanism M^D is simply (though slightly abusively) a sequence of functions $(c_t)_{t=1}^T$, with $c_t : \Delta^t \to \mathbb{R}_+$ for $t \leq T-1$ and $c_T : \Delta^{T-1} \to \mathbb{R}_+$. Since our interest is in direct mechanisms in which the agent reports truthfully in equilibrium, we narrow the usual definition of a direct mechanism, requiring the following. For a mechanism to be called direct, at any history such that the agent is indifferent over messages, truthful revelation must be a report that maximises firm profits.[8] It is then without loss of generality to restrict attention to equilibria in which the agent reports truthfully in any incentive-compatible direct mechanism.

Timing. The timing of the game can then be summarised as follows. In the initial period:

1. First, the agent learns his date-1 type $\delta_1 = \delta^{(n)}$, determining the discounting of the future relative to date 1.

break-even condition. This has the apparent benefit of aiding comparability with HK and GZ.

[6]The assumption ensures, in particular, that the terms of the contract under a given contractual offer do not depend on the set of other contractual offers he receives. Therefore, the payoffs available to the agent from a given offer will not depend on the other offers he receives.

[7]For all j and n, any collection of mechanisms $(M_1, \ldots, M_J)$, let $\sigma_{j,n}^{orig}(M_1, \ldots, M_J)$ denote the probability the agent of type $\delta_1 = \delta^{(n)}$ accepts the mechanism of Firm j in the original equilibrium. Supposing Firm 1 chooses some (possibly indirect) mechanism M_1^{orig} in the original equilibrium, define a new equilibrium in which Firm 1 chooses the outcome equivalent direct mechanism M_1^{new} (i.e., it induces, for each agent type history, the same values of consumption given that the agent reports truthfully), while all other firms' choices remain unchanged. Then define a modified agent strategy such that, for all j and n, and all $(M_1, \ldots, M_J)$, $\sigma_{j,n}^{new}(M_1, \ldots, M_J) = \sigma_{j,n}^{orig}(M_1, \ldots, M_J)$ whenever $M_1 \neq M_1^{new}$, while $\sigma_{j,n}^{new}(M_1^{new}, \ldots, M_J) = \sigma_{j,n}^{orig}\left(M_1^{orig}, \ldots, M_J\right)$. Moreover, the agent reports truthfully in Firm 1's mechanism M_1^{new}, while his reporting strategy is otherwise unchanged. Then we have a new equilibrium in which Firm 1's mechanism is direct. The same argument can be applied iteratively to all firms, establishing the claim.

[8]Put differently, if there are multiple optimal reporting strategies for the agent, truth-telling must maximise firm profits among them.

2. Next, the J firms simultaneously offer a mechanism subject to the feasibility requirements.

3. The agent of realised initial type $\delta^{(n)}$ chooses among the J offered mechanisms, accepting at most one of them, and sends an initial message.

4. The mechanism determines the agent's date-1 consumption.

In a generic period $t \geq 2$, the mechanism has already been determined and a history of reports H^{t-1} have been made. The timing is then as follows:

1. The agent learns his date t type $\delta_t \in \Delta$ (unless $t = T$, in which case he learns nothing).

2. The agent sends a message to the mechanism.

3. The mechanism provides the agent with his consumption.

Equilibrium concept. Formally, the equilibrium notion we apply is Perfect Bayesian Equilibrium, but since firms make an initial one-shot move, posterior beliefs never need to be calculated (the updating according to Bayes' rule requirement is trivially satisfied). Recall that we are restricting to equilibria where:

- The agent chooses a pure message strategy and breaks indifferences among messages in a firm's mechanism by selecting the most profitable.
- The agent reports truthfully in any incentive-compatible direct mechanism.
- Firms make pure-strategy offers of deterministic incentive-compatible direct mechanisms.

2.1 Discussion of relationship to quasi-hyperbolic discounting

An important motivation for studying models with present-biased agents is apparent "choice reversals" of the kind described by Ainslie (1991, p 334): "a majority of adults report that they would rather have \$50 immediately than \$100 in 2 years, but almost no one prefers \$50 in 4 years over \$100 in 6 years, even though this is the same choice seen at 4 years' greater distance".[9] These choice reversals can be explained using quasi-hyperbolic discounting, in which the agent is viewed as maximising preferences at each date t given by

$$u(c_t) + \beta\bar{\delta} \sum_{\tau > t} \bar{\delta}^{\tau-1-t} u(c_\tau)$$

where $\bar{\delta} \in (0,1)$ represents long-run discounting and $\beta \in (0,1)$ represents present bias.

But quasi-hyperbolic discounting is not the only model of preferences that can account for the choice reversals above, and in fact our model can do so. To see this, consider our model with two

[9] Ainslie's example comes from research by Ainslie and Haendel (1983).

discount factors ($N = 2$), and with $\delta^{(1)} = 0.4$ and $\delta^{(2)} = 0.9$. Suppose initially that $q_2 = p_2 = 1/4$ so that $q_1 = p_1 = 3/4$. Suppose the agent, having learned his discount factor δ_t at date t, has to choose between utility of consumption equal to 50 at date t or utility equal to 100 at date $t + 1$ (Problem A). From the same perspective (i.e., at date t and given knowledge of δ_t), for some $s \geq 1$, the agent has to choose between utility of 50 at date $t+s$ or utility of 100 at date $t+s+1$ (Problem B). Then, the agent chooses the immediate reward 50 in Problem A when $\delta_t = 0.4$ and hence with probability 3/4. But he always chooses the later reward of 100 in Problem B. The possibility that the agent overestimates future patience strengthens the reversal in choice: e.g. if instead $p_2 = 0$ and $p_1 = 1$, then the agent certainly has the low discount factor at date t ($\delta_t = 0.4$) and so the agent chooses the immediate reward for sure in Problem A, while it does not affect his choice in Problem B. Conversely, increasing q_2 strengthens the agent's preference for the delayed reward in Problem B.

It helps to understand our model of preferences through the lens of Lu and Saito (2018), which provides an axiomatic foundation for a "Random Discounting Representation" underlying consumer choice. They focus on a one-shot "ex ante" decision problem, with stochastic choice over possibly random consumption streams over an infinite horizon. Extending our model of preferences to the infinite horizon, and considering choice at any fixed date t, our model of agent preferences generates stochastic choice that is representable by random discounting in the sense of Lu and Saito.

For choice at date t, we need to define a distribution over discount factors $D(\tau)$, for $\tau \geq t$, where $D(\tau)$ denotes the total discounting of date-τ utility and $D(t) = 1$. For $i \in \{1, 2\}$, let $D^{(i)}(t) = 1$; and, for $s \geq 1$, let

$$D^{(i)}(t+s) = \beta^{(i)} \bar{\delta}^s,$$

where $\bar{\delta} = q_1\delta^{(1)} + q_2\delta^{(2)}$ and $\beta^{(i)} = \frac{\delta^{(i)}}{\bar{\delta}}$. Let $D^{(i)}$ be the discounting that occurs with probability p_i. The date-t choice behaviour of our agent – restricted to choices over consumption streams as in Lu and Saito (2018) – is generated by this stochastic discounting representation.

The representation here is reminsicent of what Lu and Saito define (p 787) as a "random quasi-hyperbolic representation", involving a distribution over deterministic quasi-hyperbolic discounting functions. However, if $p_2 \in (0, 1)$, then their definition is not satisfied, as $\beta^{(2)} > 1$. This conclusion extends immediately to the case of $N > 2$ discount factors when each occurs with positive probability (i.e., $p_n > 0$ for all n). In contrast, in Section 3, we assume that $p_2 = 0$, and in this case date-t choice *is* representable by deterministic quasi-hyperbolic discounting as $\beta^{(1)} < 1$.

Relative to the above discussion, it is worth emphasizing that the focus of Lu and Saito (2018) is one-shot choice over streams of (possibly random) consumption. In our contracting model, by contrast, the agent makes decisions that affect future choices under later discount factor realisations. An important determinant of this dynamic setting is the agent's beliefs about how discounting will evolve and affect future behaviour. Therefore, agent behaviour in our model would generally differ if beliefs about future discounting were instead taken directly from the stochastic discounting representation above. For this reason, even when the stochastic discounting representation introduced above is quasi-hyperbolic (as in Section 3), our model differs from quasi-hyperbolic models in the contracting literature.

Finally, consider $N = 2$ and suppose, contrary to what we have assumed, that $q_2 = 1$. Then if, in addition, $p_1 = 1$, agent preferences and beliefs coincide with those of the "fully naive" agent in the baseline environment of GZ. Therefore, quasi-hyperbolic discounting in the sense understood by the contracting literature does emerge as a limiting case of our model.

3 Agent with degenerate impatience

The central objective of this section is to argue that our model with stochastic discount factors can accommodate a logic close to the one articulated in HK and GZ: the agent is offered the possibility of strongly backloaded consumption plans which he may believe he will likely use, but it turns out that he in fact opts for more front-loaded options. The difference in our setting is that the agent is uncertain about future levels of patience and understands that both patient and impatient future states are possible. In our model, there appears to be no obvious notion of the preferences of a more patient "long-run self", and we discuss the implications of using realised preferences (equivalently, the firm's beliefs which are understood to be correct) as the welfare benchmark.

Our analysis in this section makes several restrictions of the model to bring it closest to the earlier work. There are two types: $N = 2$. The agent is certainly impatient: $p_1 = 1$. As noted above, agent utility satisfies $u(0) = 0$. (Other restrictions on the environment and on equilibria are as above.)

In this setting, firms compete for an agent whose initial type δ_1 is certainly $\delta^{(1)}$. There is then no loss of generality in defining direct mechanisms to omit the date-1 report. The possible message histories in a direct mechanism are then $\mathcal{H}^R$, all t-tuples $(\delta_1, \delta_2, ..., \delta_t)$, $t \leq T-1$, such that $\delta_1 = \delta^{(1)}$.[10]

Suppose the maximum expected discounted payoff that can be delivered to the agent at date 1 in an incentive-compatible direct mechanism that generates non-negative profits for the firm exists and is finite (as will be shown below – see Lemma 2). Then, by a familiar undercutting argument, in any equilibrium, the agent must accept with probability one a mechanism that solves this optimisation. Moreover, an equilibrium exists. A full proof follows the arguments in the proof of Proposition 5.[11]

Understanding equilibrium therefore reduces to solving the problem (call it Problem I) of max-

[10] Noting that agent beliefs have full support, mechanisms are thus assumed to be defined at all type histories the agent believes possible.

[11] The key arguments are as follows. Maintaining the equilibrium restrictions in Section 2, now with direct mechanisms omitting date-1 reports, it is easy to see that no firm can earn strictly positive equilibrium profits. Indeed, the firm earning the lowest profit could deviate by offering a mechanism that is accepted with positive probability, but with slightly higher date-1 consumption. The agent must then earn the maximum expected payoff in an incentive-compatible direct mechanism generating non-negative profits for the firm (see, e.g., Problem I below). Otherwise, given continuity of the value to this problem in the income I_T (from Lemma 2 and the Theorem of the Maximum), a firm can deviate to a mechanism that provides the agent with an expected payoff slightly smaller than the maximised value, while earning positive expected profit.

imising by choice of $c_t\left(H^t\right) \geq 0$, $1 \leq t \leq T$ and $H^t \in \mathcal{H}^R$, the agent expected utility

$$\mathbb{E}_A\left[u\left(c_1\left(\delta^{(1)}\right)\right)+\delta^{(1)}\sum_{t=2}^{T}\Pi_{s=2}^{t-1}\tilde{\delta}_s u\left(c_t\left(\delta^{(1)},\tilde{\delta}_2,\ldots,\tilde{\delta}_t\right)\right)\right]$$

(where we follow the convention that $\Pi_{s=2}^{1}\delta_s \equiv 1$) subject to agent incentive compatibility constraints and to the firm break-even condition (introduced formally below).[12]

Now consider the agent incentive constraints. These are constraints at dates $t \in \{2,\ldots,T-1\}$ at histories in $H^{t-1} \in \mathcal{H}^R$. The first kind of constraint ensures that high types do not imitate low types:

$$\begin{aligned}
&u\left(c_t\left(H^{t-1},\delta^{(2)}\right)\right)+\delta^{(2)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s u\left(c_\tau\left(H^{t-1},\delta^{(2)},\tilde{H}_{t+1}^{\tau}\right)\right)\right]\\
\geq &u\left(c_t\left(H^{t-1},\delta^{(1)}\right)\right)+\delta^{(2)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s u\left(c_\tau\left(H^{t-1},\delta^{(1)},\tilde{H}_{t+1}^{\tau}\right)\right)\right].
\end{aligned}$$

The second kind ensures that low types do not imitate high types:

$$\begin{aligned}
&u\left(c_t\left(H^{t-1},\delta^{(1)}\right)\right)+\delta^{(1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s u\left(c_\tau\left(H^{t-1},\delta^{(1)},\tilde{H}_{t+1}^{\tau}\right)\right)\right]\\
\geq &u\left(c_t\left(H^{t-1},\delta^{(2)}\right)\right)+\delta^{(1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s u\left(c_\tau\left(H^{t-1},\delta^{(2)},\tilde{H}_{t+1}^{\tau}\right)\right)\right].
\end{aligned} \tag{1}$$

These incentive constraints are necessary for an incentive-compatible direct mechanism.[13] That they are also sufficient follows a standard backward induction argument. If the agent knows he will be truthful at all histories at dates $t+1,\ldots,T-1$, then the specified incentive constraints ensure he is also willing to be truthful at date t.

Now, for $1 \leq t \leq T-1$, let $L^t = \left(\delta^{(1)},\delta^{(1)},\ldots,\delta^{(1)}\right)$ be the $t-$tuple comprising only elements equal to $\delta^{(1)}$. Let $L^T = L^{T-1}$. Because the firm has degenerate beliefs on histories $H^t = L^t$, the non-negative profit constraint is

$$\sum_{t=1}^{T}\frac{c_t\left(L^t\right)}{R^{t-1}} \leq I_T.$$

Following the approach also to be used in Section 4, it convenient to state Problem I as maximising instead over the utilities, i.e. the values $v_t\left(H^t\right) \equiv u\left(c_t\left(H^t\right)\right) \geq u(0)$, $1 \leq t \leq T$, where $H^t \in \mathcal{H}^R$. Letting $\phi = u^{-1}$ denote the inverse of u, a direct mechanism can now be written as the

[12]The expectation $\mathbb{E}_A$ is the expectation given the agent beliefs, i.e. placing probability q_n on each type $\delta^{(n)}$.

[13]Noting that agent beliefs have full support, an incentive-compatible direct mechanism is defined to provide incentives for truth-telling at all histories the agent believes possible. From the usual replication argument, this is without loss of generality.

collection of functions $M^D = (\phi(v_t))_{t=1}^T$. Since a mechanism can be represented by the utilities it provides, we will generally also write simply $M^D = (v_t)_{t=1}^T$.

The updated (but equivalent) problem is now Problem II, where the objective is

$$\mathbb{E}_A \left[v_1 \left(\delta^{(1)} \right) + \delta^{(1)} \sum_{t=2}^{T} \Pi_{s=2}^{t-1} \tilde{\delta}_s v_t \left(\delta^{(1)}, \tilde{\delta}_2, \ldots, \tilde{\delta}_t \right) \right]$$

and the incentive constraints are (all $2 \leq t \leq T-1$, all H^{t-1})

$$\begin{aligned} & v_t \left(H^{t-1}, \delta^{(2)} \right) + \delta^{(2)} \mathbb{E}_A \left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau \left(H^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau} \right) \right] \\ \geq & v_t \left(H^{t-1}, \delta^{(1)} \right) + \delta^{(2)} \mathbb{E}_A \left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau \left(H^{t-1}, \delta^{(1)}, \tilde{H}_{t+1}^{\tau} \right) \right], \end{aligned} \tag{2}$$

and

$$\begin{aligned} & v_t \left(H^{t-1}, \delta^{(1)} \right) + \delta^{(1)} \mathbb{E}_A \left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau \left(H^{t-1}, \delta^{(1)}, \tilde{H}_{t+1}^{\tau} \right) \right] \\ \geq & v_t \left(H^{t-1}, \delta^{(2)} \right) + \delta^{(1)} \mathbb{E}_A \left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau \left(H^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau} \right) \right]. \end{aligned} \tag{3}$$

The firm's non-negative profit constraint is

$$\sum_{t=1}^{T} \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}} \leq I_T.$$

In relation to this problem, it is worth noting that ϕ inherits convenient properties from u. It is strictly convex, strictly increasing and twice continuously differentiable on $\mathbb{R}_+$, with $\lim_{v \to \infty} \phi'(v) = \infty$.

In solving the above problem, a key observation is that we can focus on mechanisms such that (3) holds with equality. In particular, we show the following.

