Catchment Areas, Stratification and Access to Better Schools

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Abstract

School Choice programs are intended to offer students access to better schools than their neighborhood school. This is of paramount importance for students coming from disadvantaged areas, for stratified districts with unambiguously bad schools. So Access to Better Schools is a matter of efficiency yet also a matter of fairness. We illustrate with a simple theoretical model and with complementary numerical simulations that Top Trading Cycles provides more access to better schools in general and particularly for disadvantaged students, as compared to Deferred Acceptance. The intuition is twofold: 1) the well-known interrupters problem overwhelms real choice under DA for realistically large market sizes, 2) in TTC, disadvantaged students have better chances for "leftover" seats at underprioritized good schools.

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1 Introduction

A large fraction of OECD countries have expanded school choice in the last two decades. In the past every child was assigned a spot in a school in the neighborhood, but as noted in the Friedman Foundation for Educational Choice's website: "School Choice is a public policy that allows parents/guardians to choose a school regardless of residence and location". Hence, a basic idea behind the implementation of choice, is that it shall facilitate access to other schools than the default school. Or in other words, school choice's subsumes providing *access to better schools* (ABS) for families, beyond their defaults school, that is, their neighborhood school.

ABS could be regarded as an aggregate count of efficiency of the final allocation. Nevertheless, one could consider ABS for particular groups, particularly for a set of disadvantaged students. In such a case, ABS becomes also a measure of fairness.

This paper shows that, under some rather common features of the market and the choice mechanisms, this essential requirement of a school choice program may not hold. In particular in large markets with school stratification and where priority is given to residents in the catchment area of the school, we show that Deferred Acceptance, the most popular assignment mechanisms substituting the formerly popular Boston Mechanism, may provide close to zero ABS. In the case of the Boston mechanism, the existence of a bad school leads parents to play the safest strategy of applying for the school they have highest priority for. For DA, the existence of a set of schools that are generally perceived as worse by all applicants will lead to an over-assignment of children to neighborhood schools.

We compare Gale-Shapley Deferred Acceptance (DA) to Top Trading Cycles (TTC),[1] both introduced in the school choice literature by Abdulkadiroğlu and Sönmez (2003). Demand that exceeds school capacity is resolved by ordering applicants according to priorities and random lotteries. We consider the case with coarse priorities defined by residence in the neighborhood of the school. We introduce stratification between schools: there is a *bad* school, a school that all families believe is the worst. We study the extent to which families can move away from their neighborhood school, the school they are given priority for by the authorities and the default school when there is no school choice. For this purpose we define *Access to Better School* (ABS), which is *the expected fraction of individuals who are allocated a school that is better*

[1] We also evaluate ABS under the Boston Mechanism in the discussion section.

than their neighborhood school.

We show that priorities for neighborhood and stratification may limit ABS drastically under DA. When all good schools are weakly *overprioritized* (i.e. they have no less students living in the neighborhood than capacity), *all* students will, at best, be allocated to their neighborhood school, independently of their preferences. To understand why DA fails in this case, consider the simple example where all schools have equal capacity and equal mass of prioritized students, that is 1/3. Clearly, no student in the catchment area of a good school can end up in the bad school under DA (access to neighborhood school is guaranteed in any round). In other words, all student living in the catchment area of the bad school are condemned to stay there, regardless of their tie-breaking lottery number. Students with priority in the different good schools may want to "exchange" their slots. Nevertheless, when applying for their preferred school, since they do not have priority for it, they will have to get a higher lottery number than any of the individuals living in the bad neighborhood. In a reasonably big economy, the individual with highest lottery number in the bad neighborhood would almost systematically win, blocking ABS for individuals in the good neighborhoods. We illustrate that this well-known problem of the interrupters (Kesten, 2010) has overwhelming consequences for rather realistic school capacities. Therefore, stability brings low ABS here– see Roth (2008) for more on the limitations imposed by stability. [2] The moral is: a stable allocation that gives no chances for improvement to disadvantaged students will be an assignment with low access to betters schools in general, and vice versa.

Our base model provides a wide characterization of ABS, where not all scenarios are doomed. For instance, we allow for the possibility of having *underprioritized* good schools. However, we also illustrate that TTC becomes a good alternative to DA. We show that TTC dominates DA in terms of ABS. Under TTC, individuals preferring each others' schools can always trade (this is precisely how the mechanism operates). Intuitively, the interrupter problem does not arise in TTC. From Gale and Shapley (1964) we know that TTC guarantees Pareto-optimality of the final allocation. Yet

[2] This result is very related to the results in Combe, Tercieux and Terrier (2017) where they show that DA for teaching assignment to schools allows for limited number of reassignments if teachers cannot be forced to move out of their current school. Having priority for the school you are currently assigned before you apply for a new school needs to be guaranteed for the mechanism to be individually rational. But then under DA there is limited access to a different school, which is a similar notion to ABS. Hence, their result is a special case of ours, where the number of prioritized seats is identical to the number of available seats in the schools.

this does not imply that TTC Pareto-dominates DA. Consequently, the finding that TTC ABS-dominates DA is not an obvious implication of what we already knew from the literature.

Perhaps more strikingly, we find that disadvantaged students with priority at bad schools always have higher chances to access better schools under TTC than under DA. TTC is not only more efficient than DA according to the ABS count, it is also fairer!

This finding appears somewhat unintuitive at first glance. TTC works on the basis of trading ideas, and no one wants to "trade priority rights" with a disadvantaged students. Hence, a disadvantaged student will have weakly lower chances at any good school than any other student. This is not the case of DA, where all nonprioritized students have equal chances at a given school.

The point we make is that, by allowing for trades, disadvantaged students obtain an advantage in further steps of the TTC allocation algorithm. When the time arrives to distribute "leftovers", that is, remaining seats in underprioritized schools, and conditional on the trades made, disadvantaged students have better lottery numbers, and hence better access chances.

Notice that we do not claim the TTC allocation to be weakly preferable to the DA allocation for all disadvantaged students. TTC gives more chances to access *a* school better than the worst school, yet maybe less chances to access the student's favorite school.

Empirical basis. The key feature in our model is the **vertical differentiation of schools**. In the US, for instance, the concept of "failing school" is prevalent in the policy arena and in the media and it refers to the schools that have had poor performance for two years in a row, constituting 10% of the schools in the US.[3] The media emphasizes how some families have a hard time moving out of their neighborhood failing schools, how choice does not necessarily improve opportunities for disadvantaged families. However, empirical evidence on families preferences coinciding at the bottom of the ordering is scarce. The main challenge is that preferences are unobservable and that the mechanisms that elicit them often do so through a

[3]In the US the requirement of the federal No Child Left Behind Public Choice Program requires that local school districts allow students in academically unacceptable schools (F-rated schools) to transfer to higher performing, non-failing schools in the district– if there is capacity available. See See Title I Public School Choice for schools identified as Low Performing: http://www.ncpie.org/nclbaction/publicchoice.html.

manipulable mechanism. Hence, empirical evidence on this comes from structurally estimated preferences. He (2017) structurally estimates preferences over 4 colleges in Beijing under the Boston mechanism and finds that one of the schools is surely ranked fourth for at least 58% of individuals.[4] This is also the worse school in terms of average academic performance. He (2017) also finds that only 5% of students rank it as their first choice. Calsamiglia, Fu and Güell (2020) show that 44% of schools in Barcelona are filled up in the first round and 40% are never filled up. Similarly, Table 4 in Agarwal and Somaini (2016) shows that one of the pre schools in Cambridge (USA) King Open Ola, is ranked in the submitted list only by five families, while the next best schools already has 51 applicants. Combe, Tercieux and Terrier (forthcoming) analyze teacher assignment to schools in France, where there are specific schools that no teachers wants to be assigned to. The lack of desire for such schools is a problem that has been partially resolved by guaranteeing future priority to move to better preferred schools to teachers previously assigned to such schools. Hence, market stratification is not only common in school choice by students but also by teachers.

An aspect that makes our results more drastic is schools being **overprioritized**, which means that the number of applicants with priority for that school is larger or equal than the number of seats. This is true in markets like the teachers' market in France or the school choice market in cities where individuals have a guaranteed spot in their neighborhood, such as in the Charlotte-Mecklenburg Public School District (see Hastings and Weinstein, 2008). In most school choice markets there is a transition from a neighborhood based assignment to a centralized assignment with priorities for neighborhood, where the previously assigned school becomes the prioritized school. Also administrations want to guarantee that families have access to a close-by school. Hence weak overprioritization is not unlikely even in the cases where it is not imposed through providing a default school assignment.

Our results show that under both DA and with neighborhood priorities for all seats a large fraction of families will be assigned their neighborhood school. This is also the case of other mechanisms such as the Boston Mechanisms, which we also comment on the discussion section. Calsamiglia and Güell (2018) show that in Barcelona priorities play a large role in determining the list submitted by parents under the BM. They

[4]For the remaining families this fact cannot be proved to be true, but is not rejected either.

exploit a change in the definition of neighborhoods in the city of Barcelona to identify that a large fraction of parents apply for the neighborhood school, independently of their preferences. On the other hand, Calsamiglia, Fu and Güell (2020) also perform counterfactual analysis of the allocation that would result if DA or TTC were implemented instead. In Barcelona around 40% of families prefer a school outside of their neighborhood. Table 19 in their paper shows that for families whose favorite school is *not* their neighborhood school, both BM and DA assign them to their favorite school less often than TTC, the proportion of assignment to out of neighborhood school being 47.2%, 41.8% and 58.9% for BM, DA and TTC respectively. For DA, despite the fact that families can be truthful and reveal that they want to move out of their neighborhood, the mechanism assigns them most often to the neighborhood school. On the other hand we also see that TTC clearly facilitates families moving out of their neighborhood, more than DA and BM. Hence, the loss due to overassignment to neighborhood school induced by both DA and BM limits the power of families' preferences to determine the allocation of students to schools.