Lemma 1. *Consider any direct mechanism $M^D = (v_t)_{t=1}^T$ that, if chosen by the agent, results in non-negative profits for the firm. Then there is another mechanism $\bar{M}^D = (\bar{v}_t)_{t=1}^T$ that gives the firm non-negative profits in which, for all $t \geq 2$ and all H^{t-1}, (3) holds as equality. Moreover, the agent's expected payoff is no lower than under M^D.*

The perturbation to the mechanism M^D to arrive at $\bar{M}^D$ involves decreasing the agent's utility $v_t\left(H^{t-1}, \delta^{(1)}\right)$ and increasing $v_t\left(H^{t-1}, \delta^{(2)}\right)$ whenever (3) is slack at H^{t-1} $(2 \leq t \leq T-1)$. This is done in such a way as to leave agent expected payoffs, from the perspective of dates before t, unchanged. We then find that firm profits increase if H^{t-1} is the sequence of $\delta^{(1)}$ realisations and are unchanged otherwise. The increase in profits can be transferred back to the agent through an increase in $v_1(H_1)$. Hence, in any solution to Problem II, the inequality (3) must hold as equality when $H^{t-1} = L^{t-1}$ (for $2 \leq t \leq T-1$).

Our next result writes the objective in Problem II after substituting in the incentive constraints that hold with equality.

Lemma 2. *Any solution $M^D = (v_t)_{t=1}^{T}$ to Problem II must be such that, for all $t \in \{2, 3, \ldots, T-1\}$, $v_t\left(L^{t-1}, \delta^{(2)}\right) = u(0)$. Moreover, for $t \in \{1, 2, \ldots, T\}$, the values $v_t\left(L^t\right)$ are the unique maximisers of the expression*

$$\sum_{t=1}^{T-1} \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t\left(L^t\right) + \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} v_T\left(L^T\right) \tag{4}$$

subject to the firm breaking even. The maximised value represents the agent's equilibrium payoff. Moreover, any mechanism the agent accepts in equilibrium yields him the aforementioned utilities at the specified histories (i.e., respectively, at $\left(L^{t-1}, \delta^{(2)}\right)$ with $t \in \{2, 3, \ldots, T-1\}$, and at L^t with $t \in \{1, 2, \ldots, T\}$).

The result can be compared to the findings of GZ, especially their Lemma 2 (which applies given that u is unbounded, see Citanna et al. (2023)), which indeed follows from a similar argument. Although we deliberately exclude this case, our arguments and findings apply also when $q_2 = 1$:[14] indeed, agent preferences and beliefs then coincide with those of the "fully naive" agent in GZ. In this sense, our findings can be viewed as generalising GZ's for the case where the agent is fully naive.

It is worth highlighting that the values $v_t\left(L^t\right)$ found in Lemma 2 vary continuously in q_2 by an application of the Theorem of the Maximum, and hence our predictions for the realised agent utilities approach the ones predicted by GZ for the limiting case as $q_2 \to 1$. Moreover, the mechanism has the same structure in which, at histories $\left(L^{t-1}\right)$ with $t \in \{2, 3, \ldots, T-1\}$, the agent has the option of highly backloaded consmption, with zero consumption in the present period. This shows how our model can approximate the behavioural predictions of GZ, but in a setting where the agent assigns positive probability to being impatient.

The form of the agent's payoff derived in Lemma 2, suggests that the timing of consumption will often be close to a benchmark contract that maximises the agent's payoff subject to the firm breaking even, and where the agent's commonly known discount factor is deterministic and constant at $q_1\delta^{(1)} + q_2\delta^{(2)}$. That is, the timing of consumption is similar in equilibria of our model with mis-specified beliefs to what would be predicted in a simpler setting with correct beliefs where all parties know the agent's discount factor is $q_1\delta^{(1)} + q_2\delta^{(2)}$. Notably, the agent is more patient in the benchmark than the true preferences of the agent in our model, whose intertemporal preferences turn out to be given by the discount factor $\delta^{(1)}$.

[14] Excluding this case clarifies the distinction of our model from GZ, with interpretation discussed in the Introduction.

To make the above argument precise, we can follow the argument in the Theorem 1 of Citanna et al. (2023) as follows. Recall that I_T is increasing and converges to I_∞. We say that the problem of maximising the agent's payoff with the benchmark discount factor (i.e., $q_1\delta^{(1)}+q_2\delta^{(2)}$) is "well-posed" if:

$$\sup_{(v_t)_{t=1}^{\infty}} \left\{ \sum_{t=1}^{\infty} \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t : \sum_{t=1}^{\infty} \frac{\phi(v_t)}{R^{t-1}} \leq I_\infty \right\} < \infty.$$

If the length of the horizon is T, then let W_T^A denote the maximum agent welfare, subject to the firm breaking even, and given that the agent has the benchmark discount factor. We have

$$W_T^A = \max_{(v_t)_{t=1}^{T}} \left\{ \sum_{t=1}^{T} \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t : \sum_{t=1}^{T} \frac{\phi(v_t)}{R^{t-1}} \leq I_T \right\}.$$

Let $v_t^{E,T}\left(L^t\right)$, $t = 1, \ldots, T$, be the values referred to in Lemma 2 and let

$$W_T^E = \sum_{t=1}^{T} \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t^{E,T}\left(L^t\right).$$

Then W_T^E is the agent's discounted payoff in the setting with the benchmark discount factor, but evaluated at the equilibrium choices of our original model, as characterised in Lemma 2. We have the following result:

Proposition 1. *Suppose that the problem of maximising the agent's payoff with the benchmark discount factor $q_1\delta^{(1)} + q_2\delta^{(2)}$ is well-posed. Then $\lim_{T\to\infty}\left(W_T^A - W_T^E\right) = 0$.*

Proposition 1 generalises in a specific way the finding of GZ and Citanna et al. (2023) by providing a benchmark discount factor ($q_1\delta^{(1)} + q_2\delta^{(2)}$) against which equilibrium consumption streams perform well, at least when the horizon is sufficiently long. (Their result implies the statement in the proposition for $q_2 = 1$.) As noted, this benchmark is higher than the agent's true discount factor $\delta^{(1)}$. As noted in the Introduction, use of the latter would appear consistent with a number of papers on models with mis-specified beliefs, so we now investigate a comparison with this benchmark in detail.

We now term the "efficient policies" those which maximise the agent's payoff subject to firm break-even when the agent's discount factor is commonly known to be equal to $\delta^{(1)}$. We index these policies with a B (for the second *benchmark*), so that the efficient path of utilities is given by $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^{T}$. Using standard Lagrangian arguments and the result in Lemma 2, we find that the equilibrium policies are more backloaded than the efficient policies in a sense made precise in the following proposition.

Proposition 2. *Consider the efficient utility path $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^{T}$ and equilibrium utility path $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$. Then $v_1^{B,T}(L) \geq v_1^{E,T}(L)$. Moreover, if $v_t^{E,T}\left(L^t\right) \geq v_t^{B,T}\left(L^t\right)$ and $v_t^{E,T}\left(L^t\right) > 0$ for some t, then*

$v_s^{E,T}(L^s) \geq v_s^{B,T}(L^s)$ for all $s > t$. The inequality is strict where $t \leq T-2$ and $v_s^{E,T}(L^s) > 0$. In addition, if $v_t^{B,T}(L^t) > v_t^{E,T}(L^t)$ for some t, then $v_{t'}^{B,T}\left(L^{t'}\right) < v_{t'}^{E,T}\left(L^{t'}\right)$ for some $t' > t$, and hence indeed $v_s^{E,T}(L^s) \geq v_s^{B,T}(L^s)$ for all $s \geq t'$.

The characterisation is more straightforward when utilities are always strictly greater than zero.

Corollary 1. *Suppose that $v_t^{B,T}(L^t) > 0$ for all t, which is guaranteed if $u'_+(0) = \infty$. Then there is a $t^* \in \{2, 3, \ldots, T-1\}$ such that $v_t^{E,T}(L^t) < v_t^{B,T}(L^t)$ for all $t < t^*$ and $v_t^{E,T}(L^t) > v_t^{B,T}(L^t)$ for all $t > t^*$.*

Proposition 2 and Corollary 1 appears in conflict with HK, who suggest that equilibrium involves overborrowing. Naturally, the source of this conflict is the different welfare criterion, which is based on the agent's realised preferences (equivalently, firms' beliefs about those preferences). It is also suggests a potential conflict with GZ, who as noted above find approximate efficiency in long horizons.

To demonstrate the divergence from GZ, let

$$V_T^B = \max_{(v_t)_{t=1}^T} \left\{ \sum_{t=1}^T \left(\delta^{(1)}\right)^{t-1} v_t : \sum_{t=1}^T \frac{\phi(v_t)}{R^{t-1}} \leq I_T \right\} = \sum_{t=1}^T \left(\delta^{(1)}\right)^{t-1} v_t^{B,T}\left(L^t\right)$$

be the maximised (efficient) surplus: i.e. the agent's maximal utility subject to the firm breaking even given true discount factor $\delta^{(1)}$. Let

$$V_T^E = \sum_{t=1}^T \left(\delta^{(1)}\right)^{t-1} v_t^{E,T}\left(L^t\right),$$

i.e. the value of surplus evaluated with the discount factor $\delta^{(1)}$ for the utilities realised in equilibrium. To simplify, we restrict attention to the case where $\delta^{(1)}R \leq 1$, and hence (from Equation (21) in the proof of Proposition 2) $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^T$ is a weakly decreasing sequence. We also assume that, as T increases, I_T grows according to a fixed additional income $w > 0$ per period; i.e., $I_T = \sum_{t=1}^T \frac{w}{R^{t-1}}$. We show the following.

Proposition 3. *Suppose that $\delta^{(1)}R \leq 1$ and $\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R\phi'\left(u\left(I_\infty\right)\right) > \phi'_+(0)$, and that I_T is determined by a constant income w. Then there is $\varepsilon > 0$ and $\bar{T} > 0$ such that $V_T^B - V_T^E \geq \varepsilon$ for all $T \geq \bar{T}$.*

The above result show that although the agent's initial discount factor is commonly known to be $\delta^{(1)}$, and although equilibrium involves a contract that maximises the agent's expected payoff conditional on the firm breaking even, equilibrium often involves inefficiency. In particular, the agent consumes inefficiently late. While the divergence with the earlier literature reflects the choice of benchmark, it is worth commenting on the reason for our result, focusing on the case where q_2 is arbitrarily close to 1, i.e. close to the special case of GZ. A firm's offer aims at increasing the agent's anticipated payoff, and hence increasing the date-2 continuation payoff in case $\delta_2 = \delta^{(2)}$, to which the agent attaches high probability. However, this is subject to the agent's upward date-2 incentive constraint, which requires the agent to value sufficiently his anticipated consumption when $\delta_2 = \delta^{(1)}$. Anticipated consumption comes largely from two channels: date-2 utility $v_2\left(L^2\right)$ and the expected continuation utility from date 3 given that $\delta_3 = \delta^{(2)}$. Raising the latter is subject to the date-3 upward incentive constraint at history L^2, and in turn calls for a higher value of $v_3\left(L^3\right)$, explaining why consumption is more back-loaded than efficient between dates 2 and 3 given $\delta_2 = \delta^{(1)}$.

4 Findings for the richer model

While Section 3 deliberately considers an environment where the screening logic is close to HK and GZ, we now consider the results for richer settings. There are now $N \geq 2$ possible discount factors. Firm beliefs $(p_n)_{n=1}^{N}$ (i.e., true probabilities) are assumed to have full support. We allow that either $u(0)$ is finite or $\lim_{c\to 0} u(c) = -\infty$. While the backloading logic in the model of Section 3 necessitates the corner solution of zero consumption in some states (see Lemma 2), fully interior equilibrium mechanisms now become possible and indeed will be guaranteed in case $\lim_{c\to 0} u(c) = -\infty$. Interiority is an important restriction for some results.

We begin by briefly characterising and commenting on what may be viewed as efficient consumption in this environment. Since we view firms as having the correct beliefs, this maximises

$$\mathbb{E}_F\left[\sum_{t=1}^{T} \Pi_{s=1}^{t-1}\tilde{\delta}_s u\left(c_t\left(\tilde{H}^t\right)\right)\right],$$

where $\mathbb{E}_F$ is the expectation under firm beliefs, subject to the firm break-even condition

$$\mathbb{E}_F\left[\sum_{t=1}^{T} \frac{c_t\left(\tilde{H}^t\right)}{R^{t-1}}\right] \leq I_T.$$

We assume that the solution to this problem is interior, i.e. consumption is always strictly positive, which is guaranteed for example if $\lim_{c\to 0} u(c) = -\infty$ or, if $u(0)$ is finite, by the Inada condition $u'_+(0) = \infty$. Following the treatment in the previous section, we write the characterisation in terms of the utilities: for all t, all H^t, $v_t\left(H^t\right) = u\left(c_t\left(H^t\right)\right)$.

Proposition 4. *The efficient policy $\left(v_t^{B,T}\left(H^t\right)\right)_{t=1}^{T}$ exists and is unique. If the solution is interior then, for each $t \geq 2$, $v_t^{B,T}\left(H^t\right)$ is strictly increasing in each of the first $t-1$ arguments.*

Proposition 4 has the immediate implication, of interest in light of the discussion in Section 3, that the efficient policy is not implementable by a dynamic mechanism. In any mechanism seeking to implement the efficient policy, the agent will simply send the messages intended for the highest realisations of the discount factor. Thus inefficiency appears "hard-wired" given that the true discount factor realisations are now uncertain.

Now consider equilibrium mechanisms. A direct mechanism gives rise to specific outcomes $\left(c_t\left(\delta_1, H_2^t\right)\right)_{t=1}^T$ for an agent with initial type δ_1 and continuation histories H_2^t. It will be seen that, for each initial agent type δ_1, in any equilibrium, these outcomes must maximise

$$\mathbb{E}_A\left[u\left(c_1\left(\delta_1\right)\right)+\delta_1 \sum_{t=2}^T \Pi_{s=2}^{t-1} \tilde{\delta}_s u\left(c_t\left(\delta_1, \tilde{H}_2^t\right)\right)\right]$$

subject to the firm break-even condition

$$\mathbb{E}_F\left[\sum_{t=1}^T \frac{c_t\left(\delta_1, \tilde{H}_2^t\right)}{R^{t-1}}\right] \leq I_T$$

and to incentive constraints for the truthful reporting of δ_t at all $t \in\{2, \ldots, T-1\}$ and all histories $\left(\delta_1, H_2^{t-1}\right)$. In particular, for each such history, each δ_t and each $\hat{\delta}_t$, we must have

$$\begin{aligned}
& u\left(c_t\left(\delta_1, H_2^{t-1}, \delta_t\right)\right)+\delta_t \mathbb{E}_A\left[\sum_{\tau=t+1}^T \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s u\left(c_\tau\left(\delta_1, H_2^{t-1}, \delta_t, \tilde{H}_{t+1}^\tau\right)\right)\right] \\
\geq & u\left(c_t\left(\delta_1, H_2^{t-1}, \hat{\delta}_t\right)\right)+\delta_t \mathbb{E}_A\left[\sum_{\tau=t+1}^T \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s u\left(c_\tau\left(\delta_1, H_2^{t-1}, \hat{\delta}_t, \tilde{H}_{t+1}^\tau\right)\right)\right] .
\end{aligned} \tag{5}$$

Call this Problem III. That the incentive constraints ensure incentive compatibility from date 2 onwards follows by the same backwards induction argument as for the previous section.

The following result confirms that Problem III is central to understanding equilibrium.

Proposition 5. *For each δ_1, there is a unique solution to Problem III. This solution represents the outcomes in the dynamic mechanism accepted by an agent with initial type δ_1 in any equilibrium. An equilibrium with such outcomes for any δ_1 exists.*

In equilibrium, each initial type of agent δ_1 has his expected payoff maximised subject to incentive compatibility from date 2 onwards, and subject to firm break-even. In particular, the agent's consumption process is given by the solution to Problem III. Moreover, there exists an equilibrium in which all firms offer the direct mechanism defined by solving Problem III for every initial type δ_1. Note that, while Problem III does not make reference to incentive compatibility of the agent's date-1 report, this is satisfied because the constraints of the maximisation for each initial type δ_1 are identical. Because a firm earns zero discounted profits in the solution to Problem III (as defined for any initial type δ_1), and because these profits do not depend on the agent's true initial type, incentive compatibility of the date-1 report follows in particular from the following observation. If a type $\delta_1=\delta^{(n)}$ can gain by deviating to mimic some initial type $\delta^{(n')}$, then the solution to Problem III for

$\delta^{(n')}$ must represent an improvement on the solution to Problem III as parameterised by $\delta_1 = \delta^{(n)}$, a contradiction.

As in the proof of Proposition 5, we can rewrite Problem III so that the problem is to determine agent utilities $v_t\left(\delta_1, H_2^t\right) = u\left(c_t\left(\delta_1, H_2^t\right)\right)$, call this Problem IV. Let $\left(v_t^{E,T}\left(\delta_1, H_2^t\right)\right)_{t=1}^T$ denote the solution. Our objective is to characterise this solution.

We begin by showing that optimal policies satisfy a particular form of monotonicity, and that only local incentive constraints are relevant. By local incentive constraints, we mean, at any date $t \in \{2, \ldots, T-1\}$, any history $\left(\delta_1, H_2^{t-1}\right)$, the agent with discount factor $\delta_t = \delta^{(n)}$ prefers to report truthfully (and then follow truthful reporting thereafter), rather than report an adjacent discount factor (i.e., $\delta^{(n-1)}$ if $n \geq 2$ or δ^{n+1} if $n \leq N-1$).

Lemma 3. *For any date $t \geq 2$, any history $\left(\delta_1, H_2^{t-1}\right)$, $v_t^{E,T}\left(\delta_1, H_2^{t-1}, \delta_t\right)$ is non-increasing in δ_t, while*

$$\mathbb{E}\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau^{E,T}\left(\delta_1, H_2^{t-1}, \delta_t, \tilde{H}_{t+1}^\tau\right)\right]$$

is non-decreasing. The solution to Problem IV is unchanged if we omit all incentive constraints other than local incentive constraints.

The monotonicity finding in Lemma 3 follows from incentive compatibility of the solution to Problem IV. For higher values of the discount factor, the agent relatively favours higher future consumption streams, and is willing to give up current consumption. Thus the monotonicity properties are crucial to sorting (in fact, they are necessary for incentive compatibility in general, not only a feature of the optimal mechanism).

Now considering Problem IV but only with local incentive constraints, we show that upward incentive constraints hold with equality.

Lemma 4. *In the solution to Problem IV, all local upward incentive constraints hold with equality.*

That upward incentive constraints hold with equality in the solution to Problem IV can be explained by the value of increasing the payoffs of higher types. In particular, from the perspective of date 1, the agent particularly values consumption in the periods following high discount factor realisations. But consumption after such realisations can be increased only subject to incentive constraints, and in particular that the agent does not want to mimic an upward adjacent discount factor. Higher types are thus given as much consumption as possible, subject to upward incentive constraints.

We now consider how contractual terms vary with changes in the agent's date-t discount factor δ_t. In particular, we are interested in how the firm's expected discounted continuation cost of the agent's consumption, at any date $t \in \{2, \ldots, T-1\}$ and any $\left(\delta_1, H_2^{t-1}\right)$, varies with the agent's realized date-t discount factor. This cost is given by

$$\mathbb{E}_F\left[\sum_{\tau=t}^{T}\frac{\phi\left(v_\tau\left(\delta_1,H_2^{t-1},\delta_t,\tilde{H}_{t+1}^{\tau}\right)\right)}{R^{t-1}}\right]. \tag{6}$$

In providing the result, we use the notion of the "continuation policy at history $\left(\delta_1,H_2^{t-1}\right)$ for discount factor $\delta^{(n)}$" to mean the sequence $\left(v_\tau\left(\delta_1,H_2^{t-1},\delta^{(n)},\cdot\right)\right)_{\tau=t}^{T}$. The continuation policies at history $\left(\delta_1,H_2^{t-1}\right)$ for two distinct discount factors $\delta^{(n')}$ and $\delta^{(n'')}$ are distinct if the continuation policies are different: $\left(v_\tau\left(\delta_1,H_2^{t-1},\delta^{(n')},\cdot\right)\right)_{\tau=t}^{T}\neq\left(v_\tau\left(\delta_1,H_2^{t-1},\delta^{(n'')},\cdot\right)\right)_{\tau=t}^{T}$.

Corollary 2. *In Problem IV, for any date $t\in\{2,\ldots,T-1\}$, any history $\left(\delta_1,H_2^{t-1}\right)$, the firm's expected discounted continuation cost (as given by Equation (6)) is weakly increasing in δ_t. For two discount factors $\delta_t=\delta^{(n')}$ and $\delta_t=\delta^{(n'')}$, $\delta^{(n')}<\delta^{(n'')}$, if the continuation policies at $\left(\delta_1,H_2^{t-1}\right)$ are distinct, then the firm's expected discounted continuation cost is strictly higher for $\delta_t=\delta^{(n'')}$.*

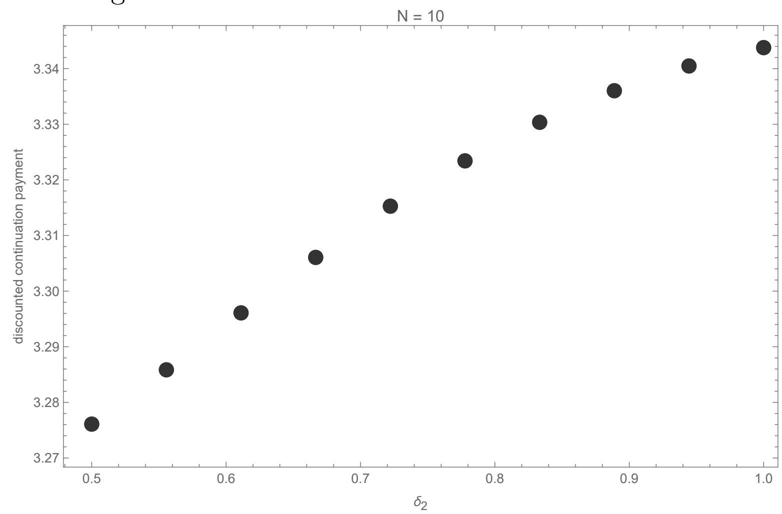
The result shows that when the agent realises a lower discount factor, at dates $t\in\{2,\ldots,T-1\}$, then he receives worse terms in the sense of a lower expected NPV payout, where the NPV is calculated according to the firm gross interest rate R. The result has value when there is at least some separation in the terms received by the agent when realising different discount factors from date 2 onwards. The following example illustrates in a tractable setting that indeed full separation can occur.

Example 1. Let $T=3$ and $u(c)=\sqrt{c}$. Suppose that discount factors are commonly believed to be uniformly distributed over values $\delta^{(n)}=\underline{\delta}+\frac{(n-1)}{N-1}\left(\bar{\delta}-\underline{\delta}\right)$, for $n=1,\ldots,N$, where $0<\underline{\delta}<\bar{\delta}$. Hence, $q_n=p_n=1/N$ for all n. If N is sufficiently large, then for all δ_1, $v_2(\delta_1,\delta_2)$ is strictly positive and strictly decreasing in δ_2, while $v_3(\delta_1,\delta_2)$ is strictly positive and strictly increasing in δ_2.

The example confirms that, for a plausible setting with agreement on beliefs – a "neoclassical" setting in the language of HK – the agent finds reporting to be less patient, and thus taking earlier consumption, more expensive. This higher expense is judged using firms' intertemporal preferences as captured by the gross interest rate R. As noted in the Introduction, this differs from the implications discussed by HK of a possible alternative neoclassical model (p 2300).

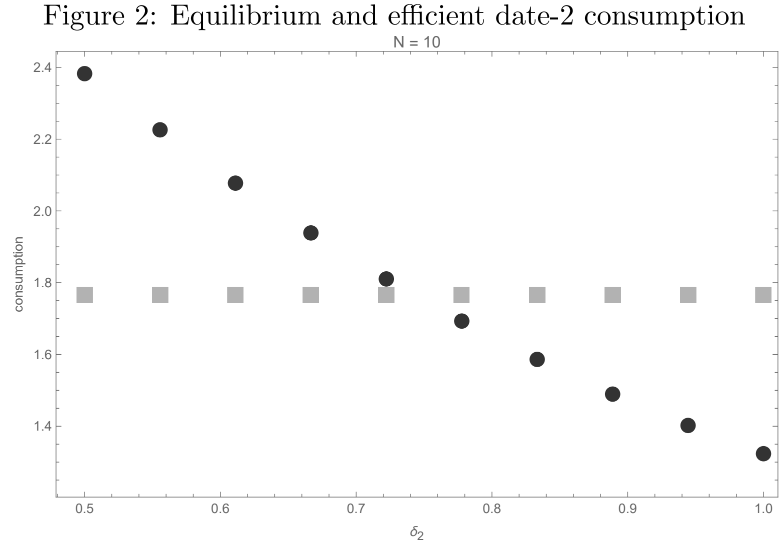
To illustrate Example 1, we compute numerically a particular case, displayed in Figures 1-3. Figure 1 shows that when the agent has a higher realised discount factor δ_2, he receives in equilibrium better terms in the sense that the NPV of consumption from date-2 onwards, using the firm's gross interest rate R, is higher. Thus, if the agent chooses front-loaded consumption at date 2 (as he does when δ_2 is small), he is penalised through worse terms (according to the gross interest rate R). Consumption is naturally then lower at date 3. Compared to the efficient policy (as studied at the beginning of this section), the dispersion in consumption is reduced due to the need to respect incentive constraints. In other words, higher types consume too little and lower types consume too much. This reflects that incentive compatibility limits the extent to which the firm can favour more patient types (as demanded by the efficient policy, see Proposition 4).

Figure 1: NPV of firm's date-2 continuation cost



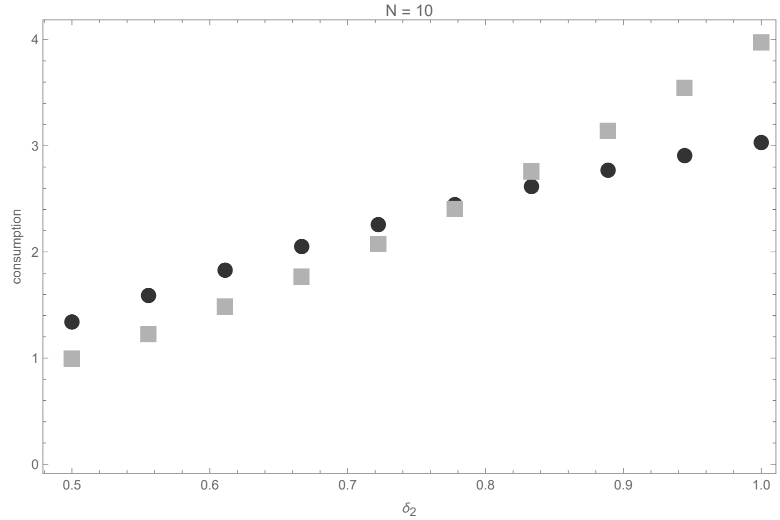
NPV of date-2 continuation cost $c_2(1, \delta_2) + \frac{c_3(1,\delta_2)}{R}$ as function of δ_2 for Example 1. Parameter values: $\underline{\delta} = 0.5$, $\bar{\delta} = 1$, $N = 10$, $R = 3/2$, $I_3 = 3$.

Figure 2: Equilibrium and efficient date-2 consumption



Date-2 consumption $c_2(1, \delta_2)$ as function of δ_2: efficient (grey) and equilibrium (black). Parameterisation as in Example 1 with $\underline{\delta} = 0.5$, $\bar{\delta} = 1$, $N = 10$, $R = 3/2$, $I_3 = 3$.

Figure 3: Equilibrium and efficient date-2 consumption



Date-3 consumption $c_3(1, \delta_2)$ as function of δ_2: efficient (grey) and equilibrium (black). Parameterisation as in Example 1 with $\underline{\delta} = 0.5$, $\bar{\delta} = 1$, $N = 10$, $R = 3/2$, $I_3 = 3$.

One important feature of the contracts predicted in HK is the large magnitude of the penalty for electing earlier consumption. They emphasise their prediction of "large penalties for falling behind [the] front-loaded repayment [i.e., delayed consumption] schedule" (p 2280). In our setting where agent and firm beliefs agree, and with many types, the equilibrium mechanism is flexible in the sense that a small reduction in patience corresponds to a small change in contract terms. Large or discontinuous changes in payment terms are possible with belief agreement, but then seem to be a feature of specifications with a small number of types. For instance, for the parameterisation considered in the figures, but setting $N = 2$, the firm's discounted continuation cost of agent consumption after an initial discount factor $\delta_1 = 1$, i.e. $c_2(1, \delta_2) + \frac{c_3(1,\delta_2)}{R}$, moves from 3.57 if $\delta_2 = 1$ to 3.10 if $\delta_2 = 0.5$, a 13 per cent fall in the NPV of consumption. The key reason for HK's finding of large penalties for delayed repayment (which can be read as early consumption in our model) is different and seems sourced in the agent's misprediction of his future preferences and behaviour. Because the specification in Section 3 replicates the logic in the models with quasi-hyperbolic discounting, our model can also account for situations where small differences in preferences (e.g., where $\delta^{(2)} - \delta^{(1)}$ is small in Section 3) is associated with discretely different consumption behaviour, with a large effective penalty (evaluated according to the gross interest rate R) for early consumption.

To further analyse the structure of equilibrium repayment, we refer to the argument in the proof of Example 1, where we derive necessary conditions for optimality in Problem IV when $T = 3$ generally. These conditions apply provided the solution is interior (consumption is strictly positive) and separating (higher values of δ_2 receive lower date-2 consumption and higher date-3 consumption). Let λ be the multiplier on the firm's break-even condition, let v_1 be the date-1 utility, $(w_n)_{n=1}^N$ be the date-2 utilities at each discount factor $\delta_2 = \delta^{(n)}$, and let $(z_n)_{n=1}^N$ be the corresponding date-3 utilities. Let $\bar{Q}_n = \sum_{m=n}^{N} q_n$. The necessary conditions can be written:

$$1 = \lambda \phi' (v_1) ; \tag{7}$$

$$\delta_1 = \frac{\lambda}{R} \sum_{n=1}^{N} p_n \phi' (w_n) ; \tag{8}$$

for $n \geq 2$,

$$\begin{aligned} 0 = & \left(\delta^{(n)} - \delta^{(n-1)}\right) \left(\delta_1 \bar{Q}_n - \frac{\lambda}{R} \sum_{m=n}^{N} p_m \phi' (w_m)\right) \\ & + \frac{\lambda p_n \delta^{(n)}}{R} \phi' (w_n) - \frac{\lambda p_n}{R^2} \phi' (z_n) ; \end{aligned} \tag{9}$$

and

$$\delta^{(1)} R \phi' (w_1) = \phi' (z_1) . \tag{10}$$

One set of insights suggested by these conditions relates to distortions in the timing of date 2 and date 3 consumption. In particular, note that for a fixed amount of consumption (in NPV terms, according to the firm's gross discount factor R) to be efficiently distributed between date 2 and date 3 for a given $\delta_2 = \delta^{(n)}$ requires

$$\delta^{(n)} R \phi' (w_n) = \phi' (z_n) .$$

This condition is necessary and sufficient for the agent's payoff not to be increased by any movement of consumption between dates 2 and 3 in state $\delta_2 = \delta^{(n)}$, subject to such change not reducing the firm's profit. Then Equation (10) provides a sense in which there are "no distortions at the bottom". Using Equation (8), and the assumption that agent beliefs (q_n) first-order stochastically dominate firm beliefs (p_n), the first term on the right side of Equation (9) is strictly positive, for $n \geq 2$. This implies

$$\delta^{(n)} R \phi' (w_n) < \phi' (z_n) \tag{11}$$

and so consumption is "too backloaded" relative to efficiency. These observations can be summarised as follows.

Proposition 6. *Let $T = 3$ and fix δ_1. Suppose the solution to Problem IV is interior (all consumption is strictly positive) and that the mechanism is separating (higher values of δ_2 receive strictly lower date-2 consumption but strictly higher date-3 consumption). Then for all values $\delta_2 = \delta^{(n)}$, $n \geq 2$, the agent's date-2 continuation payoff given δ_2 can be increased in the absence of incentive constraints by shifting consumption earlier (i.e., to date 2), while leaving the NPV of firm profits unchanged. In this sense, equilibrium consumption at dates 2 and 3 is inefficiently backloaded.*

Proposition 6 provides a sense in which inefficient backloading of consumption can occur in equilibrium, including when the agent and firms have common beliefs ($q_n = p_n$ for all n). Notice that the statement is conditional on the realisation of the discount factor δ_2 and so the statement is not sensitive to the beliefs over discount factors one may apply to evaluate welfare. It is also worth observing that the distortion described in Proposition 6 will often vanish for the highest type $\delta_2 = \delta^{(N)}$ as the number of types N becomes large (we might say there is "approximate efficiency at the top"). The calculations in the proof of Example 1 imply that, at least in this example, we have $\phi'(z_N) - \delta^{(N)} R \phi'(w_N) \to 0$, and so there is approximate effiency in the timing of consumption for the highest type.

The reason consumption is more backloaded than efficient in the sense of Proposition 6 is simply that increasing the date-3 utility of a given value of $\delta_2 > \delta^{(1)}$ relaxes the upward incentive constraint pointing to this type. Indeed, higher types have a relative preference for more backloaded consumption, compared to those types below. The effect can be understood by considering an envelope representation of the agent's date-2 continuation payoff:

$$w_n + \delta^{(n)} z_n = w_1 + \delta^{(1)} z_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)} \right) z_i.$$

This expression is derived using that upward incentive constraints bind. Increases in the date-3 consumption z_i, for some $i \geq 2$, increase the date-2 continuation utility $w_n + \delta^{(n)} z_n$ of all higher types $\delta_2 = \delta^{(n)} \geq \delta^{(i)}$, while leaving the utilities of types below $\delta^{(i)}$ unchanged. This increases the extent to which the mechanism can favour higher realisations of δ_2 at date 3 (which, as we have seen, is associated with greater efficiency) while respecting incentive compatibility. This effect on the dispersion of date-2 continuation payoffs does not apply however for $\delta_2 = \delta^{(1)}$: an increase in z_1 raises the date-2 continuation utility of all types equally. This helps to explain why we predict the efficent level of backloading between dates 2 and 3, conditional on $\delta_2 = \delta^{(1)}$.

Another useful perturbation that can shed light on the dynamics of equilibrium and efficient consumption is to consider shifting utility into the subsequent period by a constant that does not respond to type. This has the advantage of preserving incentive constraints and is therefore a relevant perturbation both in Problem IV and in studying the efficient policy. For $T \geq 3$, we consider such perturbations at dates $t \in \{2, \ldots, T-1\}$ conditional on history $\left(\delta_1, H_2^{t-1}\right)$, as well as between the first and second period. We find the following.