The provided evidence is no direct proof of our assumptions or conclusions, but is suggestive that our analysis can shed light on why we may have limited ABS in some cities where school choice is implemented in a stratified school system and where neighborhood priorities are warranted.

Literature. The literature has emphasized different properties of the norms characterizing assignment mechanisms: strategy-proofness, stability and efficiency. The first property consists of providing incentives to reveal true preferences independently of what other families do, referred to as the mechanism being strategy proof. The Gale-Shapley Deferred Acceptance algorithm (DA) does have this property and therefore elicit true preferences. This greatly simplifies matters for families. DA is also valued because its resulting allocation is *stable*. Stability requires the final allocation to be such that we cannot simultaneously have 1) an individual who prefers a given school to her assigned school, and 2) the preferred school has another individual admitted with lower priority than she has for that school. Importantly, the results on DA in this paper apply to any stable mechanism. But the DA allocation is not Pareto efficient except for some specific priority structures (Ergin, 2002). Pareto-efficiency is defined as the lack of an alternative allocation that makes an individual better off without making another individual worse off. The Top Trading Cycles mechanism (TTC), also described in the next section, is strategy proof and efficient, but is not stable. There is no mechanism that has the three properties (Kesten, 2010). But the efficiency costs of DA, as measured in experiments, such as Chen and Sönmez (2006), are small and so DA has actually been adopted in cities like New York and Boston, substituting the former mechanism, referred to as the Boston Mechanism.[5]

Both DA and BM, or a combination of the two (see Chen and Kesten (2013)) are *by far the most debated alternatives*.[6] TTC has been used in New Orleans, and for some time in San Francisco, as far as the authors are aware.[7] This paper suggests that the choice between Deferred Acceptance and the Boston Mechanism may be less important, given that in both cases the final allocation of students is largely determined by (neighborhood) priority rules.

An important reference is the seminal paper Kesten (2010), which shows that for any vector of school capacities and any set of students, there are priority structures *and* individual preferences such that the stable-optimal allocation gives each student one of her two worst options.[8] Kesten solves this problem by introducing the Efficiency-Adjusted Deferred Acceptance Mechanism (EADAM) in which each student previously consents on waiving priority rights that have no impact on her final allocation, while they may harm other students' final placement. Our paper points at a similar direction, however with some important differences. We instead show that, under the existence of bad schools, a common and simple catchment area priority structure determines the allocation to a large extent, *regardless* students' preferences for all schools except the worse schools, for which preference order need to be lowest. Moreover, we show that the interrupter problem signalled by Kesten has an enormous bite in our model for reasonable capacity numbers.

The present paper does not analyze mechanisms not used in current practice which are susceptible of manipulation. Instead, it takes strategy-proofness as a value to keep, and compares the mechanisms suggested in the seminal paper by Abdulkadiroglu and

[5] Experiments evaluating the efficiency cost have been done in the lab, and the simulated environments used did not contain *bad schools*, as we model them here or are found in the data. This paper suggests that under the presence of bad schools efficiency losses may be very large, since preferences may have a rather small effect on the final allocation.

[6] Important contributions to this debate include Abdulkadiroğlu and Sönmez (2003); Abdulkadiroğlu, Pathak, Roth and Sönmez (2006); Ergin and Sönmez (2006); Miralles (2008); Pathak and Sönmez (2008); Abdulkadiroğlu, Che and Yasuda (2011)

[7] Actually, the mechanism used in San Francisco was a manipulable variant of TTC. It was substituted by Deferred Acceptance in 2019.

[8] Following Kesten, there is recent literature advocating for a relaxation of the stability concept (Ehlers and Morrill, 2019; Troyan et. al., 2020.)

Sonmez (2003.)

Our paper considers reasonable coarse priority structures[9] lying in between two rather extreme models in the school choice literature: the strict priority model (e.g. Ergin and Sönmez, 2006; Pathak and Sönmez, 2008) and the no-priorities model (Abdulkadiroğlu, Che and Yasuda, 2011; Miralles, 2008). Of course, all these papers discuss their models beyond the adopted extreme assumption, yet their most illustrative proofs rely on their chosen assumption. Ergin and Erdil (2008) constitutes an exception that formally analyzes weak priority structures. Methodologically, Ergin and Erdil focus on improving the assignment after ties have been broken in some given way, whereas our paper considers the randomness of tie-breakers explicitly.

The results of this paper are also important for the empirical literature in the economics of education that evaluates the impact of school choice on school outcomes– see Lavy (2010), Hastings, Kane and Staiger (2010). This literature assumes that implementing choice implies that preferences will affect the allocation of children to schools. But these empirical papers ignore how allocation mechanisms are affected by the priority structure and therefore they may be attributing the effects to the wrong source of variation.

The results in the current paper are complementary of a recent insight by Abdulkadiroglu et al. (2020). The authors determine that, among strategy-proof and Pareto-optimal mechanisms, Top-Trading Cycles is the one minimizing justified envy. Or, in other words, the main advantage of Deferred Acceptance over nonstable mechanisms is minimized if one uses TTC.

Outline of the paper. Section 2 clarifies the concept of Access to Better Schools while discussing its alternatives. Section 3 introduces the mechanisms we compare in this paper. Section 4 contains two examples illustrating the main driving forces by which the different mechanisms obtain different ABS outcomes. The results in this paper are presented in a rather stylized model for ease of exposition. This model is introduced in Section 5. The model should facilitate understanding the intuition for the results without harming its perceived robustness. Similarly to Miralles (2008), Abdulkadiroğlu, Che and Yasuda (2014), we follow Aumann (1964) and assume an assignment problem with a continuum of individuals to be allocated to a finite num-

[9]These can be enlarged to include sibling priorities and low-income priorities with no qualitative harm on our results, upon request.

ber of schools. Example 1 in Section 3 and simulations in a further section indicate that the insights from the continuum model hold for reasonable finite school capacities. Section 6 extends results and makes observations to models with more schools. Numerical simulations illustrate the generality of the results. Section 7 discusses the Boston Mechanism and the stratification assumption. Section 8 concludes. Appendix A contains a discussion of the discrete model example from Section 4. Appendix B contains all proofs. An Online Appendix contains similation codes and detailed results.

2 Access to Better Schools

Access to Better Schools (**ABS**) is *the expected fraction of individuals who are assigned to a school preferred to their neighborhood school.*

According to this definition, the benchmark to measure the success of a school choice program is neighborhood assignment. Prior to the implementation of a centralized school choice program, children were assigned to their neighborhood school when possible. Hence ABS gives us the proportion of students that have improved their situation thanks to the school choice program.

The concept of ABS can also apply to specific groups of students. Particular attention deserves a group of students considered as disadvantaged, perhaps because their catchment areas are associated to bad schools. Therefore ABS could also be regarded as a concept of fairness.

Low ABS is an indicator of the value of neighborhood priorities. When ABS is close to zero, students' choices are effectively limited to their neighborhood schools. As ABS becomes larger, their neighborhood priorities are less decisive on the students' assignments. Some families may try to move into the neighborhood of their favorite school. Since such an option to "buy priorities" are available only for rich families. Large ABS seems to make entire competition more equal.[10]

A fair objection to the concept of Access to Better Schools is that it ignores the assignment of those who did not improve upon their catchment area school. If one understands "choice" as a measure of how students stand with respect to their catchment areas, should not we also take count of those students who actually end

[10] We thank a referee for the ideas in this paragraph.

worse-off ? This count, that could be named Access to Worse Schools (AWS), would serve to calculate a “net" ABS, $NABS = ABS - AWS$.

We point out, however, that this critique is not out of question either. Consider a stratified model of school choice, as seen in the following sections, in which there is a bad neighborhood with a bad school everyone dislikes. If one wants to minimize AWS, one could perversely suggest that students from such bad neighborhood should have no chances at other schools but that bad school. If such a student gained access to a better school, some other student from a better neighborhood with a better school had to end up assigned to the bad school. The net count would be zero. In other words, NABS gives no merit to mechanisms that allow children to escape from ghettos, due to a crowding-out effect. Yet, according to the No Child Left Behind initiative, access to better schools is particularly important for families from disadvantaged neighborhoods.

Such NCLB ideas are taken into account to the extent that, in some school districts, students from the bad school catchment areas have priority *at all schools* over students from good school catchment areas. For instance, the San Francisco Unified School District gives highest priority in all schools to families living in areas with “bad schools" (the lowest 20% percentile of average test scores).[11]

In order to better take all these arguments into account, one could suggest an aggregated welfare indicator of the type

$$W = ABS - \delta AWS + \gamma ABS_w$$

where δ is a penalty factor for each student who obtains an allocation worse than her priority-giving school, and γ is a redistribution premium for each student with priority at a worst school who obtains a better allocation. Therefore, for each student who obtains a worse placement than her priority-giving school, we require a compensation of δ students from good catchment areas obtaining a better placement, or $\frac{\delta}{1+\gamma}$ if the students improving their positions come from disadvantaged areas. $NABS$ would be a special case of this formula with $\delta = 1$ and $\gamma = 0$.