Proposition 7. *Consider any $T \geq 3$ and suppose both equilibrium and efficient consumption are interior – i.e., all consumption values are strictly positive. For any $t \in \{2, \ldots, T-1\}$, any history $\left(\delta_1, H_2^{t-1}\right)$, the equilibrium utilities satisfy*

$$\mathbb{E}_F \left[\phi' \left(v_{t+1}^{E,T} \left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1} \right) \right) \right] = R \mathbb{E}_A \left[\tilde{\delta}_t \right] \mathbb{E}_F \left[\phi' \left(v_t^{E,T} \left(\delta_1, H_2^{t-1}, \tilde{\delta}_t \right) \right) \right]$$

while the efficient policies satisfy

$$\mathbb{E}_F \left[\phi' \left(v_{t+1}^{B,T} \left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1} \right) \right) \right] = R \mathbb{E}_F \left[\tilde{\delta}_t \right] \mathbb{E}_F \left[\phi' \left(v_t^{B,T} \left(\delta_1, H_2^{t-1}, \tilde{\delta}_t \right) \right) \right].$$

Similarly,

$$\mathbb{E}_F\left[\phi'\left(v_2^{E,T}\left(\delta_1,\tilde{\delta}_2\right)\right)\right]=R\delta_1\phi'\left(v_1^{E,T}\left(\delta_1\right)\right)$$

and

$$\mathbb{E}_F\left[\phi'\left(v_2^{B,T}\left(\delta_1,\tilde{\delta}_2\right)\right)\right]=R\delta_1\phi'\left(v_1^{B,T}\left(\delta_1\right)\right).$$

We have the following corollary to Proposition 7.

Corollary 3. *Suppose $T\geq 3$. If $u(c)=\ln(c)$, the interiority assumption in Proposition 7 holds and we have $\phi'(u(c))=c$. Therefore, for $t\in\{1,2,\ldots,T-1\}$, the ratio $\frac{\mathbb{E}_F\left[c_{t+1}\left(\delta_1,\tilde{H}_2^{t+1}\right)\right]}{\mathbb{E}_F\left[c_t\left(\delta_1,\tilde{H}_2^{t+1}\right)\right]}$ is weakly larger for equilibrium consumption than efficient consumption. The ratio is the same in both cases if firm and agent beliefs agree. For $t\geq 2$, the ratio is strictly larger for equilibrium consumption than efficient consumption if the agent is strictly more optimistic about patience than the firm.*

These results give a sense of how agent beliefs affect the rate of growth, or rate of decline, of equilibrium consumption relative to efficiency. The corollary shows that it is possible to compare the rates of growth (or decline) of expected consumption in the logarithmic case, with equilibrium consumption more backloaded than for the efficient policy, strictly so if the agent is more optimistic than firms. The reason for this result is that, when the agent is more optimistic about future discount factors, he attaches more weight to future consumption. In equilibrium, firms pander to the agent's beliefs by delaying consumption, thus raising the agent's expected payoffs from the contract at the time of contracting.

It is worth noting here that 6 and 7 represent distinct though related exercises, calling for careful interpretation. Consider the case of $T=3$ and the case of common beliefs. Suppose both results apply to the equilibrium contract, i.e. the equilibrium consumption values are interior and the contract fully separates types δ_2, as in the case of Example 1 for large enough N. Then, from the arguments surrounding Proposition 6 (see Equation (11), we have

$$R\sum_{n=1}^{N}p_n\delta^{(n)}\phi'\left(v_2^{E,3}\left(\delta_1,\delta^{(n)}\right)\right)<\sum_{n=1}^{N}p_n\phi'\left(v_3^{E,3}\left(\delta_1,\delta^{(n)}\right)\right). \tag{12}$$

From Proposition 7, we have

$$R\left(\sum_{i=1}^{N}p_i\delta^{(i)}\right)\sum_{n=1}^{N}p_n\phi'\left(v_2^{E,3}\left(\delta_1,\delta^{(n)}\right)\right)=\sum_{n=1}^{N}p_n\phi'\left(v_3^{E,3}\left(\delta_1,\delta^{(n)}\right)\right). \tag{13}$$

The difference in (12) and (13) is explained by the fact that $v_2^{E,3}\left(\delta_1,\delta^{(n)}\right)$ is decreasing in $\delta^{(n)}$. Thus, in Proposition 6, we find that consumption is inefficiently backloaded to 3 from date 2, while in Proposition 7 we find that the ratio of the firm's expected marginal cost of date-3 utility to the expected marginal cost of date-2 utility is the same as for the efficient policy. These findings are not contradictory.

We now further interpret Equation (13), and because the point is more general, consider now the case with $T \geq 3$. Suppose still that the agent has correct beliefs. Then, for $t \in \{2, \ldots, T-1\}$,

$$\mathbb{E}_F\left[\phi'\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1}\right)\right)\right] = R\mathbb{E}_F\left[\tilde{\delta}_t\right]\mathbb{E}_F\left[\phi'\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t\right)\right)\right].$$

Consider changing date-$t+1$ utility by ε and date-t utility by $-\varepsilon\mathbb{E}_F\left[\tilde{\delta}_t\right]$, following history $\left(\delta_1, H_2^{t-1}\right)$. This keeps agent expected payoffs unchanged conditional on history $\left(\delta_1, H_2^{t-1}\right)$. The corresponding change in firm payments conditional on history $\left(\delta_1, H_2^{t-1}\right)$ is

$$\mathbb{E}_F\left[\phi\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t\right) - \varepsilon\mathbb{E}_F\left[\tilde{\delta}_t\right]\right)\right] + \mathbb{E}_F\left[\phi\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1}\right) + \varepsilon\right)\right]$$
$$- \left(\mathbb{E}_F\left[\phi\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t\right)\right)\right] + \mathbb{E}_F\left[\phi\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1}\right)\right)\right]\right).$$

The expression is convex in ε, showing that the cost-minimising choice of ε over feasible values (i.e., such that the resulting utilities are interior) is $\varepsilon = 0$. It is then immediate that, considering shifting utilities at date t and date-$t+1$ after history $\left(\delta_1, H_2^{t-1}\right)$ by a constant independent of $\left(\tilde{\delta}_t, \tilde{\delta}_{t+1}\right)$ cannot raise the agent's expected payoff without increasing the expected cost to the firm.

Referring back to Equations (12) and (13) for the case of $T = 3$ and $t = 2$, the above implies that, in case solutions are interior and under full separation, utilities are too backloaded state by state (i.e., for each value of δ_2), but neither too backloaded nor too frontloaded if we instead consider shifting per-period utilities by a constant independent of the state. In the above exercise, taking $\varepsilon < 0$ and hence shifting utility earlier, we see that the change results in a compression of expected discounted payoffs and a relative worsening of the terms of higher types in terms of the NPV of firm expenditure. This helps to explains why shifting utility by a constant yields a different effect compared to shifting utility state by state.

Away from the case where agent beliefs are correct, suppose that the agent is more optimistic than the principal. Then, for $T \geq 3$ and $t \in \{2, \ldots, T-1\}$,

$$\mathbb{E}_F\left[\phi'\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t, \tilde{\delta}_{t+1}\right)\right)\right] > R\mathbb{E}_F\left[\tilde{\delta}_t\right]\mathbb{E}_F\left[\phi'\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \tilde{\delta}_t\right)\right)\right].$$

It then follows that decreasing utilities by a constant at date $t+1$ and increasing utilities by a constant at date t, following history $\left(\delta_1, H_2^{t-1}\right)$, such that agent expected payoffs under correct beliefs are unchanged, decreases the NPV of firm expected payments. Similarly, shifting utility earlier in this way permits an increase in agent expected utility under correct beliefs that holds firm expected profits constant.

5 Conclusions

A body of work in behavioural contract theory studies present-biased agents by specifying models with quasi-hyperbolic discounting. This paper suggests an alternative framework based on stochastic discount factors that could be a useful complement to the existing literature. Given that the fully naive agent with quasi-hyperbolic discounting arises as a limiting case of our model, our framework can be viewed as an extension (in a particular direction) of the workhorse model in the literature. This extension includes an agent with correct beliefs. It is then of interest to understand which intuitions and logic from models of quasi-hyperbolic discounting with naive agents survive as we move towards the case of correct beliefs.

In the present paper we have applied our alternative framework to a model of competitive credit markets, following closely the work of HK and GZ. In Section 3, we studied a setting with only two realisations of the discount factor, where firms know that the low value always occurs. This specialisation highlights that a similar logic to the existing literature must apply, although the agent believes the discount factor is stochastic. While the forces shaping equilibrium contracts are essentially the same, a natural choice of welfare benchmark implies that consumption is our model is inefficiently delayed. In Section 4, we consider a setting which allows more values of the discount factor and assumes full support over these values, nesting the case of correct agent beliefs. We then show that the agent being penalised for choosing options with earlier consumption is a robust feature of equilibrium mechanisms, including in the case where the agent has correct beliefs. We also provide results showing how equilibrium consumption is often inefficiently backloaded – we show that this holds in a three-period model when the contract involves the appropriate separation of types. Separation is verified in a natural special case with correct beliefs. While contractual distortions do arise in the case with correct agent beliefs, regulatory intervention cannot improve agent welfare. In this case, the appropriate beliefs with which to measure welfare are unambiguous, and it is not possible to raise agent expected payoffs without violating either firm participation or incentive-compatibility constraints. From the perspective of a regulator that does not know the true model, observing that credit contracts penalise consumers for early consumption (or late repayment) does not imply that regulatory intervention can improve consumer welfare.

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Appendix: Omitted proofs

Proofs of results in Section 3

Proof of Lemma 1. We begin by observing that in an incentive-compatible mechanism, for any $2 \leq t \leq T-1$, and any $H^{t-1} \in \mathcal{H}^R$,

$$v_t\left(H^{t-1}, \delta^{(1)}\right) \geq v_t\left(H^{t-1}, \delta^{(2)}\right) \tag{14}$$

and

$$\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau}\right)\right] \geq \mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(1)}, \tilde{H}_{t+1}^{\tau}\right)\right], \tag{15}$$

with both inequalities strict if and only if at least one of (2) and (3) hold as a strict inequality. To see this, first we can write (3) as

$$v_t\left(H^{t-1}, \delta^{(1)}\right) - v_t\left(H^{t-1}, \delta^{(2)}\right) \geq \delta^{(1)} \begin{pmatrix} \mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ -\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \end{pmatrix}. \tag{16}$$

If the inequality (15) fails to hold, then

$$v_t\left(H^{t-1}, \delta^{(1)}\right) - v_t\left(H^{t-1}, \delta^{(2)}\right) > \delta^{(2)} \begin{pmatrix} \mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ -\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t-1}, \delta^{(1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \end{pmatrix} \tag{17}$$

and so (2) is violated. So we can conclude (15) is satisfied. Then if (14) fails to hold, both types strictly prefer to report $\delta^{(2)}$.

Now, note that, when both of (14) and (15) hold as equality, both types are indifferent across both reports; i.e., both incentive constraints (2) and (3) hold as equality. It is easy to see we cannot have exactly one of these two inequalities hold as equalities, and if both are strict inequalities then at least one of the two incentive constraints is strict. (For instance, if (3) holds as equality and hence equivalently (16) holds as equality, then the reverse strict inequality as in inequality (17) holds, i.e. (2) holds strictly.)

Suppose now that (3) holds as a strict inequality at date t s.t. $2 \leq t \leq T-1$, and history $H^{t-1} \in \mathcal{H}^R$, with the difference between left and right sides equal to x. Increase $v_t\left(H^{t-1}, \delta^{(2)}\right)$ by $x\left(1-q_2\right)$ and decrease $v_t\left(H^{t-1}, \delta^{(1)}\right)$ by xq_2. Now, (3) holds as an equality and since (15) holds strictly, we have (2) holds strictly. Moreover, for any period $t' < t$, the expected agent continuation payoff from $t'+1$ onwards conditional on having reported $H^{t'} \in \mathcal{H}^R$, as determined by

$$\mathbb{E}_A\left[\sum_{\tau=t'+1}^{T} \Pi_{s=t'+1}^{\tau-1} \tilde{\delta}_s v_\tau\left(H^{t'}, \tilde{H}_{t'+1}^{\tau}\right)\right],$$

remains unchanged. Hence all incentive constraints at dates before t are unaffected, and the same is clearly true for incentive constraints after t. Thus, it is easy to see that all incentive constraints remain intact.

Finally, note that the agent's expected payoff has not changed and firm profits are either unchanged or increase if H^{t-1} is the sequence comprising only $\delta^{(1)}$. The change can be implemented at every t, $2 \leq t \leq T-1$, and H^{t-1} such that (3) holds as a strict inequality.

$\square$

Proof of Lemma 2. Suppose now that all incentive constraints (3) hold with equality. By substituting this equality for $t = 2$ into the agent's expected payoff, we obtain

$$
\begin{aligned}
& v_1\left(\delta^{(1)}\right) + \delta^{(1)}\mathbb{E}_A\left[v_2\left(\delta^{(1)}, \tilde{\delta}_2\right) + \sum_{t=3}^{T} \Pi_{s=2}^{t-1}\tilde{\delta}_s v_t\left(\delta^{(1)}, \tilde{\delta}_2, \ldots, \tilde{\delta}_t\right)\right] \\
=& v_1\left(\delta^{(1)}\right) - q_2\left(\delta^{(2)} - \delta^{(1)}\right) v_2\left(\delta^{(1)}, \delta^{(2)}\right) + q_2\delta^{(2)}\left[\begin{array}{c} v_2\left(\delta^{(1)}, \delta^{(2)}\right) \\ +\delta^{(1)}\mathbb{E}_A\left[\sum_{t=3}^{T} \Pi_{s=3}^{t-1}\tilde{\delta}_s u\left(c_t\left(\delta^{(1)}, \delta^{(2)}, \tilde{\delta}_3, \ldots, \tilde{\delta}_t\right)\right)\right] \end{array}\right] \\
& + q_1\delta^{(1)}\left(v_2\left(\delta^{(1)}, \delta^{(1)}\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=3}^{T} \Pi_{s=3}^{t-1}\tilde{\delta}_s v_t\left(\delta^{(1)}, \delta^{(1)}, \tilde{\delta}_3, \ldots, \tilde{\delta}_t\right)\right]\right) \\
=& v_1\left(\delta^{(1)}\right) - q_2\left(\delta^{(2)} - \delta^{(1)}\right) v_2\left(\delta^{(1)}, \delta^{(2)}\right) + q_2\delta^{(2)}\left[\begin{array}{c} v_2\left(\delta^{(1)}, \delta^{(1)}\right) \\ +\delta^{(1)}\mathbb{E}_A\left[\sum_{t=3}^{T} \Pi_{s=3}^{t-1}\tilde{\delta}_s u\left(c_t\left(\delta^{(1)}, \delta^{(1)}, \tilde{\delta}_3, \ldots, \tilde{\delta}_t\right)\right)\right] \end{array}\right] \\
& + q_1\delta^{(1)}\left(v_2\left(\delta^{(1)}, \delta^{(1)}\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=3}^{T} \Pi_{s=3}^{t-1}\tilde{\delta}_s v_t\left(\delta^{(1)}, \delta^{(1)}, \tilde{\delta}_3, \ldots, \tilde{\delta}_t\right)\right]\right) \\
=& v_1\left(\delta^{(1)}\right) - q_2\left(\delta^{(2)} - \delta^{(1)}\right) v_2\left(\delta^{(1)}, \delta^{(2)}\right) + \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) v_2\left(\delta^{(1)}, \delta^{(1)}\right) \\
& + \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)\delta^{(1)}\mathbb{E}_A\left[\sum_{t=3}^{T} \Pi_{s=3}^{t-1}\tilde{\delta}_s v_t\left(\delta^{(1)}, \delta^{(1)}, \tilde{\delta}_3, \ldots, \tilde{\delta}_t\right)\right].
\end{aligned}
$$

We then observe that, for $\tau \in \{3, 4, \dots, T-1\}$,

$$
\begin{aligned}
&\delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau}^{T} \Pi_{s=\tau}^{t-1}\tilde{\delta}_s v_t\left(L^{\tau-1}, \tilde{\delta}_\tau, \dots, \tilde{\delta}_t\right)\right] \\
=&q_2\delta^{(2)}\left(v_\tau\left(L^{\tau-1}, \delta^{(2)}\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau+1}^{T} \Pi_{s=\tau+1}^{t-1}\tilde{\delta}_s v_t\left(L^{\tau-1}, \delta^{(2)}, \tilde{\delta}_{\tau+1}, \dots, \tilde{\delta}_t\right)\right]\right) \\
&-\left(\delta^{(2)} - \delta^{(1)}\right) q_2 v_\tau\left(L^{\tau-1}, \delta^{(2)}\right) \\
&+ q_1\delta^{(1)}\left(v_\tau\left(L^\tau\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau+1}^{T} \Pi_{s=\tau+1}^{t-1}\tilde{\delta}_s v_t\left(L^\tau, \tilde{\delta}_{\tau+1}, \dots, \tilde{\delta}_t\right)\right]\right) \\
=&q_2\delta^{(2)}\left(v_\tau\left(L^\tau\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau+1}^{T} \Pi_{s=\tau+1}^{t-1}\tilde{\delta}_s v_t\left(L^\tau, \tilde{\delta}_{\tau+1}, \dots, \tilde{\delta}_t\right)\right]\right) \\
&-\left(\delta^{(2)} - \delta^{(1)}\right) q_2 v_\tau\left(L^{\tau-1}, \delta^{(2)}\right) \\
&+ q_1\delta^{(1)}\left(v_\tau\left(L^\tau\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau+1}^{T} \Pi_{s=\tau+1}^{t-1}\tilde{\delta}_s v_t\left(L^\tau, \tilde{\delta}_{\tau+1}, \dots, \tilde{\delta}_t\right)\right]\right) \\
=&\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)\left(v_\tau\left(L^\tau\right) + \delta^{(1)}\mathbb{E}_A\left[\sum_{t=\tau+1}^{T} \Pi_{s=\tau+1}^{t-1}\tilde{\delta}_s v_t\left(L^\tau, \tilde{\delta}_{\tau+1}, \dots, \tilde{\delta}_t\right)\right]\right) \\
&-\left(\delta^{(2)} - \delta^{(1)}\right) q_2 v_\tau\left(L^{\tau-1}, \delta^{(2)}\right).
\end{aligned}
$$