Our simulations in Section 6 yield that, even if we suppress the redistribution premium ($\gamma = 0$) and we introduce a penalty factor $\delta = 2$, TTC W-dominates DA in

[11]http://www.sfusd.edu/en/assets/sfusd-staff/enroll/files/2012-13/annual_report_march_5_2012_FINAL.pdf, page 81.

all of the environments we consider. A factor $\delta = 4$ suffices for TTC to W-dominate DA in 90% of the scenarios we consider in which TTC and DA do not yield identical assignments.

There are obviously other ways to compare the families' satisfaction generated by (different) school choice mechanisms. Pareto domination is the natural guide for comparison. An alternative criterion is rank domination (Featherstone, 2014). A mechanism outcome rank-dominates another mechanism's outcome if for every position n, the percentage of students allocated to the n-th ranked school or better is higher under the first outcome. Pareto-dominance implies rank dominance, which implies weakly higher ABS, yet the converse implications are not forcefully true.

A caveat of such alternative domination criteria is that they often do not allow us to unambiguously rank different mechanisms, above all with the usual presence of weak priorities. ABS is a measure that always allows for comparison, instead.

3 The Mechanisms

The mechanisms we compare are the Deferred Acceptance (DA) and the Top-Trading Cycles (TTC). In all these mechanisms, parents (students) are requested to submit a ranked list of schools. The student's strategy space is the set of all rankings among the schools. Each student may belong to the catchment area of a school. Belonging to a school's catchment area is the main priority criterion when resolving excess demands. Additionally, a unique lottery number per agent breaks any other eventual tie. The outcome of the lottery is uncertain at the moment students submit their lists.

Deferred Acceptance (DA):

- In every round, each student applies for the highest school in her submitted list that has not rejected her yet.
- For every round k, $k \geq 1$: Each school tentatively assigns seats to the students that apply to it or that were preaccepted in the previous round following its priority order (breaking ties through a fair lottery)[12]. When the school capacity

[12] We assume that there is a single tie-breaker that serves to break ties when necessary at all schools. In the absence of priorities, a single-tie breaker guarantees ex-post efficiency, while a separate tie-breaker per school cannot guarantee such a property (Abdulkadiroğlu, Che and Yasuda, 2014.)

is attained the school rejects any remaining students that apply to it in that round.

- The DA mechanism terminates when no student is rejected. The tentative matching becomes final.[13]

Top-Trading Cycles (TTC):

- In each round, we find a cycle as follows. Taking a school s with remaining seats, we choose the first student in its priority list, i. This student points at her most preferred school s', which points at its highest-priority student i', etc. A cycle is always found because there are a finite number of schools.
- We assign to each student of the cycle a slot of the school she points at. We remove these students and slots.
- We repeat the process round by round (having erased completely filled schools from students' lists and assigned students from schools' priority lists) until we have assigned all the students.[14]

4 Two examples and two intuitions

4.1 The interrupter in moderately large markets

Our main model uses a continuum economy for ease of exposition. In this section we illustrate one insight of the paper through an example with a finite set of individuals.

[13] Abdulkadiroğlu, Che and Yasuda (2014) show that this algorithm converges to an assignment in big continuum economies, even though not necessarily in finite time.

[14] TTC converges in the continuum (Leshno and Lo, 2021). An idea is to discretize both the mass of applicants and school capacity and to show that the discrete version converges as the size of the units goes to 0. In order to do this on the demand side, define a given type t by individuals with particular preferences and priorities (before ties are broken). Since the set of priorities and schools is finite, so is the set of orderings and types. Next, divide each type into *units* of size $1/n$. Let n be a natural number such that each type and each school capacity is larger than $1/n$, so that each type and school is composed of at least one unit u_n. However, each type and school capacity may not be divisible by an integer number of units. Note that the total mass of leftovers on the demand side is divisible by an integer number of units, since the total mass is of unit 1. Similarly for the supply side. Now define the preference ordering for the leftover units on demand side as a random preference ordering of the leftover types in that unit. Similarly distort capacities so that the remaining seats are all of one of the schools. We can now run TTC on units of individuals and schools. The assignment is distorted by mass of the leftovers. But one can show that the mass of leftovers on both sides goes to 0 as n goes to infinity.

We have three neighborhoods with n families living in each of them, and each with a school of capacity n. Let $i \in \{i_1, i_2, i_3\}$ denote that the individual lives in the neighborhood of school $s \in \{1, 2, 3\}$, and therefore has priority at school s. Ties in priorities given by residence are broken through a unique fair lottery.

Preferences are as follows. Everyone ranks school 3 as the worst one. Student i_1 has school 2 as favorite, whereas the other students prefer school 1.

In Deferred Acceptance, students with priority at a good school have guaranteed assignment to a good school. This implies that all students of the i_3 type are eventually assigned to the worst school. Suppose $n = 1$. Students i_1 and i_2 would like to "exchange" their guaranteed slots. But under DA, since they do not have priority for their preferred school, this "exchange" will only happen if both individuals i_1 and i_2 get a better lottery number than the i_3-student with the highest lottery number. The student i_3 blocks such trade with probability 2/3. This is the well-known *interrupter* problem (Kesten, 2010.)

Appendix A shows how, for $n > 1$, the probability of blocking the x-th exchange rapidly increases with $x = 1, .., n$, since not doing so requires both x-th best lottery numbers in i_1 and i_2 to beat the best lottery number in i_3.

Table 1 presents the results from calculating the expected proportion of students that obtain access to a better school than the catchment area school. This percentage rapidly decreases to zero as n grows large. Even with n being small, the percentage is dramatically low (1.77% with 20 students per school).[15]

This trade-blocking cannot happen under TTC. School 1 points at a student with type i_1, who points at school 2, which points at a student with type i_2, who points at school 1, and the cycle is closed. Students from types i_1 and i_2 are assigned to a school that is better than their priority-giving schools regardless of how lucky students of type i_3 are with their lottery numbers.

Table 1 summarizes the expected number of students who obtain a better placement than the school for which they had priority (weighted by the size $3n$ of the market).

[15] Table A6 in the Online Appendix contains similar results with 4 good schools and one worst school.

Table 1: Expected percentage of students who get Access to Better School (ABS)

Mechanism \ n	1	2	5	10	20	∞
DA	22	13.3	6.3	3.4	1.77	0
TTC	66.7	66.7	66.7	66.7	66.7	66.7

Notice how the expected percentage of students who improve upon their catchment areas rapidly decreases to zero under DA. With ten students per school, only 3.4% of them are expected to obtain a better assignment outside their catchment areas. With twenty students per school, this percentage is 1.77%. This exemplifies that the bad results obtained by DA later in the model are not an artefact of the continuum model we illustrate in Section 4.

4.2 More access to "leftovers"

In Top-Trading Cycles, it is well understood that student's chances at a top-preferred school primarily depend on how popular to others the school that gives the student priority is. Under such observation, a disadvantaged student has always weakly lower chances at any good school than those of another student.

On the contrary, Deferred Acceptance, not being based on trading ideas, seems more egalitarian. The chances a disadvantaged student has at some good school is equal to the chances any non-prioritized student would have.

Therefore, is it true that disadvantaged students are more likely to be assigned to the worst schools under TTC than under DA? We argue it is just the opposite!

Let us observe the following example. We have four schools, two good (1 and 2), one "bad" (b) and one "worst" (w). Consider the set of good schools to be $G = \{1, 2, b\}$ in the sense that everyone wants to avoid the worst school. However, inside G, nobody likes the bad school. Indeed, everyone ranks schools b and w third and fourth, respectively.

Capacities and prioritized students are $q_1 = q_b = 6, q_2 = q_w = 8$, $n_1 = n_b = n_w = 8, n_2 = 4$. These are treated as masses rather than nondivisible units. Students with priority at school 2 prefer school 1, others prefer school 2.

The next table summarizes the mass of allocated students to each school, according to their priority group. In each cell, the first number corresponds to DA; the second number, to TTC.

Assigned to↓ Priority at→	1	2	b	w
1	6,2	0,**4**	0,0	0,0
2	$\frac{4}{3}$,**4**	4,0	$\frac{4}{3}$,2	$\frac{4}{3}$,2
b	0,0	0,0	6,6	0,0
w	$\frac{2}{3}$,2	0,0	$\frac{2}{3}$,0	6+$\frac{2}{3}$,6

In Deferred Acceptance, we start by observing that school 1 will not be accessible to nonprioritized students. To see why, we see that all students would like to enter school 2, except for prioritized students. Supposing that prioritized students take 4 slots at school 2, the remaining 4 slots have to be allocated among nonprioritized students. Under DA, all of the latter students have *equal chances* at school 2. Correspondingly, students with priority at school 1 occupy 4/3 slots of school 2. Applying to the second-ranked school in their preferences, 6 students of this group are allocated to all slots of school 1, where they have priority. This confirms that school 1 will not be accessible.

At the same time, this fact confirms that 4 slots of school 2 are to be allocated to all of its prioritized students, who have no access to their more-preferred school 1. The rest of the assignment is easy to understand. Students with priority at b occupy all the slots of that school. Finally, the remaining students are assigned to the worst school.

As for TTC, the allocation algorithm starts with a trade of slots between prioritized students by school 1 and those by school 2. Notice that half of students prioritized by school 1 obtain a slot at school 2. Were the lottery number space a segment from 0 to 1, with a "the lower the better" tie-breaking rule, the infimum lottery number among still unassigned students from the latter priority group is now 1/2. We continue the TTC algorithm by assigning the 4 leftover slots of school 2.

Now, those slots are assigned in equal shares to students with priority at schools b and w. Notice, that, as compared to DA, these latter students have an advantage at collecting these slots over those prioritized by school 1.

The TTC algorithm continues by assigning the 2 remaining slots of school 1 to prioritized students. The algorithm ends after assigning the 6 slots of school b to prioritized students and finally the slots of school w to still unassigned students.

This example illustrates several important insights regarding TTC as compared to DA:

1) Students prioritized by rather undemanded schools, including the worst school, obtain a side benefit from not interrupting trades among other students. When the time arrives to assign leftover slots from underprioritized good schools, the enjoy an advantage regarding the tie-breaker lottery.

We included school b in the example to note that students prioritized by undemanded good schools could also benefit from a higher access to leftovers.

2) Students prioritized by highly demanded schools obtain better chances to improve their assignments via trade, yet they also convey a higher risk of being assigned to the worst school.

This example also serves to identify who would prefer DA to TTC. If the student's favorite school is her priority-giving school, and that school is highly demanded by other students, the trades carried out through the TTC algorithm would convey no benefit for the student. Instead, the student is likely to suffer a higher risk of being placed into a worse school.

5 Model

We present a simple model in order to illustrate our insights.[16] We have a mass N of students $i \in I = [0, N]$, each of them to be allocated to one of three schools. I is endowed with the uniform Lebesgue measure λ. Two of the schools are "good" and one is "bad", in the sense that all students rank it as worst. Good schools are labelled 1 and 2, respectively, whereas the bad school is labelled w (as for "worst".) Schools have strictly positive capacities q_1, q_2 and q_w that add up to N. Students $i \in I$ have preferences $\succ_i$over the schools. No student is indifferent between any two schools.

There is a measurable catchment area function $\pi : I \to \{1, 2, w\}$. Each student has a unique catchment area where she has priority over students outside the catchment area. There is a mass n_1, n_2 and n_w of students for the catchment areas of schools 1, 2 and w, respectively. We denote with $\Pi_s = \{i \in I : \pi(i) = s\}$ the set of students prioritized by school s. Belonging to the catchment area of a school gives priority there over students outside its catchment area.

Student i's preferences over schools $\succ_i$could be summarized by the identity of the

[16]In previous versions of this paper we use a more general model with an arbitrary number of schools. Results are qualitatively similar to the ones we find here. For DA, there is an upper bound to ABS that collapses to zero when good schools are weakly overprioritized.

most-preferred school, since w is ranked last by everyone. Therefore, $\Pi_{ss'}$ denotes the set of students with priority at school s whose favorite school is s', and $n_{ss'}$ denotes its associated mass.

Other ties are resolved when needed using a *fair* [17] lottery outcome $l : I \rightarrow [0, 1]$ that assigns one number to each student. We apply the convention that a lower lottery number beats a higher lottery number.

For each school s, define $\rho_s = q_s/n_s$. We say school s is *overprioritized* if $\rho_s < 1$ (capacity is smaller than the number of individuals with priority in the school), and underprioritized in the opposite case. Notice that we cannot have the three schools being either all overprioritized or all underprioritized, since we have assumed that total capacity is equal to total mass of students. For two schools s and s' we say that s is more prioritized than s' if $\rho_s < \rho_{s'}$.

A *matching* [18] is a function $\mu = I \rightarrow \{1, 2, w\}$. For each matching μ we compute the mass of students who obtain a slot in a school preferred to that of their catchment areas, as a measure of students' real choice. We call this measure *Access to Better Schools*, denoted ABS. We also compute ABS for priority groups Π_s. More formally:

$$\begin{aligned} ABS(\mu) &= \lambda\left(\{i \in I : \mu(i) \succ_i \pi(i)\}\right) \\ ABS_s(\mu) &= \lambda\left(\{i \in \Pi_s : \mu(i) \succ_i \pi(i)\}\right) \end{aligned}$$

We compare two matchings, the one induced by truth-telling in Deferred Acceptance (the DA matching,) and that induced by truth-telling in Top-Trading Cycles (the TTC matching.)

We find the following result:

Proposition 1 *For every school $s \in \{1, 2, w\}$ we have $ABS_s(TTC) \geq ABS_s(DA)$.*

Note that the statement is also true for the set Π_w of disadvantaged students. Therefore TTC is more efficient than DA insofar as ABS is taken as a measure of allocative efficiency. And it is also fairer, in that it gives more chances to disadvantaged students to improve their positions.

The proof is found in Appendix B. The proof has to check a set of different cases separately. The difficulty in generalizing this result to setups with arbitrarily more

[17] "Fair" meaning that for every interval $[l', l''] \subset [0, 1]$ and every group $\Pi_{ss'}$ we have $\lambda\left(\{i \in \Pi_{ss'} : l(i) \in [l', l'']\}\right) = n_{ss'}(l'' - l')$.

[18] In this paper, we use the terms matching, assignment and allocation indistinctively.

good schools stems from the expansion of the variety of cases to consider. The next section illustrates via numerical simulations that our insights are not an artefact of the two-good-school model.

6 More good schools

6.1 Numerical simulations

We compute numerical simulations[19] in which we consider four good schools $G = \{1, 2, 3, 4\}$ and one worst school w. Each school has 20 slots and there are 100 students. Students' valuations for schools have three components: 1) an extra for neighborhood school (a *neighborhood effect* caused by geographical proximity), 2) a common value component c, and 3) an independent value component u_i,

$$v_{is} = 1\{\pi(i) = s\} + \alpha u_{is} + \beta c_s, i \in I, s \in G$$

All values v_{iw} are zeros. All c_s and u_{is}, $\forall i \in I$,$s \in G$, are independently drawn from the uniform distribution. The common vector c is then sorted so that $c_1 > c_2$... Therefore schools are numbered according to popularity.

Simulations are programmed and computed with MATLAB R2022a, with default seed for random number generation. We consider a grid of scenarios varying according to:

1) Whether the two most popular schools are less or more overprioritized than the other less popular good schools, combined with different levels of overall underprioritization of good schools (equivalently, overprioritization of the worst school,) giving rise to twelve possibilities: [40/30/20 students prioritized by the worst school] x [big/small difference in number of prioritized students among good schools[20]] x [the two most popular schools are the least/ the most overprioritized].

2) Different values for α and β so as to give more or less importance to each of the components of v_i, giving rise to nine possibilities: [balance between neighborhood effect and other sources, more weight to neighborhood effect, less weight to neighborhood effect] x [balance between common component and independent component, more weight to common component, less weight to common component].

[19]We thank Juan Sebastian Pereyra and Li Chen for the help with the simulations.

[20]We alternate values of 10, 15, 20 and 25 prioritized students.

For each scenario, 50 valuation matrices are calculated, and for each one 50 single tie-breakers are computed, giving a total of 2,500 computed assignments per scenario and mechanism (5,000 computed assignments in total.) We are considering 90 scenarios,[21] thus our simulations amount to a total of 450,000 computed assignments.

In 12 of the scenarios considered, the valuation generating formula gives so much weight to the neighborhood effect and to the common value component so as to make both DA and TTC collapse de facto into Serial Dictatorship,[22] making both matchings identical. We will focus on the remaining cases, where differences are observed.

Results are shown along the next tables. A complete deploy of all calculations and the MATLAB code are shown in the Online Appendix. For each scenario we compute ABS for all students and for each priority group $\Pi_1, ..., \Pi_4, \Pi_w$, both under DA and under TTC, measured as the fraction of the considered group who obtain a slot in a school better than the priority-giving school. We also compute Access to Worse School (AWS), the fraction of students, for each considered group, that obtain a slot in a school worse than the priority-giving school. We additionally compute, for all students and for each priority group, the fraction of students who prefer their allocations under DA than under TTC. We finally compute the opposite, that is, the fraction of students who obtain an allocation under TTC that is better than the one obtained under DA.[23]

The overall observation is that ABS under TTC is *always* superior. ABS dominance tends to be minor when ordinal preferences are highly correlated among individuals and there is a high weight of the neighborhood effect (e.g. $\alpha = 0.5, \beta = 1.5$). In such a case, the allocations under TTC and under DA tend to coincide, and to collapse into a serial dictatorship allocation.

Figure 1 summarizes the difference between ABS under TTC and under DA in the scenarios under which the mechanism deliver different matchings. Figure 2 ap-

[21] Criteria 1) and 2) should yield 108 scenarios. In using criterion set 1), we considered 2 kinds of scenarios in which all good schools have the same number of prioritized students. In such cases, the criterion of giving the highest number of prioritized seats to popular / less popular good schools does not bite. This is the reason why we have 90 scenarios to consider.

[22] For a Serial Dictatorship one needs to have a linear ordering of all students. We refer to the linear ordering in which students are first ordered according to the priority-giving school (being school 1 first, school 2 second...,) breaking ties thereafter with the lottery number. In those scenarios, all students from Π_1 preferred school 1 the most, then all students from Π_2 ranked school 2 first or immediately after school 1, etc.

[23] Standard deviations not reported yet easy to calculate. Being each variable a binary variable with parameter p, the variance is $p(1-p)$.

	Percentiles	Smallest		
1%	.00065	.00065		
5%	.00101	.00092		
10%	.00139	.001	Obs	78
25%	.03049	.00101	Sum of Wgt.	78
50%	.05654		Mean	.0916532
		Largest	Std. Dev.	.0877828
75%	.15264	.27255		
90%	.21811	.30176	Variance	.0077058
95%	.27255	.32057	Skewness	1.000891
99%	.35134	.35134	Kurtosis	3.156208

Figure 1: $ABS(TTC) - ABS(DA)$ when allocations are not identical.

proximately relates this difference to the parameters of each scenario by means of a simple linear regression. One has to regard the regressions in this Section as a way to summarize information, which would occupy lots of space otherwise (the Online Appendix shows all the results.)