Substituting iteratively, we find that

$$
\begin{aligned}
&v_1\left(\delta^{(1)}\right) + \delta^{(1)}\mathbb{E}_A\left[v_2\left(\delta^{(1)}, \tilde{\delta}_2\right) + \sum_{t=3}^{T} \Pi_{s=2}^{t-1}\tilde{\delta}_s v_t\left(\delta^{(1)}, \tilde{\delta}_2, \dots, \tilde{\delta}_t\right)\right] \\
=&\sum_{t=1}^{T-1}\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t\left(L^t\right) + \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2}\delta^{(1)} v_T\left(L^T\right) \\
&-\sum_{t=2}^{T-1}\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-2} q_2\left(\delta^{(2)} - \delta^{(1)}\right) v_t\left(L^{t-1}, \delta^{(2)}\right).
\end{aligned}
\tag{18}
$$

Now, consider a relaxed program in which we maximize (18) subject only to the constraints that the firm makes non-negative profit and that utilities are no less than $u(0)$. First note that, for any $t \in \{2, \dots, T-1\}$, the obective is decreasing in $v_t\left(L^{t-1}, \delta^{(2)}\right)$, so these must be equal to $u(0) = 0$. Second, because T is finite, satisfaction of the non-negative profit constraint requires that utilities are uniformly bounded across histories L^t. Hence, a maximum in the problem exists. It is easy to see that the firm makes zero profit. Because these will correspond to the equilibrium utilities, we index the optimal values with a superscript E. In particular, the utilities maximising (18) are given by $v_t^{E,T}\left(L^t\right)$, $t = 1, \dots, T$, and $v_t^{E,T}\left(L^{t-1}, \delta^{(2)}\right)$, $t = 2, \dots, T-1$. Using strict convexity of ϕ, these values are uniquely determined.

Now we can determine an incentive-compatible mechanism with the utilities $\hat{v}_t\left(H^t\right)$, $H^t \in \mathcal{H}^R$. For $t = 1, \ldots, T$, $\hat{v}_t\left(L^t\right) = v_t^{E,T}\left(L^t\right)$. For $t = 2, \ldots, T-1$, let $\hat{v}_t\left(L^{t-1}, \delta^{(2)}\right) = v_t^{E,T}\left(L^{t-1}, \delta^{(2)}\right) = u(0)$. We complete the specification by setting, for any period $t \in \{2, \ldots, T-1\}$ and history such that $H^t = \left(L^{t-1}, \delta^{(2)}\right)$, all subsequent utilities to be constant; i.e. for all H_{t+1}^{τ} with $\tau \geq t+1$, let $\hat{v}_\tau\left(L^{t-1}, \delta^{(2)}, H_{t+1}^{\tau}\right) = \bar{v}\left(L^{t-1}, \delta^{(2)}\right)$. Then, all incentive constraints occurring after the history $\left(L^{t-1}, \delta^{(2)}\right)$, i.e. at date $t+1$ or any later period, are satisfied trivially. We only need to ensure incentive constraints occuring at $t \in \{2, \ldots, T-1\}$ at histories $H^{t-1} = L^{t-1}$, and we do so such that the constraints (3) hold with equality. This can be achieved iteratively working backwards. For $t = T-1$, we choose $\hat{v}_T\left(L^{T-2}, \delta^{(2)}\right) = \bar{v}\left(L^{T-2}, \delta^{(2)}\right)$, with $\bar{v}\left(L^{T-2}, \delta^{(2)}\right)$ satisfying

$$v_{T-1}^{E,T}\left(L^{T-1}\right) + \delta^{(1)} v_T^{E,T}\left(L^{T-1}\right) = u(0) + \delta^{(1)} \bar{v}\left(L^{T-2}, \delta^{(2)}\right).$$

So we have (3) holds with equality at $H^{T-2} = L^{T-2}$, while (2) is satisfied because $\hat{v}_T\left(L^{T-1}\right) = v_T^{E,T}\left(L^{T-1}\right) \leq \bar{v}\left(L^{T-2}, \delta^{(2)}\right) = \hat{v}_T\left(L^{T-2}, \delta^{(2)}\right)$.

Now, for any $t \in \{2, \ldots, T-2\}$, if $\hat{v}_\tau\left(L^t, H_{t+1}^{\tau}\right)$, all $H_{t+1}^{\tau} \in \Delta^{\tau-t}$, have been determined for $\tau > t$, specify $\bar{v}\left(L^{t-1}, \delta^{(2)}\right)$ as solving

$$v_t^{E,T}\left(L^t\right) + \delta^{(1)} \mathbb{E}_A\left[\sum_{\tau'=t+1}^{T} \Pi_{s=t+1}^{\tau'-1} \tilde{\delta}_s \hat{v}_{\tau'}\left(L^t, \tilde{H}_{t+1}^{\tau'}\right)\right] = u(0) + \delta^{(1)} \sum_{\tau'=t+1}^{T} \left(q_1 \delta^{(1)} + q_2 \delta^{(2)}\right)^{\tau'-(t+1)} \bar{v}\left(L^{t-1}, \delta^{(2)}\right).$$

Thus, we have specified, for $\tau > t$, all $H_{t+1}^{\tau} \in \Delta^{\tau-t}$, $\hat{v}_\tau\left(L^{t-1}, \delta^{(2)}, H_{t+1}^{\tau}\right) = \bar{v}\left(L^{t-1}, \delta^{(2)}\right)$. And hence, $\hat{v}_\tau\left(L^{t-1}, H_t^{\tau}\right)$ is determined for all $\tau > t-1$, all $H_t^{\tau} \in \Delta^{\tau-t+1}$. Thus, proceeding iteratively backwards, we specify the entire mechanism. The mechanism is constructed to ensure constraint (3) holds with equality at histories L^{t-1}, $2 \leq t \leq T-1$. That (2) is then satisfied follows because at these histories,

$$\begin{aligned}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s \hat{v}_\tau\left(L^t, \tilde{H}_{t+1}^{\tau}\right)\right] &\leq \mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1} \tilde{\delta}_s \hat{v}_\tau\left(L^{t-1}, \delta^{(2)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ &= \sum_{\tau=t+1}^{T} \left(q_1 \delta^{(1)} + q_2 \delta^{(2)}\right)^{\tau-(t+1)} \bar{v}\left(L^{t-1}, \delta^{(2)}\right).\end{aligned}$$

This completes the proof.

□

Proof of Proposition 1. The argument exactly follows Citanna et al. (2023), reproduced here for convenience. Consider the programme:

$$V_T(\beta, I_T) = \max_{\{v_t\}_{t=1}^{T}} \left\{\sum_{t=1}^{T-1} \left(q_1 \delta^{(1)} + q_2 \delta^{(2)}\right)^{t-1} v_t + \beta \left(q_1 \delta^{(1)} + q_2 \delta^{(2)}\right)^{T-2} v_T\right\}$$

subject to

$$\sum_{t=1}^{T} \frac{\phi\left(v_t\right)}{R^{t-1}} \leq I_T.$$

By Lemma 2, $V_T\left(\delta^{(1)}, I_T\right)$ is the agent's equilibrium payoff in the T-period version of our original model when discounted income is I_T. Then, because the benchmark-discounting problem is well-posed, $\left(V_T\left(q_1\delta^{(1)} + q_2\delta^{(2)}, I_T\right)\right)_{T=3}^{\infty}$ is bounded from above and it is a non-decreasing sequence. Hence it has a finite limit and so the sequence is Cauchy:

$$\lim_{T\to\infty} \left(V_T\left(q_1\delta^{(1)} + q_2\delta^{(2)}, I_T\right) - V_{T-1}\left(q_1\delta^{(1)} + q_2\delta^{(2)}, I_{T-1}\right)\right) = 0.$$

For any $T \geq 3$, we have

$$W_T^A - W_T^E = V_T\left(\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right), I_T\right) - \sum_{t=1}^{T}\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t^{E,T}\left(L^t\right) \geq 0.$$

But then

$$\sum_{t=1}^{T}\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t^{E,T}\left(L^t\right) \geq V_T\left(\delta^{(1)}, I_T\right) \geq V_{T-1}\left(\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right), I_{T-1}\right).$$

This implies that

$$V_T\left(\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right), I_T\right) - V_{T-1}\left(\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right), I_{T-1}\right) \geq W_T^A - W_T^E \geq 0$$

as desired.

□

Proof of Proposition 2. The proof of Lemma 2 follows from an analysis of Karush-Kuhn-Tucker (KKT) conditions. First consider the programme determining the equilibrium utilities $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$. The constraints are the zero lower bounds on utility and the firm's break-even condition. That KKT conditions are valid for the optimum $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ follows from a standard constraint qualification. For instance, noting that $I_T > 0$ and hence $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ is not degenerately zero, the gradients of the active constraints evaluated at $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ are linearly independent.

The Lagrangean is

$$\mathcal{L}^{E,T} = \sum_{t=1}^{T-1} \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} v_t\left(L^t\right) + \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} v_T\left(L^T\right)$$
$$+ \lambda\left(I_T - \sum_{t=1}^{T} \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}}\right) + \sum_{t=1}^{T} \mu_t v_t\left(L^t\right).$$

Here, $\left(v_t\left(L^t\right)\right)_{t=1}^T$ are the utilities to be chosen, λ represents the multiplier on the firm's break-even condition, and $(\mu_t)_{t=1}^T$ are the multipliers on the non-negative utility constraints. Necessary conditions for an optimum are: (i) for $t \leq T-1$,

$$\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} + \mu_t = \frac{\lambda}{R^{t-1}}\phi'\left(v_t\left(L^t\right)\right),$$

(ii)

$$\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} + \mu_T = \frac{\lambda}{R^{T-1}}\phi'\left(v_T\left(L^T\right)\right),$$

(iii)

$$\lambda\left(I_T - \sum_{t=1}^{T} \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}}\right) = 0,$$

and (iv) for $t \in \{1, 2, \ldots, T\}$,

$$\mu_t v_t\left(L^t\right) = 0.$$

In addition, the constraints must be satisfied ($I_T - \sum_{t=1}^T \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}} \geq 0$ and $v_t\left(L^t\right) \geq u(0)$ for all t) and multipliers must be non-negative. We let $\left(\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^T, \lambda^{E,T}, \left(\mu_t^{E,T}\right)_{t=1}^T\right)$ denote a KKT solution and note that, by convexity of ϕ, $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^T$ maximises the Lagrangean given the multipliers $\left(\lambda^{E,T}, \left(\mu_t^{E,T}\right)_{t=1}^T\right)$. Note that (i) and (ii) above imply $\lambda^{E,T} > 0$.

Now, let ρ be the inverse of ϕ'. We can write, for $t \leq T-2$,

$$v_t^{E,T}\left(L^t\right) = \begin{cases} \rho\left(\frac{\left(q_1\delta^{(1)}+q_2\delta^{(2)}\right)^{t-1}R^{t-1}}{\lambda^{E,T}}\right) & \text{if } \frac{\left(q_1\delta^{(1)}+q_2\delta^{(2)}\right)^{t-1}R^{t-1}}{\lambda^{E,T}} \geq \phi'_+(0) \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Also,

$$v_T^{E,T}\left(L^T\right) = \begin{cases} \rho\left(\frac{\left(q_1\delta^{(1)}+q_2\delta^{(2)}\right)^{T-2}\delta^{(1)}R^{T-1}}{\lambda^{E,T}}\right) & \text{if } \frac{\left(q_1\delta^{(1)}+q_2\delta^{(2)}\right)^{T-2}\delta^{(1)}R^{T-1}}{\lambda^{E,T}} \geq \phi'_+(0) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Now consider the efficient policy, which recall solves the problem of maximising the agent's utility subject to the firm breaking even when the agent's discount factor is $\delta^{(1)}$. The problem and analysis is the same as above if we set $q_1 = 1$ and $q_2 = 0$. A corresponding KKT solution is denoted $\left(\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^{T}, \lambda^{B,T}, \left(\mu_t^{B,T}\right)_{t=1}^{T}\right)$, where $\lambda^{B,T} > 0$. The efficient policy is thus given, for $1 \le t \le T$,

$$v_t^{B,T}\left(L^t\right) = \begin{cases} \rho\left(\frac{\left(\delta^{(1)}\right)^{t-1} R^{t-1}}{\lambda^{B,T}}\right) & \text{if } \frac{\left(\delta^{(1)}\right)^{t-1} R^{t-1}}{\lambda^{B,T}} \ge \phi'(0) \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Now let us prove the claims in the proposition. Because the firm's break-even condition is the same across the two cases, and because ρ is strictly increasing, we must have $\lambda^{B,T} \le \lambda^{E,T}$ and thus $v_1^{B,T}\left(\delta^{(1)}\right) \ge v_1^{E,T}\left(\delta^{(1)}\right)$.

Consider next $t \le T-2$, and suppose $v_t^{E,T}\left(L^t\right) \ge v_t^{B,T}\left(L^t\right)$ and $v_t^{E,T}\left(L^t\right) > 0$. Then

$$\frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} R^{t-1}}{\lambda^{E,T}} \ge \frac{\left(\delta^{(1)}\right)^{t-1} R^{t-1}}{\lambda^{B,T}}$$

for $t+1 \le s \le T-1$

$$\frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{s-1} R^{s-1}}{\lambda^{E,T}} > \frac{\left(\delta^{(1)}\right)^{s-1} R^{s-1}}{\lambda^{B,T}}$$

and

$$\frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} R^{T-1}}{\lambda^{E,T}} > \frac{\left(\delta^{(1)}\right)^{T-1} R^{T-1}}{\lambda^{B,T}}.$$

This shows that, for all $s > t$, $v_s^{E,T}\left(L^s\right) \ge v_s^{B,T}\left(L^s\right)$; and $v_s^{E,T}\left(L^s\right) > v_s^{B,T}\left(L^s\right)$ if $v_s^{E,T}\left(L^s\right) > 0$. Considering $t = T-1$, $v_{T-1}^{E,T}\left(L^{T-1}\right) \ge v_{T-1}^{B,T}\left(L^{T-1}\right)$ and $v_{T-1}^{E,T}\left(L^{T-1}\right) > u(0)$ implies

$$\frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} R^{T-2}}{\lambda^{E,T}} \ge \frac{\left(\delta^{(1)}\right)^{T-2} R^{T-2}}{\lambda^{B,T}}$$

and hence

$$\frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} R^{T-1}}{\lambda^{E,T}} \ge \frac{\left(\delta^{(1)}\right)^{T-1} R^{T-1}}{\lambda^{B,T}}.$$

Therefore, $v_T^{E,T}\left(L^T\right) \ge v_T^{B,T}\left(L^T\right)$.

□

Proof of Corollary 1. Note that $u'_+(0)$ is a standard Inada condition guaranteeing interiority of solutions by a simple perturbation argument (any policy that specifies zero utility at some date can be strictly improved by slightly increasing utility at that date and decreasing it at some other date). If $v_1^{E,T}(L) \ge v_1^{B,T}(L)$, then recalling $T \ge 3$ and the Equations (19)-(21), we must have $v_t^{E,T}\left(L^t\right) > v_t^{B,T}\left(L^t\right)$ for $t > 1$. Since $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^{T}$ generates zero firm profits, $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ must generate negative firm profits, a contradiction. Therefore, $v_1^{E,T}(L) < v_1^{B,T}(L)$. Proposition 2 implies that if $v_{t'}^{E,T}\left(L^{t'}\right) \ge v_{t'}^{B,T}\left(L^{t'}\right)$ for $t' \le T-2$, then $v_s^{E,T}\left(L^s\right) > v_s^{B,T}\left(L^s\right)$ for all $s > t'$, and in this case we can take $t^* = t'$ in the corollary. From Equations (19)-(21) and the fact that the firm earns zero profits under both $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^T$ and $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^T$, we can see that the only other possibility is that $v_{T-1}^{E,T}\left(L^{T-1}\right) > v_{T-1}^{B,T}\left(L^{T-1}\right)$ and $v_T^{E,T}\left(L^T\right) > v_T^{B,T}\left(L^{T-1}\right)$ and in this case $t^* = T-1$.