The size of the ABS domination of TTC over DA becomes enormous in some cases. See for instance Table A10.c in the Appendix ($n_1 = n_2 = 10, n_3 = n_4 = 20$, particularly when $\alpha = 6, \beta = 2$.) We see differences of the order of .32 (from 12.6% under DA to 44.6% under TTC).

From the estimation, we see that increasing the level of overprioritization in either most popular or less popular good schools have a similar effect. By increasing in one person the number of prioritized students, the difference between $ABS(TTC)$ and $ABS(DA)$ slightly increases by 0.5%. This is in line with the idea that overprioritization is bad news for DA concerning ABS. The effect of α, the weight of i.i.d. preferences, is clear: an increase in one unit enlarges the difference regarding ABS by 4.85%. The coefficient associated to β, the weight of common values, has the expected negative sign. An increase in one unit shrinks the difference by 0.9%. Decreasing the weight of the neighborhood effect by increasing $\alpha + \beta$ while keeping α/β constant has the expected positive effect as long as $\alpha/\beta > 0.9/4.85 \approx 0.19$. As an example, when $\alpha = \beta$, increasing $\alpha + \beta$ by one unit enlarges the difference in ABS by approximately 2%.

We wish to stress that α is the parameter that represents taste variety. It is in the context of a high α that School Choice programs have a clearer purpose. And it is precisely in these scenarios that TTC dominates more strongly.

Source	SS	df	MS		
Model	.518584519	4	.12964613	Number of obs =	78
Residual	.074763741	73	.001024161	F(4, 73) =	126.59
				Prob > F =	0.0000
Total	.593348261	77	.007705822	R-squared =	0.8740
				Adj R-squared =	0.8671
				Root MSE =	.032

diffabs	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
n12	.0052519	.0010173	5.16	0.000	.0032244	.0072793
n34	.0049911	.0010173	4.91	0.000	.0029637	.0070185
alpha	.0485095	.0022583	21.48	0.000	.0440087	.0530103
beta	-.009126	.0020945	-4.36	0.000	-.0133003	-.0049516
_cons	-.1908442	.0337659	-5.65	0.000	-.2581395	-.1235488

Figure 2: Regression of $ABS(TTC) - ABS(DA)$ over the parameters of each scenario, when the mechanisms deliver unidentical allocations. n12 stands for n_1 and n_2, whereas n34 stands for n_3 and n_4.

We notice that, as expected, ABS under DA collapses when all good schools are weakly overprioritized, given the highly likely appearance of an interrupter, since disadvantaged students obtain no chances at good schools (Table A6 in Online Appendix.)

As for ABS for disadvantaged students (those in Π_w), it is clear that TTC is *superior in all cases*. The difference with DA tends to be minimal when either: 1) ordinal preferences are highly correlated among individuals and there is a high weight of the neighborhood effect, 2) all good schools are weakly overprioritized (no access to good schools for disadvantaged students), 3) all good schools are weakly underprioritized (the amount of disadvantaged students who obtain placement at a good school is mechanism-invariant.)

The domination of TTC over DA regarding ABS for disadvantaged students is sizable in the same cases in which the difference in overall ABS is high. In the example we were considering for the general case, disadvantaged students obtain ABS equal to 27.5% under DA and 49.7% under TTC.

Figure 3 presents a summary description of $ABS_w(TTC) - ABS_w(DA)$, exception made for the following cases: 1) when both TTC and DA deliver identical allocations coinciding to that of Serial Dictatorship, 2) when both good schools are weakly underprioritized. In the latter case, all students prioritized by a good school will certainly

	Percentiles	Smallest		
1%	0	0		
5%	0	0		
10%	0	0	Obs	32
25%	.01184	0	Sum of Wgt.	32
50%	.04134		Mean	.0712959
		Largest	Std. Dev.	.0702518
75%	.12421	.16804		
90%	.16804	.18768	Variance	.0049353
95%	.22238	.22238	Skewness	.8130252
99%	.22258	.22258	Kurtosis	2.368389

Figure 3: Summary description of $ABS_w(TTC) - ABS_w(DA)$ for cases in which this value is not trivially 0.

Source	SS	df	MS		
				Number of obs =	32
				F(4, 27) =	47.71
Model	.134031542	4	.033507885	Prob > F =	0.0000
Residual	.018963095	27	.000702337	R-squared =	0.8761
				Adj R-squared =	0.8577
Total	.152994636	31	.004935311	Root MSE =	.0265

wdiffabs	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
n12	.0001087	.0019096	0.06	0.955	-.0038096	.004027
n34	-.0000704	.0019096	-0.04	0.971	-.0039887	.0038478
alpha	.0388554	.0029206	13.30	0.000	.0328629	.0448479
beta	-.0130176	.0027296	-4.77	0.000	-.0186184	-.0074169
_cons	.0027422	.0710812	0.04	0.970	-.1431043	.1485886

Figure 4: Regression of $ABS_w(TTC) - ABS_w(DA)$ on the parameters characterizing the different scenarios, for cases in which the dependent variable is not trivially 0.

obtain a slot at a good school, under both mechanisms. The number of slots available for disadvantaged students is identical between mechanisms.

Figure 4 presents the results of a simple linear regression between $ABS_w(TTC) - ABS_w(DA)$ and the set of parameters characterizing each scenario, with a subsample that skips cases 1) and 2) above.

$ABS_w(TTC) - ABS_w(DA)$ behaves similarly to $ABS(TTC) - ABS(DA)$, although with lower intensities. The 95 percentile on ABS_w gain is around 22.2% (27.2% for general ABS). An increase in the number of prioritized students seems not to have a significant effect. An increase of one unit in α rises the difference in ABS_w by 3.88%, while an equivalent increase in β reduces this difference by 1.3%.

	Percentiles	Smallest		
1%	0	0		
5%	0	0		
10%	0	0	Obs	78
25%	0	0	Sum of Wgt.	78
50%	0		Mean	.0678943
		Largest	Std. Dev.	.104043
75%	.1391397	.269035		
90%	.2398644	.2837187	Variance	.0108249
95%	.269035	.3730301	Skewness	1.38615
99%	.4256759	.4256759	Kurtosis	4.108209

Figure 5: Description of $\frac{AWS(TTC)-AWS(DA)}{ABS(TTC)-ABS(DA)}$ for cases in which TTC and DA do not deliver identical matchings.

Decreasing the weight of the neighborhood effect by increasing $\alpha+\beta$ while keeping α/β constant has a positive effect on $ABS_w(TTC) - ABS_w(DA)$ as long as $\alpha/\beta > 1.3/3.88 \approx 0.335$. As an example, when $\alpha = \beta$, increasing $\alpha+\beta$ by one unit enlarges the difference in ABS by approximately 1,29%.

TTC obtains worse results than DA regarding AWS in general, exception being the cases where the parameters forced one of these outcomes: 1) equivalent allocations in both mechanisms, 2) weakly overprioritized good schools (no prioritized student could bear a risk of obtaining a worse placement in either of the mechanisms).

It is however noticeable that, *in all cases considered*, the unfavorable difference for TTC in AWS is lower than the favorable difference in ABS. Figure 5 summarizes the ratio $\frac{AWS(TTC)-AWS(DA)}{ABS(TTC)-ABS(DA)}$ for cases in which the mechanisms do not deliver identical matchings.

In more than half of the cases considered, DA and TTC deliver identical AWS. This is not that surprising because of the scenarios in which all good schools are weakly overprioritized. In such scenarios, no student prioritized by a good school could ever obtain a worse allocation, in either of the mechanisms considered. The 95% percentile of this ratio is roughly below 27%.

With a welfare aggregator of the type

$$W(\mu) = ABS(\mu) - \delta AWS(\mu) + \gamma ABS_w(\mu)$$

, even if we suppress the bonus to ABS_w ($\gamma = 0$,) TTC would welfare-dominate DA in

	Percentiles	Smallest		
1%	0	0		
5%	0	0		
10%	.00012	0	Obs	78
25%	.023224	0	Sum of Wgt.	78
50%	.057794		Mean	.090785
		Largest	Std. Dev.	.0893094
75%	.152402	.275814		
90%	.234668	.300872	Variance	.0079762
95%	.275814	.319954	Skewness	1.023942
99%	.354396	.354396	Kurtosis	3.146456

Figure 6: Difference between the percentage of students who prefer TTC to DA and the percentage of students who prefer DA to TTC, for cases in which TTC and DA do not trivially yield identical allocations.

more than 90% of the cases in which the allocations differ, even for penalties to AWS, δ, as big as 4. Such a value of δ indicates that in order to compensate for a student who obtains a worse placement than her priority-giving school, we need at least four students who are assigned to a better school than each one's priority-giving school. With $\delta = 2$ TTC would welfare-dominate DA in all the simulations we computed.

We also compare the percentage of students who prefer the TTC assignment to that of DA with the percentage of students who prefer the DA assignment to the TTC assignment. The difference between percentages is described in Figure 6. The results of a linear regression of such difference on the parameters of each scenario, for those scenarios where TTC and DA do not coincide with the Serial Dictatorship allocation, is represented on Figure 7. We do not extend on repetitive comments, since the behavior of this difference is so similar to that of $ABS(TTC) - ABS(DA)$.

We notice however that, *in all cases* considered here, the percentage of students who prefer the TTC assignment to that of DA is at least weakly higher than the percentage of students who prefer the DA assignment to the TTC assignment.

This statement is not true for every priority group. Interestingly, we actually find cases in which more students from Π_w prefer their DA assignment to their TTC assignment is higher than the opposite. This is not incompatible with a higher access to better schools for these students under TTC. On the contrary, it can be explained as follows. Under TTC, disadvantaged students cannot interrupt trades among other students, hence they face lower chances of accessing highly popular schools. As a

Source	SS	df	MS			
				Number of obs	=	78
				F(4, 73)	=	129.85
Model	.538481376	4	.134620344	Prob > F	=	0.0000
Residual	.075682988	73	.001036753	R-squared	=	0.8768
				Adj R-squared	=	0.8700
Total	.614164364	77	.007976161	Root MSE	=	.0322

diffbest	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
n12	.0054053	.0010235	5.28	0.000	.0033654	.0074451
n34	.0057407	.0010235	5.61	0.000	.0037009	.0077806
alpha	.0491478	.0022721	21.63	0.000	.0446194	.0536762
beta	-.0095235	.0021074	-4.52	0.000	-.0137235	-.0053236
_cons	-.2081452	.0339729	-6.13	0.000	-.275853	-.1404374

Figure 7: Regression of the variable described on the previous figure with respect to the parameters characterizing each scenario.

compensation, they have more access to good schools in general.

6.2 Other observations for arbitrarily many good schools

6.2.1 A lower bound for ABS under Deferred Acceptance

It is easy to compute a preference-independent upper bound for ABS under Deferred Acceptance. It is clear that, from each priority group $\Pi_s, s \in G$, the minimum mass of students in the group obtaining a slot at a good school is $\min\{q_s, n_s\}$. So the maximum mass of good school slots available to disadvantaged students is $\sum_{s\in G} \max\{0, q_s - n_s\}$. The infimum lottery number among students from Π_w that are assigned to the worst school, namely l_w, is not higher than $\frac{1}{n_w}\sum_{s\in G} \max\{0, q_s - n_s\}$.

Here is where stability imposes its cost to ABS. Since the allocation is stable, every student who obtains a better allocation than her priority-giving school must at the very least have a lottery number below l_w. Hence:

Proposition 2 $ABS(DA) \leq \frac{N}{n_w}\sum_{s\in G} \max\{0, q_s - n_s\}$.

Corollary 1 $ABS(DA) = 0$ *if all schools are weakly overprioritized.*

Another corollary, not stated formally, is that the more relatively important the mass of disadvantaged students is in relation to the mass of leftover slots from underprioritized good schools, the lower the proportion of students from a good school that would be able to obtain access to a better school.

We conclude this subsection by noting that the TTC allocation, not being necessarily stable, is not constrained by such upper bound.

6.2.2 Accessibility

Another perspective on access to better schools would consist on measuring to what extent a school gives slots to nonprioritized students who like it more than their corresponding priority-giving schools. Under such approach, we regard a school as *accessible* under some matching if that matching gives a strictly positive mass of slots to such nonprioritized students. A school is *more accessible* under some matching against another matching if it assigns a higher mass of slots to such nonprioritized students under the former matching than under the latter. We make a couple of observations about accessibility.

Proposition 3 *If the worst school is weakly underprioritized, there is at least one good school that is not accessible under DA.*

We provide an informal proof here. Suppose all good schools are accessible under DA. This implies all students prioritized by some good school must be allocated to a good school. Again, this is a feature of stability: a prioritized student that is assigned to the worst school blocks accessibility. Therefore the number of slots at good schools remaining for disadvantaged students is $\sum_{s\in G}(q_s - n_s) = n_w - q_w$. This makes the infimum lottery number among students from Π_w that are assigned to the worst school, namely l_w, equal to $1 - \rho_w$. Since $ABS(DA) \leq Nl_w$, $\rho_w \geq 1$ gives us a contradiction.

Proposition 4 *There is at least a school that is weakly more accessible under TTC than under DA.*

Here the proof is based on the Pareto-optimality of the TTC assignment. By Pareto-optimality, there must be a school s that gives all its slots only to students regarding that school as favorite. This implies that, among students from Π_s, only students from Π_{ss} could obtain a slot at s. Now, suppose school s is accessible under DA, otherwise we would be trivially done. This means that $q_s > n_{ss}$, since all students from Π_{ss} must be assigned to s in order to allow for accessibility (again this being a consequence of stability.) But then, the mass of slots that school s gives to nonprioritized students wishing to be assigned there under TTC is $q_s - n_{ss}$, a number that cannot be exceeded by DA.

7 Discussion

7.1 The Boston Mechanism

Previous versions of the current manuscript put a stronger accent on the comparison between the Boston Mechanism and Deferred Acceptance, at the time the two mostly used mechanisms in practice. The main purpose was to argue that both mechanisms could fail to provide access to better schools for different reasons. In the current version, we just make a few comments about ABS under the Boston Mechanism, leaving formal proofs outside of the main text.[24]

The Boston Mechanism works quite similarly as Deferred Acceptance, yet with one important difference in the assignment algorithm. While all acceptances are tentative under DA, BM forces each acceptance to be definitive along the allocation process. Among other consequences, this makes BM manipulable.

In previous versions of the paper, we argued that, if students have a moderately good valuation for their priority-giving school, Access to Better Schools will collapse. We could have a unique Nash equilibrium in which all students could play the safe strategy of ranking the priority-giving school in first position, despite preferring some other school. As a result, good schools would be minimally accessible, if accessible at all.

Note that the line of reasoning is so different from that of DA. DA may fail to provide high access to better schools because of stability. BM, instead, potentially fails because it may induce safe strategies to risk-averse families, in line with the empirical findings of Calsamiglia and Guell (2018).

7.2 The need for stratification

A reasonable question at this point is whether results are sensitive to a relaxation of the fundamental assumptions of the model. Mainly, one could wonder if stratification is not too a strong assumption. Seemingly, some students from disadvantaged areas could have some preference for the worst school, provided its geographical proximity.

A first, trivial observation is that we do not need the worst school to actually be the least preferred one. It is enough that the "worst" school is not better than

[24]We are thankful to referees that suggested a change of focus of the paper.

the priority-giving school for anyone, to make DA have a poor performance regarding ABS.

If we disregard this assumption, one could construct an example in which a nonstratified district yields results opposite to what here exposed: DA could ABS-dominate TTC. Consider this finite economy example in which DA actually ABS-dominates TTC even when the allocation economy is replicated arbitrarily many times.

Example 1 *There are three schools a, b and w with one slot each. There are three students i, j and k with schools ranked as in the table below. The superscript* π *indicates where the student has priority. The subscript TTC indicates the allocated slot under such mechanism.*

	$1st$	$2nd$	$3rd$
i	b	a^{π}_{TTC}	w
j	$\mathbf{w}_{TTC}$	b^{π}	a
k	b_{TTC}	a	w^{π}

$ABS(TTC) = 2/3$ *(replica-invariant)*

In this example, school w is better than the priority-giving school for one student. DA yields the allocation $(i \to b, j \to w, k \to a)$ *(ABS=1) with probability 1/2 (i's lottery number better than k's) and the TTC allocation in all other cases. Hence* $ABS(DA) = 5/6$ *(even with any number of replicas.)*

We nevertheless argue that it is rather easy to come up with a widened definition of stratification, and still find large economies in which ABS collapses under DA. We say that the school district is *weakly stratified* when there is a set G of "good" schools for which the union of all of its prioritized students prefer all schools in G to all other schools. Notice that hierarchies are not that clear here. We allow for (some) nonprioritized students to actually dislike schools in G.

Consider a large economy (with a continuum of agents), not necessarily with a worst school. But there is a group of "good" schools G for which, for prioritized students, these are actually superior to a set of "bad schools" B. However, for students from B, other factors (eg. geographical proximity) might make some schools outside G preferred to those on G.

We say that $s \in G$ is *chain-linked* to another set B with $G \cap B = \varnothing$ if there is an array of positive-measured sets of agents $I_0, I_1,..., I_K$ and schools $\{s_1, ..., s_K\} \subset G$ such that: 1) $I_0 \subset \Pi_B = \cup_{s \in B} \Pi_s$, 2) $I_k \subset \Pi_{s_k}$ $\forall k = 1, ..., K$, 3) $s_1 \succ_i s'$ $\forall s' \notin G, \forall i \in I_0$, 4) $s_k \succ_i s_{k-1}$ $\forall i \in I_{k-1}, \forall k = 2, ..., K$, and 5) $s_K = s$.

Proposition 5 *Consider a continuum weakly stratified school district with a set of weakly overprioritized good schools G. Assume that every $s \in G$ is chain-linked to a set of schools B with $G \cap B = \varnothing$. Then $ABS_s(DA) = 0$ for all $s \in G$.*

The proof is rather quick. Stability and 1) imply that no student in Π_B can obtain a slot in G. But then, since $\inf \{l(i) : i \in I_0, DA(i) \neq s_1\} = 0$ and by stability, no slot at s_1 is occupied but for prioritized students. Recursively, we find that $DA^{-1}(s_k) = \Pi_{s_k}$ for all $k = 2, ..., K$.

8 Conclusions

Since Abdulkadiroğlu and Sönmez (2003) the Boston Mechanism has been widely criticized in the school choice literature. Since then many cities around the world have substituted this mechanism by the Gale Shapley Deferred Acceptance mechanism.[25] Deferred Acceptance has been adapted from matching theory as a good alternative, since it is not manipulable, it protects nonstrategic parents and provides more efficient assignments in setups with strict priorities. The debate between these two mechanisms was based upon models that did not incorporate some important realities about the schools system, such as the vertical differentiation among schools. We solve a simple model of school choice with coarse residential priorities and vertical differentiation separating good from bad schools. We show that if school choice aims to improve *access to better schools* than the neighborhood school, then Deferred Acceptance is likely to perform very poorly. We illustrate that the priority structure, under the presence of a stratified school system, can determine the final allocation to a great extent in both of these mechanisms.