□

Proof of Proposition 3. We begin with a preliminary observation: *If $\delta^{(1)}R = 1$, then, for any T, $v_t^{B,T}\left(L^t\right) = w$ for all $1 \leq t \leq T$. If $\delta^{(1)}R < 1$, then $v_t^{B,T}\left(L^t\right)$ converges pointwise as $T \to \infty$ to a weakly decreasing sequence $\left(v_t^{B,\infty}\left(L^t\right)\right)_{t=1}^\infty$.*

The observation for $\delta^{(1)}R = 1$ is because, for each T, efficient consumption is constant over time with the firm earning zero profit. Consider then the case where $\delta^{(1)}R < 1$. There exists a $\underline{\lambda}^B > 0$ such that $\lambda_T^B \geq \underline{\lambda}^B$ for every T-period efficient problem. In particular, we have

$$\phi'\left(u\left(I_\infty\right)\right) \geq \phi'\left(v_1^B\left(L\right)\right) = \frac{1}{\lambda^{B,T}}$$

implies

$$\lambda^{B,T} \geq \frac{1}{\phi'\left(u\left(I_\infty\right)\right)},$$

as period 1 consumption does not exceed I_∞.

This implies that utility in period t, $v_t^{B,T}\left(L^t\right)$, for any horizon T, is bounded above by

$$\begin{cases} \rho\left(\frac{\left(\delta^{(1)}\right)^{t-1}R^{t-1}}{\underline{\lambda}^B}\right) & \text{if } \frac{\left(\delta^{(1)}\right)^{t-1}R^{t-1}}{\underline{\lambda}^B} \geq \phi'_+\left(0\right) \\ 0 & \text{otherwise.} \end{cases}$$

There is then $\bar{T}$ such that this bound is 0 for all $t \geq \bar{T}$; i.e., there is no consumption after $\bar{T}$. It is easy to see that, at each horizon length T, the budget constraint is satisfied with equality, hence $\lambda^{B,T}$ is in fact uniquely determined and must eventually be decreasing with T. Hence, for each t, $v_t^{B,T}\left(L^t\right)$ is non-decreasing in T for all T sufficiently large.

Moreover, noting that $v_1^{B,T}\left(L\right) > 0$ for all T, we have $\lambda^{B,T} = \frac{1}{\phi'\left(v_1^{B,T}\left(L\right)\right)}$. As $T \to \infty$, we must have $\lambda^{B,T} \to \lambda^{B,\infty}$ for some $\lambda^{B,\infty} > 0$, and utility converges pointwise to

$$v_t^{B,\infty}\left(L^t\right) = \begin{cases} \rho\left(\frac{\left(\delta^{(1)}\right)^{t-1}R^{t-1}}{\lambda^{B,\infty}}\right) & \text{if } \frac{\left(\delta^{(1)}\right)^{t-1}R^{t-1}}{\lambda^{B,\infty}} \geq \phi'_+\left(0\right) \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

In order that the firm earns zero profit for $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^T$ for large T, and because $I_T \to I_\infty$ as $T \to \infty$, we must have zero profit also in the infinite-horizon problem: i.e., $\sum_{t=1}^\infty \frac{\phi\left(v_t^{B,\infty}\left(L^t\right)\right)}{R^{t-1}} = I_\infty$.

Therefore, $v_1^{B,\infty}(L) > w$.

We now prove the result in three cases, according to the value of $\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)$.

Case 1: $\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R < 1$**.** In this case, by the same argument as argued in the observation above, the utility $v_t^{E,T}\left(L^t\right)$ converges to

$$v_t^{E,\infty}\left(L^t\right) = \begin{cases} \rho\left(\frac{\left(q_1\delta^{(1)}+q_2\delta^{(2)}\right)^{t-1} R^{t-1}}{\lambda^{E,\infty}}\right) & \text{if } \frac{\left(\delta^{(1)}\right)^{t-1} R^{t-1}}{\lambda^{E,\infty}} \geq \phi'_+(0) \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

as $T \to \infty$, where $\lambda^{E,\infty} = \lim_{T\to\infty} \frac{1}{\phi'\left(v_1^{E,T}(L)\right)} > 0$. As above, utilities $\left(v_t^{E,\infty}\left(L^t\right)\right)_{t=1}^{\infty}$ generate zero profit for the firm over the infinite horizon: $\sum_{t=1}^{\infty} \frac{\phi\left(v_t^{E,\infty}\left(L^t\right)\right)}{R^{t-1}} = I_\infty$.

We now show it is not the case that $v_1^{E,\infty}(L) = u(I_\infty)$ by supposing that it were. Then $\lambda^{E,\infty} = \frac{1}{\phi'(u(I_\infty))}$ and so

$$\begin{aligned} \phi'\left(v_2^{E,\infty}\left(L^2\right)\right) &= \frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R}{\lambda^{E,\infty}} \\ &= \left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R\phi'\left(u\left(I_\infty\right)\right) \\ &> \phi'_+(0), \end{aligned}$$

where the inequality is assumed in the statement of the proposition. Hence, $v_2^{E,\infty}\left(L^2\right) > 0$ and so the firm earns strictly negative profit from $\left(v_t^{E,\infty}\left(L^t\right)\right)_{t=1}^{\infty}$ in the infinite horizon, in contradiction to our previous claim. We have therefore shown that $v_1^{E,\infty}(L) < u(I_\infty)$. By Equation (23) and because the firm earns zero profit in the infinite horizon problem with utilities $\left(v_t^{E,\infty}\left(L^t\right)\right)_{t=1}^{\infty}$, we have $v_2^{E,\infty}\left(L^2\right) > 0$.

That the firm earns zero profits in the infinite horizon under both $\left(v_t^{E,\infty}\left(L^t\right)\right)_{t=1}^{\infty}$ and $\left(v_t^{B,\infty}\left(L^t\right)\right)_{t=1}^{\infty}$, together with Equations (22) and (23), then imply that $\lambda^{E,\infty} > \lambda^{B,\infty}$ and in particular, $v_1^{E,\infty}(L) < v_1^{B,\infty}(L)$. As a consequence, there exists a $\varepsilon > 0$ and $\bar{T}$ sufficiently large that, for all $T \geq \bar{T}$,

$$\begin{aligned} v_1^{E,T}(L) - \lambda^{B,T}\phi\left(v_1^{E,T}(L)\right) &\leq v_1^{B,T}(L) - \lambda^{B,T}\phi\left(v_1^{B,T}(L)\right) + \varepsilon \\ &= \max_{v_1(L)}\left\{v_1(L) - \lambda^{B,T}\phi\left(v_1(L)\right)\right\} + \varepsilon \end{aligned} \tag{24}$$

where we use strict convexity of ϕ and the fact that $v_1^{B,T}(L)$ maximises $v_1(L) - \lambda^{B,T}\phi(v_1(L))$, while $v_1^{E,T}(L)$ remains bounded away from $v_1^{B,T}(L)$ for all large enough T.

Therefore, for all $T \geq \bar{T}$,

$$\begin{aligned}
\sum_{t=1}^{T} \left(\delta^{(1)}\right)^{t-1} v_t^{B,T}\left(L^t\right) &= \max_{\left(v_t\left(L^t\right)\right)_{t=1}^{T}} \left\{ \sum_{t=1}^{T} \left(\delta^{(1)}\right)^{t-1} v_t\left(L^t\right) + \lambda^{B,T}\left(I_T - \sum_{t=1}^{T} \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}} \right) \right\} \\
&= \max_{\left(v_t\left(L^t\right)\right)_{t=1}^{T}} \left\{ \sum_{t=1}^{T} \left(\left(\delta^{(1)}\right)^{t-1} v_t\left(L^t\right) - \lambda^{B,T} \frac{\phi\left(v_t\left(L^t\right)\right)}{R^{t-1}} \right) + \lambda^{B,T} I_T \right\} \\
&\geq \sum_{t=1}^{T} \left(\left(\delta^{(1)}\right)^{t-1} v_t^{E,T}\left(L^t\right) - \lambda^{B,T} \frac{\phi\left(v_t^{E,T}\left(L^t\right)\right)}{R^{t-1}} \right) + \lambda^{B,T} I_T + \varepsilon \\
&= \sum_{t=1}^{T} \left(\delta^{(1)}\right)^{t-1} v_t^{E,T}\left(L^t\right) + \lambda^{B,T}\left(I_T - \sum_{t=1}^{T} \frac{\phi\left(v_t^{E,T}\left(L^t\right)\right)}{R^{t-1}} \right) + \varepsilon \\
&= \sum_{t=1}^{T} \left(\delta^{(1)}\right)^{t-1} v_t^{E,T}\left(L^t\right) + \varepsilon,
\end{aligned}$$

where the maximisation is over $v_t\left(L^t\right) \geq 0$. The first equality holds because $\left(v_t^{B,T}\left(L^t\right)\right)_{t=1}^{T}$ maximises the Lagrangian expression above, with multiplier $\lambda^{B,T}$, with the zero-profit condition satisfied with equality. The inequality follows by the inequality (24). The fourth equality holds because $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ satisfies the zero-profit condition with equality. This shows precisely the claim that $V_T^B - V_T^E \geq \varepsilon$ for all $T \geq \bar{T}$.

Case 2: $\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R = 1$. In this case, for each T, we have

$$\phi'\left(v_t^{E,T}\left(L^t\right)\right) = \frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{t-1} R^{t-1}}{\lambda^{B,T}} = \frac{1}{\lambda^{B,T}}$$

for $t \leq T-1$, while

$$\begin{aligned}
\phi'\left(v_T^{E,T}\left(L^T\right)\right) &= \max\left\{ \frac{\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right)^{T-2} \delta^{(1)} R^{T-1}}{\lambda^{B,T}}, \phi'_+(0) \right\} \\
&= \max\left\{ \frac{\delta^{(1)}}{q_1\delta^{(1)} + q_2\delta^{(2)}} \frac{1}{\lambda^{B,T}}, \phi'\left(u\left(0\right)\right) \right\}.
\end{aligned}$$

Then, because $\left(v_t^{E,T}\left(L^t\right)\right)_{t=1}^{T}$ satisfies the zero-profit condition with equality, we must have $\phi\left(v_t^{E,T}\left(L^t\right)\right) > w$ for all $t \leq T-1$. Let $\check{u}^T \equiv v_t^{E,T}\left(L^t\right)$ for $t \leq T-1$. Then

$$\sum_{t=1}^{T-1} \frac{\phi\left(\check{u}^T\right)}{R^{t-1}} \leq \sum_{t=1}^{T} \frac{w}{R^{t-1}}.$$

We conclude that $\phi\left(\check{u}^T\right) \rightarrow w$ as $T \rightarrow \infty$.

However, as noted above, given $\delta^{(1)}R < 1$, we have $\phi\left(v_1^{B,\infty}(L)\right) > w$. This shows that there is a $\varepsilon > 0$ and large enough $\bar{T}$ such that, for $T \geq \bar{T}$, the inequality (24) is satisfied. The rest of the argument is then as in Case 1.

Case 3. $\left(q_1\delta^{(1)} + q_2\delta^{(2)}\right) R > 1$**.** In this case, there is $\eta > 0$ and $\bar{T}$ sufficiently large that, for all $T \geq \bar{T}$, $v_1^{E,T}(L) < w - \eta$. Otherwise, there is a subsequence (T_m) along which $v_1^{E,T_m}(L) \geq w - \frac{1}{m}$. For m sufficiently large (for instance, using the expressions for $v_t^{E,T}\left(L^t\right)$ in Equations (19) and (20)), the policy $\left(v_t^{E,T_m}\left(L^t\right)\right)_{t=1}^{T_m}$ violates the zero-profit constraint. The remainder of the argument is then as in Case 1 and Case 2. In particular, using $\phi\left(v_1^{B,\infty}(L)\right) \geq w$, we have that there is a $\varepsilon > 0$ and large enough $\bar{T}$ such that, for $T \geq \bar{T}$, the inequality (24) is satisfied. Then the rest of the argument follows Case 1.

□

Proofs of results in Section 4

Proof of Proposition 4. In utility space, the problem can be written as maximising

$$\mathbb{E}_F\left[\sum_{t=1}^{T} \Pi_{s=1}^{t-1}\tilde{\delta}_s v_t\left(\tilde{H}^t\right)\right]$$

subject to

$$\mathbb{E}_F\left[\sum_{t=1}^{T} \frac{\phi\left(v_t\left(\tilde{H}^t\right)\right)}{R^{t-1}}\right] \leq I_T. \tag{25}$$

First consider existence. Utilities are bounded above owing to the firm's break-even condition. If $u(0)$ is finite, the problem amounts to maximising a continuous function over the compact set of values $v_t\left(H^t\right) \geq u(0)$ (where $t \in \{1, 2, \ldots, T\}$ and $H^t \in \Delta^t$) satisfying the zero-profit condition (25). In the alternative case where $\lim_{c\to 0} u(c) = -\infty$, we may without loss of optimality consider the $v_t\left(H^t\right)$ to be bounded below by some suitably small lower bound, so again we are maximising over a compact set.

Uniqueness is similarly straightforward using strict convexity of ϕ. Were there two optimal policies, a convex combination would yield the same agent expected utility but a strictly positive firm profit. This policy could be strictly improved by increasing the payment to the agent at any date and history, contradicting the optimality of the original policies.

Assuming interiority, the optimum — call it $\left(v_t^{B,T}\left(H^t\right)\right)_{t=1}^T$ — maximises the Lagrangian

$$\sum_{t=1}^{T}\sum_{n_1=1}^{N}\cdots\sum_{n_t=1}^{N}\left(\Pi_{s'=1}^{t}p_{n_{s'}}\right)\left(\Pi_{s''=1}^{t-1}\delta^{(n_{s''})}\right)v_t\left(\delta^{(n_1)},\ldots,\delta^{(n_t)}\right)$$
$$+\lambda^{B,T}\left(I_T-\sum_{t=1}^{T}\sum_{n_1=1}^{N}\cdots\sum_{n_t=1}^{N}\left(\Pi_{s=1}^{t}p_{n_s}\right)\frac{\phi\left(v_t\left(\delta^{(n_1)},\ldots,\delta^{(n_t)}\right)\right)}{R^{t-1}}\right)$$

for some multiplier $\lambda^{B,T}>0$. The stationarity condition is then, for each t, each $\left(\delta^{(n_1)},\ldots,\delta^{(n_t)}\right)$,

$$\phi'\left(v_t^{B,T}\left(\delta^{(n_1)},\ldots,\delta^{(n_t)}\right)\right)=\frac{R^{t-1}\left(\Pi_{s=1}^{t-1}\delta^{(n_s)}\right)}{\lambda^{B,T}}.$$

Because ϕ' is increasing, it follows immediately from this expression that $v_t^{B,T}\left(\delta^{(n_1)},\ldots,\delta^{(n_t)}\right)$ is strictly increasing in each index n_s, $1\leq s\leq t$.

□

Proof of Proposition 5. Given δ_1, Problem III can be written as a choice of utilities $\left(v_t\left(H^t\right)\right)_{t=1}^T$ maximising

$$\mathbb{E}_A\left[v_1\left(\delta_1\right)+\delta_1\sum_{t=2}^{T}\Pi_{s=2}^{t-1}\tilde{\delta}_s v_t\left(\delta_1,\tilde{H}_2^t\right)\right]\tag{26}$$

subject to the firm break-even condition

$$\mathbb{E}_F\left[\sum_{t=1}^{T}\frac{\phi\left(v_t\left(\delta_1,\tilde{H}_2^t\right)\right)}{R^{t-1}}\right]\leq I_T$$

and to incentive constraints for the truthful reporting of δ_t at all $t\geq 2$ and all histories $\left(\delta_1,H_2^{t-1}\right)$. (In the main text, we subsequently refer to the problem in utility space as "Problem IV", but in the proof below we refer to this still as Problem III.) In particular, for each such history, each δ_t and each $\hat{\delta}_t$, we must have

$$v_t\left(\delta_1,H_2^{t-1},\delta_t\right)+\delta_t\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1,H_2^{t-1},\delta_t,\tilde{H}_{t+1}^{\tau}\right)\right]$$
$$\geq v_t\left(\delta_1,H_2^{t-1},\hat{\delta}_t\right)+\delta_t\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1,H_2^{t-1},\hat{\delta}_t,\tilde{H}_{t+1}^{\tau}\right)\right].$$

Either $u\left(0\right)$ is finite or $\lim_{c\to 0}u\left(c\right)=-\infty$. In the latter case, it is easy to see that the optimal utilities must be bounded below. We are therefore optimising a continuous function over a compact set, and a maximum exists. Moreover, the optimum is unique: were there two optima, the convex combination would satisfy the incentive constraints and satisfy the break-even condition strictly. It would then be possible to slightly raise $v_1(\delta)$ yielding a strictly higher agent expected payoff – this contradicts the optimality of the original policies.

Now consider why any equilibrium dynamic mechanism accepted by the agent with initial type δ_1 must solve Problem III. Throughout the argument, we can associate with each date-1 message a uniquely determined continuation mechanism (a mapping from messages to consumption), and assume an agent message strategy that maximises firm profits among those optimal for the agent. Hence, because the firm anticipates i.i.d. types, there is associated with the date-1 message a unique value of discounted expected profits (independent of the agent's true initial type).