We have analyzed a natural alternative in this debate, which is Top-Trading Cycles. TTC is more immune to the priority structure because prioritized students at good schools are allowed to trade their slots with no interferences from students of a bad school's catchment area. Top-Trading Cycles obtains higher access to better schools than Deferred Acceptance. It therefore constitutes a safe mechanism with respect to both the Boston Mechanism and Deferred Acceptance, in school choice problems where coarse zone priorities exist.

[25] See Pathak and Sönmez (2013) for evidence on the number of cities around the word where the Boston mechanism has been banned.

Our paper finds that TTC provides disadvantaged students with better chances at good schools than Deferred Acceptance. Therefore, TTC outperforms DA in terms of efficiency and in terms of fairness, under the ABS measure. Our model is a model without minority reserves as proposed by Hafalir et al. (2013.) Comparing both mechanisms with and without reserves deserves special attention for future research.

More generally this paper puts forth the extreme relevance that neighborhood priorities can have on the final allocation of students to schools, inhibiting the role that preferences may have in determining the final allocation. The literature has deemed these priorities as exogenous, but ultimately they constitute a key feature of the final assignment that the administration can and does change whenever needed.[26] Future work should incorporate the design of these priorities as a fundamental part of the mechanism design problem.

9 Appendix

9.1 Appendix A: ABS under DA in the finite economy example.

As said in the main text, students with priority at different good schools would like to "exchange" their guaranteed slots, yet then the students from the bad school catchment area may block this trade. We want to derive the chances of exactly a number x of exchanges occurring. In order to gain more understanding we illustrate a simple case where $n = 2$ and $x = 1$. We calculate all the cases in which this event happens. It could be that the two top-ranked students in the tie-breaking lottery are one student of type i_1 and another one of type i_2, and the third-ranked student is i_3. We could have picked $\binom{2}{1} = 2$ students from each type, and the order between types i_1 and i_2 does not matter (there are $2! = 2$ ways to arrange them). There are also $(6-3)!$ ways to arrange the remaining students among themselves. Hence we find $2 \cdot 2 \cdot 2 \cdot 2! \cdot 3! = 96$ lottery outcomes satisfying this condition. But we have not covered all cases. It could also be that two students of type i_1 and another one of type i_2 occupy the first three positions in the lottery ranking, while the fourth position is occupied by an i_3 student. In this case there is only one way, or $\binom{2}{2}$, to pick two

[26] In cities such as Madrid, Barcelona, Boston, San Francisco and New Orleans, among others, priorities have changed over the last decade.

students out of the two existing i_1 students. We could still pick $\binom{2}{1} = 2$ students from each of the other types. The way we arrange the two i_1 students and the i_2 student does not matter (there are 3! combinations). There are $(6-4)!$ ways to arrange the remaining students. We have found other $1 \cdot 2 \cdot 2 \cdot 3! \cdot 2! = 48$ such lottery outcomes. This number has to be multiplied by 2, to cover the final yet symmetric case in which two students of type i_2 and another one of type i_1 occupy the first three positions in the lottery ranking, while the fourth position is occupied by an i_3 student. We obtain a total of 192 favorable cases out of $6! = 720$ possible lottery outcomes. The probability of exactly one exchange with two students per school is $P(1,2) = \frac{4}{15}$. More generally

$$\begin{aligned} P(x,n) &= \frac{1}{(3n)!}[\binom{n}{x}\binom{n}{x} n(2x)!(3n-2x-1)! + \\ &\quad +2\sum_{i=x+1}^{n}\binom{n}{x}\binom{n}{i} n(x+i)!(3n-x-i-1)!] \\ &= \binom{n}{x}\left[\frac{n}{3n-2x}\frac{\binom{n}{x}}{\binom{3n}{2x}} + 2\sum_{i=x+1}^{n}\frac{n}{3n-x-i}\frac{\binom{n}{i}}{\binom{3n}{x+i}}\right] \end{aligned}$$

Let $X(n)$ denote the expected percentage of students that obtain a slot in a school better than their catchment area school under DA, when each school has n slots and n prioritized students. Then

$$X(n) = \frac{2}{3}\frac{1}{n}\sum_{x=1}^{n} xP(x,n)$$

The $\frac{2}{3}$ fraction appears because one third of students (those with priority at the bad school) have no chance to escape from the bad school. Values for $X(n)$ are reported in Table 1 (main text). It can be shown that $X(n) \to 0$, in fact quite fast (e.g. $X(20) = 0.0177$).[27]

[27] In a previous version of this paper we show that if we fix a proportion of agents wishing to exchange good school slots, the probability they *all* do so shrinks to zero at factorial speed as n grows.

9.2 Appendix B: Proof of Proposition 1

We will divide the main theoretical result of the current paper into two parts. In the first one, we show that TTC ABS-dominates DA for disadvantaged students. In the second part, we show that the same result follows for all priority groups. The proof of the latter part uses the former one. That is the reason why we split the proof.

We will use the notation $k_{ss'}^{s''} = \sup\{l(i) : i \in \Pi_{ss'}, TTC(i) = s''\}$ as the maximum (worst) lottery allowing for a student with priority at s and preferred school s' to obtain a slot at s''.

Remark 1 $k_{ss'}^{s''} \geq k_{ws'}^{s''}$ *for every* $s'' \in G$.

We do not provide a formal proof, since this remark stems from a trivial observation. Students from $\Pi_s, s \in G$ could get access to another good school though trading of preexisting priorities or either through being pointed by means of the lottery number. Students from Π_w only count on the latter source, if any, for being assigned to a good school.

We regard school $s \in G$ as *accessible* under matching μ if $\lambda(\{i \notin \Pi_s, s \succ_i \pi(i) : \mu(i) = s\}) > 0$. Otherwise we regard it as inaccessible under μ.

Proposition 6 *When* $|G| = 2$, $ABS_w(TTC) \geq ABS_w(DA)$.

Proof. There are three cases to consider, of which the first two are immediate.

Case 1) No good schools are accessible under DA. It obviously yields $ABS_w(DA) = 0 \leq ABS_w(\mu)$ for every other μ.

Case 2) Both good schools are accessible under DA. For every accessible school, it must be the case that all of its prioritized students obtain a slot at a good school, by means of stability. This means that $q_1 + q_2 \geq n_1 + n_2$ and that the amount of good slots available to students from Π_w is simply $q_1 + q_2 - n_1 - n_2$. No other matching could yield less good slots available to such students when $q_1 + q_2 \geq n_1 + n_2$, therefore $ABS_w(DA) \leq ABS_w(\mu)$ for every other μ.

Case 3) One good school (say school 1) is inaccessible under DA while the other one (say school 2) is accessible. Notice first that it must be the case that $q_1 \leq n_1$ and that $q_2 > n_2$. If it where the case that $q_1 > n_1$, school 1 would be accessible. Provided that school 1 is not accessible and that school 2 is, all students prioritized by school 2 must end up assigned there and besides some other slots should be available to others, concluding that $q_2 > n_2$.

We calculate $c_2 \equiv \sup\{l(i) : i \notin \Pi_2,\ DA(i) = 2\}$. Let $\tilde{c}_1 \equiv \sup\{l(i) :\ i \in \Pi_1,\ DA(i) = 1\}$. If $\tilde{c}_1 \geq c_2$, students from Π_{11} that do not obtain a slot at school 1 would not have a chance at school 2. Therefore $c_2 = \frac{q_2 - n_2}{n_{12}+n_w}$, where the numerator takes into account that a mass n_2 of slots at school 2 are assigned to prioritized students. If $\tilde{c}_1 \leq c_2$, then no students from Π_{12} that do not obtain a slot at school 2 could have a chance at school 1. This yields $\tilde{c}_1 = q_1/n_{11}$. Moreover, we have $q_2 - n_2 = c_2(n_{12} + n_w) + (c_2 - \tilde{c}_1)n_{11}$, or $c_2 = \frac{q_1+q_2-n_2}{n_1+n_w}$. Indeed,

$$c_2 = \min\left\{\frac{q_1 + q_2 - n_2}{n_1 + n_w}, \frac{q_2 - n_2}{n_{12} + n_w}\right\}$$

Note that $ABS_w(DA) = c_2\lambda(\Pi_w)$.

We proceed to calculate $ABS_w(TTC)$. We aim to obtain the value of k^2_{w2}. Since $ABS_w(TTC) \geq k^2_{w2}\lambda(\Pi_w)$, we just need to show that $k^2_{w2} \geq c_2$. We run the TTC algorithm by letting school 2 be the first in pointing a student, as long as it keeps available slots (such choice does not alter the final allocation, which is unique.) This does not necessarily imply that school 2 assigns all slots before school 1 does so. School 2 could point at a student that points at school 1, creating a cycle in which school 1 gives a slot.

We denote with t a mass of slots of school 2 assigned using this approach, which we use as a measure of time. If slots at school 1 are still unassigned when school 2 has filled capacity, we continue our measure of time from then on by means of counting assigned slots of school 1.

We use the notation $l_s(t)$ for the infimum lottery number among students from Π_s that are still unassigned at time t. There is $t > 0$ for which $l_w(t) < l_s(t)\ \forall s \in G$, since the first cycles involve students from Π_1 and Π_2 only.

We finally use t_s for the moment at which school s has given all of its slots, also called school s termination time. Clearly, $t_2 = q_2$. We consider subcases depending on whether termination time for school 1 is below or above t_2.

Case 3.1: $t_1 < t_2$.

Suppose that $l_1(t) > l_w(t)$ for all $t \in (0, t_1)$. This means that students from Π_1 have been so far allocated though cycles in which school 1 pointed them, implying $l_1(t_1) = \rho_1$.

All the students from Π_2 that are still unassigned (if any) are now assigned to school 2. There remains a mass $q_2 - n_2$ of slots of school 2 to allocate, along with the slots of the bad school. Since school 2 is the last good school still assigning slots, and thus all unassigned students point at it, we have $k^2_{ws} = k^2_{1s'}$ for every $s, s' \in G$. In such a case, we have $q_2 - n_2 = k^2_{w2} n_w + \max\{0, k^2_{w2} - l_1\} n_1$, or

$$\begin{aligned} k^2_{w2} &= \min\left\{\frac{q_2 - n_2 + q_1}{n_1 + n_w}, \frac{q_2 - n_2}{n_w}\right\} \\ &\geq \min\left\{\frac{q_2 - n_2 + q_1}{n_1 + n_w}, \frac{q_2 - n_2}{n_{12} + n_w}\right\} = c_2 \end{aligned}$$

Suppose instead that we reach a point in time $t' < t_1$ in which $l_1(t') = l_w(t')$. Once this equality arises, it holds for the rest of the assignment algorithm, since school 2 does not discriminate among nonprioritized students other than by the lottery number. This implies that $k^2_{1s} = k^2_{ws'} = k$ for every s, $s' \in G$. Therefore we have the simple feasibility equation

$$q_1 + q_2 = n_2 + k(n_1 + n_w)$$

yielding $k^2_{w2} = \frac{q_2 - n_2 + q_1}{n_1 + n_w} \geq c_2$.

Case 3.2: $t_2 \leq t_1$.

Since $q_2 > n_2$, all students from Π_2 have been assigned already. At the point at which the last student from Π_2 has been assigned, a mass n_{21} of slots of school 2 have been assigned to students from Π_{12}, thus $l_1(n_2) = n_{21}/n_{12}$. Note $l_w(n_2) = 0$ (no disadvantaged students have been assigned so far.) We study the continuation of the TTC algorithm from then on, with a mass $q_2 - n_2$ of pending slots of school 2 to be assigned.

By school 2 being the first in giving all slots, we have $k^2_{h1} = 0$ for all schools h. If a student from Π_{h1} were pointed by school 2, this student would point at school 1, which has available slots.

We calculate the value of k^2_{w2}. Let $\delta \in [0, q_2 - n_2]$ be a mass of pending slots from school 2. While $l_w(n_2 + \delta) < l_1(n_2 + \delta)$, we have

$$\begin{aligned} l_1(n_2 + \delta) &= \frac{n_{21} + \delta \frac{n_{w1}}{n_w}}{n_{12}} \\ l_w(n_2 + \delta) &= \frac{\delta}{n_w} \end{aligned}$$

While $l_w(n_2 + \delta) < l_1(n_2 + \delta)$, each remaining slot of school 2 is assigned through cycles where school 2 points at a student from Π_w, yet with probability $\frac{n_{w1}}{n_w}$ it points to a student from Π_{w1} thus yielding a cycle where the slot is assigned to a student from Π_1.

Since $l_w(n_2) < l_1(n_2)$, school 2 starts pointing at a student from Π_w, and keeps doing so until l_w and l_1 coincide (if it does.) If $l_w(q_2) \leq l_1(q_2)$, then all $q_2 - n_2$ remaining slots are assigned through cycles in which a student from Π_w is involved, hence:

$$k_{w2}^2 = \frac{q_2 - n_2}{n_w} \geq \frac{q_2 - n_2}{n_{12} + n_w} \geq c_2$$

Let us instead suppose that there is $\delta^* \in [0, q_2 - n_2)$ such that $l_w(n_2 + \delta^*) = l_1(n_2 + \delta^*) = l^*$, or $\delta^* = \frac{n_{21} n_w}{n_{12} - n_{w1}}$ and $l^* = \frac{n_{21}}{n_{12} - n_{w1}}$. From then on, there remains a mass $q_2 - n_2 - \frac{n_{21} n_w}{n_{12} - n_{w1}}$ slots of school 2 to assign. Notice that, in this case, $k_{w2}^2 = k_{12}^2$ (once lottery numbers are tied, they keep tied until school 2 fills capacity.) Since $k_{h1}^2 = 0$ for all h, and since all students from Π_2 are already assigned,

$$q_2 - n_2 - \frac{n_{21} n_w}{n_{12} - n_{w1}} = \left(k_{w2}^2 - \frac{n_{21}}{n_{12} - n_{w1}} \right) (n_{w2} + n_{12})$$

giving

$$k_{w2}^2 = \frac{q_2 - n_{22}}{n_{w2} + n_{12}} \geq \frac{q_2 - n_2}{n_{w2} + n_{12}} \geq c_2$$

■

Proposition 7 *When $|G| = 2$, $ABS_s(TTC) \geq ABS_s(DA)$ for every $s \in G$.*

Proof. Also here we consider three cases, of which two of them are immediate:

Case 1) No good school is accessible under DA. Trivially, TTC cannot provide less access to better schools.

Case 2) One good school (say school 1) is not accessible and the other (school 2) is, under DA. In such a case we have $c_1 = 0$ so obviously TTC cannot provide less access to school 1. As for school 2, we use the result of Proposition 6 ($k_{w2}^2 \geq c_2$.) Since $k_{12}^2 \geq k_{w2}^2$ (Remark 1,) TTC provides more access to school 2 from any other school.

Case 3) Both good schools are accessible under DA. Let us assume without loss of generality that $c_1 \leq c_2$. Since all good schools are accessible, all students prioritized there obtain a slot at a good school, leaving exactly a mass $q_1 + q_2 - n_1 - n_2 = n_w - q_w$

of slots for students from the bad school. We then have

$$c_2 = \frac{n_w - q_w}{n_w} = 1 - \rho_w$$

As for c_1, it is calculated using

$$q_1 - n_{11} - (1 - c_2)n_{12} = c_1(n_{21} + n_{w1})$$

or

$$c_1 = \frac{q_1 - n_{11} - \rho_w n_{12}}{n_{21} + n_{w1}}$$

Notice that

$$c_2 \leq \frac{q_2 - n_{22} - \rho_w n_{21}}{n_{12} + n_{w2}}$$

as an implication of $c_1 \leq c_2 = 1 - \rho_w$. In other words, for each $s \in G$:

$$c_s = \min\left\{1 - \rho_w, \frac{q_s - n_{ss} - \rho_w n_{ss'}}{n_{s's} + n_{ws}}\right\}$$

As for the comparison to TTC: We analyze $k^s_{s's}$, where s and s' are both good schools. Along the proof we assume that $k^s_{s's} < 1$, otherwise we would be trivially done.

We model the TTC algorithm as always making school s be the first in pointing at a student, as long as it has available slots. This is innocuous in that it does not affect the final allocation.

We denote with t a mass of slots of school s assigned using this approach, which we use as a measure of time. If some slots at school s' are still unassigned when school s has filled capacity, we continue our measure of time from then on by means of counting assigned slots of school s'.

We finally use t_s for the moment at which school s has given all of its slots, also called school s termination time. Clearly, $t_s = q_s$. We consider subcases depending on whether termination time for school s' is below or above t_s.

Case 3.1: $t_s < t_{s'}$.

This implies that $k^s_{hs'} = 0$ for all schools h. If school s points at a student from $\Pi_{hs'}$, she will point at school s', since it still has available slots.

But then, $q_s = n_{ss} + k^s_{s's} n_{s's} + k^s_{ws} n_{ws}$, and $k^s_{s's} \geq k^s_{ws}$ implies

$$k^s_{s's} \geq \frac{q_s - n_{ss}}{n_{s's} + n_{ws}} \geq \frac{q_s - n_{ss} - \rho_w n_{ss'}}{n_{s's} + n_{ws}} \geq c_s$$

Case 3.2: $t_{s'} < t_s$.

This implies $q_{s'} < n_{s'}$ provided $k^s_{s's} < 1$. (If $q_{s'} \geq n_{s'}$ we have that every student from $\Pi_{s'}$ is assigned to a good school, and since $t_{s'} < t_s$ we have $k^{s'}_{s's} = 0$ -a student from $\Pi_{s's}$ cannot point at s' since she prefers s which has pending unassigned slots- and then $k^s_{s's} = 1$.) Since $\max\{c_1, c_2\} = 1 - \rho_w > 0$, or $q_w < n_w$, we must have $q_s > n_s$ (provided $\sum_h q_h = \sum_h n_h$.)

Because s is the last good school in assigning remaining slots to students, $q_s > n_s$ implies that school s eventually points at students according only to their lottery number. Consequently, $k^s_{wh} \geq k^{s'}_{wh}$, for both $h \in G$. Because of the lottery number criterion, $k^s_{ws} = k^s_{ws'}$. With this and Proposition 6 implying $ABS_w(TTC) \geq ABS_w(DA) = n_w (1 - \rho_w)$, we must have that $k^s_{ws} = \max_{g,h \in G} k^g_{wh} \geq 1 - \rho_w$. But then, by Remark 1, $k^s_{s's} \geq k^s_{ws} \geq 1 - \rho_w \geq c_s$.

This completes all relevant cases (the remaining knife-edge case in which slots from school 1 and 2 are exhausted simultaneously can be treated as part of subcases 3.1 or 3.2, indistinctively.) ■

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