The central part of the argument is that no firm earns positive discounted expected profit in equilibrium. Suppose not. It is enough to show that any firm can deviate to a mechanism that generates arbitrarily close to the sum of expected profits across all firms. For each type $\delta_1 = \delta^{(n)}$ that accepts with positive probability a mechanism of a Firm $j(n)$ and sends a message $\hat{\delta}_1 = \delta^{(n)}$ in the putative equilibrium, outcomes will be determined by utilities $\left(v_t^{j(n)}\left(\delta^{(n)}, H_2^t\right)\right)_{t=1}^{T}$. In case of agent randomisation, consider only the firm $j(n)$ for which firm expected discounted profits are highest (and take the smallest firm index if multiple). Then consider the mechanism defined by

$$v_t\left(\delta^{(n)}, H_2^t\right) = v_t^{j(n)}\left(\delta^{(n)}, H_2^t\right)$$

for the initial types $\delta_1 = \delta^{(n)}$ that accept some mechanism with positive probability in the putative equilibrium (thus the date-1 reports in the new mechanism are a subset of Δ). This mechanism is incentive compatible in the following restricted sense. Any agent type that accepted a mechanism with positive probability in the putative equilibrium is willing to report truthfully if accepting the new mechanism. Now make two changes: (i) omit any report δ_1 that fails to generate strictly positive expected profit, and (ii) increase the initial utility $v_1(\delta_1)$ by an arbitrarily small constant amount, uniform over the initial messages δ_1. Now, for all values of the initial type δ_1 that accept a mechanism with positive probability in the putative equilibrium but not omitted in (i), the agent strictly prefers to accept this mechanism than any of the others offered, and is willing to report truthfully. This shows that the profit from each such type is at least arbitrarily close to that generated from this type by all firms in the putative equilibrium.[15] Other agent types may accept the mechanism and choose some (untruthful) report or not accept the mechanism. However, irrespective of what the agent's strategy specifies for the other types, the new mechanism obtains arbitrarily close to the expected total profit among all firms. This is a profitable deviation at least for a firm earning the lowest expected profit.

We now note that if a type $\delta_1 = \delta^{(n)}$ accepts a firm's mechanism with positive probability in

[15] While the agent is willing to report truthfully, according to the prescribed strategy, he may not do so if there is some other message that generates strictly higher profits for the firm. However, this only raises the profitability of the constructed deviation.

equilibrium, then the consumption following truthful reporting (and given the future i.i.d. evolution of types as anticipated by each firm) must generate exactly zero expected profit. Suppose instead that a strict loss is generated. Then the firm must be generating positive expected profits from the acceptance of the mechanism by some other type $\delta_1 \neq \delta^{(n)}$. There is then a profitable deviation. In particular, similar to above, the firm can omit any initial reports δ_1 not generating strictly positive expected profit, while increasing the date-1 utility for the remaining reports by an arbitrarily small and constant amount. Any agent type that was generating positive expected profits for the firm continues to generate at least arbitrarily close to this amount, while any negative profits are eliminated.

We now argue that any initial type δ_1 has equilibrium outcomes given by the solution to Problem III. Note that each type must be obtaining an (agent)-anticipated discounted payoff equal to the value in Problem III. Obtaining more would imply negative profit for the report δ_1 (which we saw in the previous paragraph was not possible in equilibrium). If a given agent type obtains less, then a firm can earn strictly positive profit by offering a mechanism that is the solution to Problem III for this agent type (requiring no date-1 message), for a value of income I_T that is below but arbitrarily close to the true value. Indeed, by the Theorem of the Maximum (using that the constraint set can be considered compact-valued, as argued above, and because it is continuous in I_T), the value of Problem III is continuous in I_T. Given that we argued equilibrium must yield firms zero profits, this is a profitable deviation. We can conclude that each initial type δ_1 anticipates discounted payoffs equal to the value of Problem III, while generating any firm whose mechanism the agent accepts zero profits. By uniqueness of the solution to Problem III, the equilibrium outcomes of type δ_1 are then unique.

Finally, it is an equilibrium for each firm to symmetrically offer the mechanism $\left(c_t\left(\delta_1, H_2^t\right)\right)_{t=1}^T$, with $\left(\delta_1, H_2^t\right) \in \Delta^t$ for each $t \leq T$ and $\left(\delta_1, H_2^T\right) \in \Delta^{T-1}$, with values uniquely solving Problem III, and for the agent to randomise symmetrically over the equilibrium mechanisms and to report truthfully. Agent play following any off-path selection of mechanisms satisfies the equilibrium requirements: the agent follows a deterministic message strategy that maximises the firm's discounted profits among those optimal for the agent (since there are finitely many messages in each period, such a strategy exists). When the mechanism satisfies our notion of a direct mechanism (see the model set-up), the agent reports truthfully. Then firms have no incentive to deviate: if a firm offers a mechanism where discounted profits associated with a given date-1 message are strictly positive, the agent does not send this message; indeed, he obtains a lower expected payoff than the value of Problem III associated with his true initial type δ_1, while this value in Problem III is available from other firm(s). Agent incentive compatibility in the equilibrium mechanism from date-2 onwards is implied by the incentive constraints (5), as argued in the main text. Agent incentive compatibility in the equilibrium mechanism at date 1 is satisfied because each initial type δ_1 receives, by reporting truthfully, outcomes that solve Problem III, where this problem is to maximise the agent's discounted payoff given δ_1 subject to constraints that do not depend on δ_1.

□

Proof of Lemma 3. To see the monotonicity in the lemma, consider any history $\left(\delta_1, H_2^{t-1}\right)$, and any adjacent discount factors $\delta^{(n)}$ and $\delta^{(n+1)}$, $n \leq N-1$. Analogous to Equations (2) and (1), we have

$$\begin{aligned} &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) + \delta^{(n+1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ \geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n+1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right], \end{aligned} \tag{27}$$

and

$$\begin{aligned} &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ \geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right]. \end{aligned} \tag{28}$$

Then as in Equations (14) and (15) in the proof of Lemma 1, we conclude that, for any $t \in \{2, \ldots, T-1\}$, any $\left(\delta_1, H_2^{t-1}\right)$,

$$v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) \geq v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) \tag{29}$$

and

$$\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \geq \mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right], \tag{30}$$

with both inequalities strict if and only if at least one of (27) and (27) hold as a strict inequality.

Now let us show that local incentive constraints imply the remaining incentive constraints. To see that downward incentive constraints hold, argue as follows. Consider any $n > 2$ and suppose that downward incentive constraints hold for all lower types. In particular, for any $n' < n-1$, we have

$$\begin{aligned} &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}\right) + \delta^{(n-1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\ \geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n-1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right]. \end{aligned}$$

By the monotonicity finding above,

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

By the local incentive constraint,

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n-1)}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

Therefore, indeed

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

We conclude that downward incentive constraints hold for all types $\delta^{(n)}$ and lower, and hence by induction all downward incentive constraints hold.

To see that upward incentive constraints hold, argue as follows. Consider any $n < N-1$ and suppose that upward incentive constraints hold for all higher types. In particular, for any $n' > n+1$, we have

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) + \delta^{(n+1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n+1)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

By the aforementioned monotonicity, we have

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T}\Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

By the local incentive constraint, we have

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n+1)}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&v_t\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \tilde{H}_{t+1}^{\tau}\right)\right] \\
\geq &v_t\left(\delta_1, H_2^{t-1}, \delta^{(n')}\right) + \delta^{(n)}\mathbb{E}_A\left[\sum_{\tau=t+1}^{T} \Pi_{s=t+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\delta_1, H_2^{t-1}, \delta^{(n')}, \tilde{H}_{t+1}^{\tau}\right)\right].
\end{aligned}$$

This establishes that incentive constraints for $\delta^{(n)}$ are satisfied upwards, and by induction upwards incentive constraints are satisfied for all types.

□

Proof of Lemma 4. Suppose for a contradiction there is $\bar{t} \in \{2, \ldots, T-1\}$, and any history $\left(\bar{\delta}_1, \bar{H}_2^{t-1}\right)$, and some $\delta^{(\bar{n})}$ with $\bar{n} < N$ such that the local upward incentive constraint fails to hold with equality. That is,

$$\begin{aligned}
&v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n})}\right) + \delta^{(\bar{n})}\mathbb{E}_A\left[\sum_{\tau=\bar{t}+1}^{T} \Pi_{s=\bar{t}+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\bar{\delta}_1, \bar{H}_2^{t-1}, \delta^{(\bar{n})}, \tilde{H}_{\bar{t}+1}^{\tau}\right)\right] \\
= &x + v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n}+1)}\right) + \delta^{(\bar{n})}\mathbb{E}_A\left[\sum_{\tau=\bar{t}+1}^{T} \Pi_{s=\bar{t}+1}^{\tau-1}\tilde{\delta}_s v_\tau\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n}+1)}, \tilde{H}_{\bar{t}+1}^{\tau}\right)\right]
\end{aligned}$$

for some $x > 0$. We can determine a new mechanism with utilities $\left(\check{v}_t\left(\bar{\delta}_1, H_2^t\right)\right)_{t=1}^{T}$ such that the only changes to utility are at the history $\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}\right)$:

$$\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) = v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) + x\sum_{n=1}^{\bar{n}} q_n$$

for $n > \bar{n}$, and

$$\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) = v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) - x\sum_{n=\bar{n}+1}^{N} q_n$$

for $n \leq \bar{n}$. The only (adjacent) incentive constraints this affects are: (i) the upward incentive constraint for $\delta^{(\bar{n})}$ now holds with equality, and (ii) the downward incentive constraint for $\delta^{(\bar{n}+1)}$ is relaxed. In particular, note that incentive constraints are unaffected for all earlier types (as expected continuation payoffs under truthful reporting are unaffected). All other adjacent incentive constraints at history $\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}\right)$ are unaffected as a constant is added to each side of the constraint.

Note also that, by incentive compatibility, $\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right)$ remains non-increasing in n, and so the consumption levels prescribed by the new mechanism are non-negative.

Now consider the effect on firm profits. The change in the cost of the contract to the firm is proportional to

$$
\begin{aligned}
&\sum_{n=1}^{\bar{n}} p_n \left(\phi\left(v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) - x\left(\sum_{m=\bar{n}+1}^{N} q_m\right)\right)\right) + \sum_{n=\bar{n}+1}^{N} p_n \left(\phi\left(v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) + x\left(\sum_{m=1}^{\bar{n}} q_m\right)\right)\right) \\
&- \sum_{n=1}^{N} p_n \left(\phi\left(v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right)\right)\right) \\
=& - \sum_{n=1}^{\bar{n}} p_n \left(\sum_{m=\bar{n}+1}^{N} q_m\right) \int_0^x \phi'\left(v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) - r\left(\sum_{m=\bar{n}+1}^{N} q_m\right)\right) dr \\
&+ \sum_{n=\bar{n}+1}^{N} p_n \left(\sum_{m=1}^{\bar{n}} q_m\right) \int_0^x \phi'\left(v_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(n)}\right) + r\left(\sum_{m=1}^{\bar{n}} q_m\right)\right) dr \\
<& - \sum_{n=1}^{\bar{n}} p_n \left(\sum_{m=\bar{n}+1}^{N} q_m\right) \int_0^x \phi'\left(\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n})}\right)\right) dr \\
&+ \sum_{n=\bar{n}+1}^{N} p_n \left(\sum_{m=1}^{\bar{n}} q_m\right) \int_0^x \phi'\left(\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n}+1)}\right)\right) dr \\
\leq& \left(\left(\sum_{m=1}^{\bar{n}} q_m\right)\left(\sum_{n=\bar{n}+1}^{N} p_n\right) - \left(\sum_{m=\bar{n}+1}^{N} q_m\right)\left(\sum_{n=1}^{\bar{n}} p_n\right)\right) \int_0^x \phi'\left(\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \delta^{(\bar{n})}\right)\right) dr. \\
\leq& 0.
\end{aligned}
$$

The first inequality uses the definition of $\check{v}_{\bar{t}}$, that $\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \cdot\right)$ is non-increasing, and that ϕ is strictly convex. The second inequality uses that $\check{v}_{\bar{t}}\left(\bar{\delta}_1, \bar{H}_2^{\bar{t}-1}, \cdot\right)$ is non-increasing, and that ϕ is convex. The final inequality follows because the agent is weakly more optimistic about patience than the firm in the sense of first-order stochastic dominance. But we have thereby obtained a mechanism that is strictly more profitable than the original, contradicting that the original solves Problem IV.

□

Proof of Corollary 2. Consider any date $t \in \{2, \ldots, T-1\}$, any history $\left(\delta_1, H_2^{t-1}\right)$. It is enough to show that, for any $n < N$, the continuation cost in Equation (6) is strictly higher for $\delta_t = \delta^{(n+1)}$ than for $\delta_t = \delta^{(n)}$ whenever the continuation policy is distinct. Suppose not. By Lemma 4, type $\delta^{(n)}$ is willing to report $\delta^{(n+1)}$ (and report truthfully thereafter). This single change in reporting provides a different reporting strategy for the agent, inducing different outcomes — this mapping from discount factors to utilities/consumption can be induced by a revised direct mechanism. The revised mechanism is distinct from the original, gives the agent the same expected payoff, but weakly lowers the firm's expected cost. Hence, the new mechanism respects all constraints and hence represents a distinct solution to Problem IV. This contradicts uniqueness of the solution to Problem IV.

□

Proof of Example 1. Fix δ_1 and write $v_1 = v_1(\delta_1)$, while we let $w_n = v_2\left(\delta_1, \delta^{(n)}\right)$ and $z_n = v_3\left(\delta_1, \delta^{(n)}\right)$. We further introduce $U_n = w_n + \delta^{(n)} z_n$.

The firm's break-even condition is

$$\phi(v_1) + \sum_{n=1}^{N} p_n \left(\frac{\phi(w_n)}{R} + \frac{\phi(z_n)}{R^2} \right) \leq I_3,$$

while we use Lemma 4 to substitute in the incentive constraints. Given U_1 and $(z_n)_{n=2}^{N}$, we use the $N-1$ upward incentive constraints holding with equality to determine $(w_n)_{n=2}^{N}$. In particular, we observe that for $k \in \{2, \ldots, N\}$,

$$U_k = U_1 + \sum_{n=2}^{k} \left(\delta^{(n)} - \delta^{(n-1)} \right) z_n$$

and therefore

$$w_k = U_1 + \sum_{n=2}^{k} \left(\delta^{(n)} - \delta^{(n-1)} \right) z_n - \delta^{(k)} z_k.$$

Problem IV can then be written as maximising

$$v_1 + \delta_1 \sum_{n=1}^{N} q_n U_n = v_1 + \delta_1 \left(U_1 + \sum_{n=2}^{N} \bar{Q}_n z_n \left(\delta^{(n)} - \delta^{(n-1)} \right) \right),$$

where $\bar{Q}_n = \sum_{m=n}^{N} q_m$, subject to the firm's break-even condition and non-negativity constraints. Rather than imposing the negativity constraints, we study a relaxed program where all utilities may be negative, but the implied payment for an arbitrary utility v is $\phi(v) = v^2$. We then confirm that, when N is sufficiently large, all utilities are strictly positive and hence the solution to the relaxed problem is the solution to the problem of interest.

Necessary conditions for an optimum. Due to the firm's break-even condition, the feasible utilities come from a compact set. The same argument as in Proposition 5 then implies the existence of a unique solution to the relaxed problem. We can then study a Lagrangian with only the firm's break-even condition

$$\begin{aligned}\mathcal{L} =& v_1 + \delta_1 \left(w_1 + \delta^{(1)} z_1 + \sum_{n=2}^{N} \bar{Q}_n z_n \left(\delta^{(n)} - \delta^{(n-1)} \right) \right) \\ &+ \lambda \left(I_3 - \phi\left(v_1\right) - \sum_{n=1}^{N} p_n \left(\frac{\phi\left(w_1 + \delta^{(1)} z_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(n)} z_n\right)}{R} + \frac{\phi\left(z_n\right)}{R^2} \right) \right).\end{aligned} \tag{31}$$

The stationarity conditions are as follows. Optimality of v_1 yields the necessary condition

$$1 = \lambda \phi'\left(v_1\right).$$

Optimality of w_1 yields the necessary condition

$$\delta_1 = \frac{\lambda}{R} \sum_{n=1}^{N} p_n \phi' \left(w_1 + \delta^{(1)} z_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)} \right) z_i - \delta^{(n)} z_n \right). \tag{32}$$

Then, optimality of z_n for $n \geq 2$ yields

$$\begin{aligned}0 =& \left(\delta^{(n)} - \delta^{(n-1)}\right) \left(\delta_1 \bar{Q}_n - \sum_{m=n}^{N} p_m \frac{\lambda}{R} \phi' \left(U_1 + \sum_{i=2}^{m} \left(\delta^{(i)} - \delta^{(i-1)} \right) z_i - \delta^{(n)} z_n \right) \right) \\ &+ \frac{\lambda p_n \delta^{(n)}}{R} \phi' \left(U_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)} \right) z_i - \delta^{(n)} z_n \right) - \frac{\lambda p_n}{R^2} \phi'\left(z_n\right),\end{aligned} \tag{33}$$

while optimality of z_1 yields

$$\delta^{(1)} R \phi'\left(w_1\right) = \phi'\left(z_1\right). \tag{34}$$

Given the zero-profit condition is an inequality constraint, we have $\lambda \geq 0$, while stationarity requires $\lambda > 0$. Note that Equation (32) and $\phi(u) = 2u$ imply that, as $\lambda \to 0$, mean date-2 utility diverges, i.e. $\sum_{n=1}^{N} p_n w_n \to \infty$, and so there is a violation of the firm's break-even condition. Hence, λ remains bounded away from zero as $N \to \infty$.

Taking differences. We now use the specific values of q_n, p_n and ϕ. Considering $n \in \{1, \ldots, N-1\}$, taking differences of the optimality conditions for z_n (the condition for z_{n+1} less the condition for z_n), we have

$$\begin{aligned}0 =& \frac{\bar{\delta} - \underline{\delta}}{N-1} \frac{1}{N} \left(-\delta_1 + \frac{\lambda}{R} 4 \left(U_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)} \right) z_i - \delta^{(n)} z_n \right) \right) \\ &- \left(z_{n+1} - z_n\right) \left(2 \frac{\lambda \frac{1}{N}}{R^2} + 2 \frac{\lambda \frac{1}{N}}{R} \delta^{(n+1)} \delta^{(n)} \right).\end{aligned} \tag{35}$$

Considering $n \in \{1, \ldots, N-2\}$ and taking a further difference, yields

$$z_{n+2} - z_{n+1} = (z_{n+1} - z_n) \frac{\frac{1}{R} + \delta^{(n+1)}\delta^{(n)} - 2\frac{\bar{\delta}-\underline{\delta}}{N-1}\delta^{(n)}}{\frac{1}{R} + \delta^{(n+2)}\delta^{(n+1)}}. \tag{36}$$

Given N is sufficiently large, the signs of both sides are the same: they are either both strictly positive, both strictly negative, or both zero. That is, z_n is either increasing, decreasing, or constant.

Proof that z_n is strictly increasing. We now show that z_n is increasing by assuming otherwise and reaching a contradiction if N is large. We will use that, for $n \in \{1, \ldots, N-1\}$,

$$\begin{aligned} w_{n+1} - w_n =& U_1 + \sum_{i=2}^{n+1} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(n+1)} z_{n+1} \\ & - \left(U_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(n)} z_n\right) \\ =& \delta^{(n)} \left(z_n - z_{n+1}\right). \end{aligned}$$

Since z_n is weakly decreasing (either constant or decreasing) w_n is weakly increasing (constant or increasing). Therefore, recalling Equation (32),

$$\delta_1 \leq 2\frac{\lambda}{R}\left(U_1 + \sum_{i=2}^{N}\left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right),$$

with a strict inequality is z_n is strictly decreasing. Therefore, by Equation (35) evaluated at $n = N-1$,

$$\begin{aligned} 0 =& \frac{\bar{\delta}-\underline{\delta}}{N-1}\frac{1}{N}\left(\begin{array}{c} -\delta_1 + 4\frac{\lambda}{R}\left(U_1 + \sum_{i=2}^{N}\left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right) \\ +4\frac{\lambda}{R}\delta^{(N-1)}\left(z_N - z_{N-1}\right) \end{array}\right) \\ & - \left(z_N - z_{N-1}\right)\left(2\frac{\lambda\frac{1}{N}}{R^2} + 2\frac{\lambda\frac{1}{N}}{R}\delta^{(N)}\delta^{(N-1)}\right) \\ \geq& \frac{\bar{\delta}-\underline{\delta}}{N-1}\frac{1}{N}\delta_1 + \left(z_N - z_{N-1}\right)\frac{\lambda}{RN}\left(4\delta^{(N-1)}\frac{\bar{\delta}-\underline{\delta}}{N-1} - \left(\frac{2}{R} + 2\delta^{(N)}\delta^{(N-1)}\right)\right). \end{aligned}$$

Since $z_N \leq z_{N-1}$, the right side of this inequality is strictly positive for N sufficiently large. Hence, we obtain a contradiction whenever N is large enough.

Proof that z_n is strictly positive for all n. Suppose then that z_n is strictly increasing. We have by Equation (32) that

$$\delta_1 < 2\frac{\lambda}{R} w_1 = 2\frac{\lambda}{R}\left(U_1 - \delta^{(1)} z_1\right). \tag{37}$$

By Equation (34),

$$U_1 - \delta^{(1)} z_1 = \frac{z_1}{R\delta^{(1)}}$$

or

$$z_1 = \frac{U_1}{\delta^{(1)} + \frac{1}{R\delta^{(1)}}}.$$

Plugging into Equation (37) and rearranging, we have

$$U_1 > \frac{R^2 \left(\delta^{(1)}\right)^2 + R}{2\lambda} \delta_1.$$

Therefore, $z_1 > 0$, which shows that $z_n > 0$ for all n.

Proof that w_n is strictly positive for all n. Condition (34) requires that utility is efficiently distributed between date 2 and date 3 conditional on the lowest type $\delta^{(1)}$, which implies the positivity of w_1. However, w_n is decreasing, so we need to check that $w_N > 0$ also. Here, we use Equation (33), evaluated at $n = N$. If the first term is sufficiently small, then this equation also states efficient distribution of utility between date 2 and date 3 conditional on the highest type $\delta^{(N)}$, which would be enough to conclude the result. We therefore need to show that the first term in Equation (33) vanishes sufficiently fast with N. It is enough to find a lower bound on $w_N = U_1 + \sum_{i=2}^{N} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N$ that does not depend on N – doing this constitutes the bulk of the argument.

We will use from the previous step the following observations. First,

$$z_1 > \frac{\frac{R^2\left(\delta^{(1)}\right)^2 + R}{2\lambda}\delta_1}{\delta^{(1)} + \frac{1}{R\delta^{(1)}}}, \tag{38}$$

and therefore, second,

$$w_1 = U_1 - \delta^{(1)} z_1 = \frac{z_1}{R\delta^{(1)}} > \frac{R}{2\lambda}\delta_1. \tag{39}$$

We are considering the maximiser of the Lagrangean in Equation (31) and therefore the following perturbation must not increase the value in that problem: for all n, add/subtract a constant (independent of n) from the date-3 utility z_n, all else fixed. Thus w_1 is unchanged, and the implied values $(w_n)_{n=2}^{N}$ are unchanged. That this perturbation does not raise the value in Equation (31) requires

$$\delta_1 \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)} - \sum_{n=1}^{N} \frac{1}{N} \frac{\lambda}{R^2} \phi'(z_n) = 0.$$

Since z_n is increasing, this implies

$$\frac{\lambda}{R^2} \phi'(z_1) < \delta_1 \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}$$

or

$$z_1 < \frac{R^2 \delta_1}{2\lambda} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}.$$

This implies that

$$w_1 = U_1 - \delta^{(1)} z_1 = \frac{z_1}{R\delta^{(1)}} < \frac{R\delta_1}{2\lambda\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}. \tag{40}$$

From Equation (35) evaluated at $n = 1$, we have

$$\begin{aligned} z_2 - z_1 =& \frac{\frac{\bar{\delta}-\underline{\delta}}{N-1} \frac{1}{N} \left(-\delta_1 + \frac{\lambda}{R} 4 \left(U_1 - \delta^{(1)} z_1\right)\right)}{2\frac{\lambda\frac{1}{N}}{R^2} + 2\frac{\lambda\frac{1}{N}}{R} \delta^{(2)}\delta^{(1)}} \\ <& \frac{\frac{\bar{\delta}-\underline{\delta}}{N-1} \frac{1}{N} \left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{2\frac{\lambda\frac{1}{N}}{R^2} + 2\frac{\lambda\frac{1}{N}}{R} \delta^{(2)}\delta^{(1)}}, \end{aligned}$$

where the inequality follows from Equation (40). This can be used to provide an upper bound on $z_n - z_{n-1}$ also for $n \in \{3, \ldots, N\}$. In particular, from Equation (36), we have for N sufficiently large that $z_n - z_{n-1}$ has the same (positive) sign as $z_{n-1} - z_{n-2}$, but is smaller. Therefore, for $n \geq 2$,

$$\begin{aligned} & w_n - w_{n-1} \\ =& U_1 + \sum_{i=2}^{n} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(n)} z_n - \left(U_1 + \sum_{i=2}^{n-1} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(n-1)} z_{n-1}\right) \\ \geq& - \delta^{(n-1)} \left(z_2 - z_1\right) \\ >& - \delta^{(n-1)} \frac{\frac{\bar{\delta}-\underline{\delta}}{N-1} \left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{2\frac{\lambda}{R^2} + 2\frac{\lambda}{R} \delta^{(2)}\delta^{(1)}}. \end{aligned}$$

And so,

$$\begin{aligned} & w_N - w_1 \\ =& U_1 + \sum_{i=2}^{N} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N - \left(U_1 - \delta^{(1)} z_1\right) \\ \geq& - \sum_{n=2}^{N} \delta^{(n-1)} \left(z_2 - z_1\right) \\ >& - \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right) \left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{2\frac{\lambda}{R^2} + 2\frac{\lambda}{R} \delta^{(2)}\delta^{(1)}}. \end{aligned}$$

Recalling that λ remains bounded away from zero as $N \to \infty$, this provides a lower bound on w_N

that is invariant to N. In particular, we have

$$\begin{aligned}
w_N =& U_1 + \sum_{i=2}^{N} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N \\
>& U_1 - \delta^{(1)} z_1 - \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{2\frac{\lambda}{R^2} + 2\frac{\lambda}{R}\delta^{(2)}\delta^{(1)}} \\
>& \frac{R}{2\lambda}\delta_1 - \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{2\frac{\lambda}{R^2} + 2\frac{\lambda}{R}\delta^{(2)}\delta^{(1)}} \\
=& \frac{1}{\lambda}\left(\frac{R}{2}\delta_1 - \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{\frac{2}{R^2} + \frac{2}{R}\delta^{(2)}\delta^{(1)}}\right).
\end{aligned}$$

We will then use the implication that

$$\begin{aligned}
& \delta_1 - \frac{\lambda}{R}\phi'\left(U_1 + \sum_{i=2}^{N} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right) \\
<& \delta_1 - \frac{2\lambda}{R}\frac{1}{\lambda}\left(\frac{R}{2}\delta_1 - \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{\frac{2}{R^2} + \frac{2}{R}\delta^{(2)}\delta^{(1)}}\right) \\
=& \delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{\frac{2}{R^2} + \frac{2}{R}\delta^{(2)}\delta^{(1)}}.
\end{aligned} \tag{41}$$

Now note that the necessary condition in Equation (33), evaluated at $n = N$, is given by

$$0 = \begin{array}{l} \frac{\bar{\delta}-\underline{\delta}}{N-1}\frac{1}{N}\left(\delta_1 - \frac{\lambda}{R}\phi'\left(U_1 + \sum_{i=2}^{N}\left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right)\right) \\ + \frac{\lambda\frac{1}{N}\delta^{(N)}}{R}\phi'\left(U_1 + \sum_{i=2}^{N}\left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right) - \frac{\lambda\frac{1}{N}}{R^2}\phi'\left(z_N\right). \end{array}$$

By Equation (41), we then have

$$\begin{aligned}
0 <& \frac{\bar{\delta} - \underline{\delta}}{N-1}\frac{1}{N}\delta^{(N)} \frac{\left(\bar{\delta} - \underline{\delta}\right)\left(-\delta_1 + \frac{2\delta_1}{\delta^{(1)}} \sum_{n=1}^{N} \frac{1}{N} \delta^{(n)}\right)}{\frac{2}{R^2} + \frac{2}{R}\delta^{(2)}\delta^{(1)}} \\
&+ \frac{\lambda\frac{1}{N}\delta^{(N)}}{R}\phi'\left(U_1 + \sum_{i=2}^{N} \left(\delta^{(i)} - \delta^{(i-1)}\right) z_i - \delta^{(N)} z_N\right) - \frac{\lambda\frac{1}{N}}{R^2}\phi'\left(z_N\right).
\end{aligned}$$

This implies, using that z_n is increasing,

$$\begin{aligned}
&\frac{1}{R\delta^{(N)}}\phi'(z_1) - \frac{\left(\bar{\delta}-\underline{\delta}\right)^2\left(-\delta_1+\frac{2\delta_1}{\delta^{(1)}}\sum_{n=1}^{N}\frac{1}{N}\delta^{(n)}\right)}{\lambda(N-1)\left(\frac{2}{R}+2\delta^{(2)}\delta^{(1)}\right)}\\
&<\phi'\left(U_1+\sum_{i=2}^{N}\left(\delta^{(i)}-\delta^{(i-1)}\right)z_i-\delta^{(N)}z_N\right)\\
&=2w_{N.}
\end{aligned}\tag{42}$$

By Equation (38) left-hand side of the inequality in Equation (42) is greater than

$$\frac{2}{R\delta^{(N)}}\frac{\frac{R^2\left(\delta^{(1)}\right)^2+R}{2\lambda}\delta_1}{\delta^{(1)}+\frac{1}{R\delta^{(1)}}} - \frac{\left(\bar{\delta}-\underline{\delta}\right)^2\left(-\delta_1+\frac{2\delta_1}{\delta^{(1)}}\sum_{n=1}^{N}\frac{1}{N}\delta^{(n)}\right)}{\lambda(N-1)\left(\frac{2}{R}+2\delta^{(2)}\delta^{(1)}\right)}$$

which becomes strictly positive as $N\to\infty$, since the second term vanishes while the first does not. This shows that w_N is indeed strictly positive for large N as required.

□

Proof of Proposition 6. The arguments in the main text imply that, for $n\geq 2$, we have $\delta^{(n)}R\phi'(w_n)<\phi'(z_n)$. Consider reducing utility at date 3 by ε, and increasing it at date 2 by $\delta^{(n)}\varepsilon$, thus leaving discounted utility unchanged. Then this decreases the NPV of firm costs in case $\delta_2=\delta^{(n)}$ by

$$-\delta^{(n)}\phi'(w_n)\varepsilon+\frac{\phi'(z_n)}{R}\varepsilon+o(\varepsilon)$$

which is strictly positive for ε small enough. The additional profit can be used to further increase the date-2 utility, i.e. a profit-preserving change that increases the agent's payoff.

□

Proof of Proposition 7. Consider equilibrium utility, and consider any $t\in\{2,\ldots,T-1\}$, any history $\left(\delta_1,H_2^{t-1}\right)$ (the remaining case that considers shifting utility between date 1 and date 2 is analogous and omitted). Consider reducing date t utility at this history by a constant $\eta\in\mathbb{R}$ independent of δ_t and raising utility at $t+1$ by $\nu\in\mathbb{R}$. The change in expected profits conditional on $\left(\delta_1,H_2^{t-1}\right)$ is, to the first order

$$\sum_{n'=1}^{N}p_n\frac{\phi'\left(v_t^{E,T}\left(\delta_1,H_2^{t-1},\delta^{(n')}\right)\right)}{R^{t-1}}\eta-\sum_{n''=1}^{N}\sum_{m=1}^{N}p_{n''}p_m\frac{\phi'\left(v_{t+1}^{E,T}\left(\delta_1,H_2^{t-1},\delta^{(n'')},\delta^{(m)}\right)\right)}{R^t}\nu.$$

In particular, for the change to leave firm profits unchanged requires

$$\nu = \frac{\sum_{n=1}^{N} p_n \frac{\phi'\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right)\right)}{R^{t-1}}}{\sum_{n=1}^{N}\sum_{m=1}^{N} p_n p_m \frac{\phi'\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \delta^{(m)}\right)\right)}{R^{t}}}\eta + o\left(\eta\right)$$

where $o\left(\eta\right)$ represents terms that vanish faster than η as $\eta \to 0$.

Note that because utility changes by a constant, independent of δ_t and δ_{t+1}, the change leaves incentive constraints unaffected. The date-t value to the agent of the change is

$$-\eta + \sum_{n=1}^{N} q_n \delta^{(n)} \nu = \eta \left[-1 + \sum_{i=1}^{N} q_i \delta^{(i)} \frac{\sum_{n=1}^{N} p_n \frac{\phi'\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right)\right)}{R^{t-1}}}{\sum_{n=1}^{N}\sum_{m=1}^{N} p_n p_m \frac{\phi'\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \delta^{(m)}\right)\right)}{R^{t}}}\right] + o\left(\eta\right).$$

Since the change leaves both firm profits and incentive constraints unaffected, it must not raise the agent's expected payoff. Therefore, a necessary condition for optimality in Problem IV is

$$\sum_{n=1}^{N} p_n \sum_{m=1}^{N} p_m \phi'\left(v_{t+1}^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}, \delta^{(m)}\right)\right) = R \sum_{i=1}^{N} q_i \delta^{(i)} \sum_{n=1}^{N} p_n \phi'\left(v_t^{E,T}\left(\delta_1, H_2^{t-1}, \delta^{(n)}\right)\right).$$

This establishes the first equality in the proposition. The equalities relating to the efficient policy follow from analogous reasoning.

□

Proof of Corollary 3. The corollary follows by taking expectations of the equalities in the proposition over histories $\tilde{H}_2^{t-1}$, and using that agent beliefs first-order stochastically dominate the firm beliefs so that $\mathbb{E}_A\left[\tilde{\delta}_t\right] \geq \mathbb{E}_F\left[\tilde{\delta}_t\right]$, with the distributions differing and hence a strict inequality if the agent is strictly more optimistic than the firm.

□