Mixture-Dependent Preference for Commitment

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Abstract

The literature on temptation and self-control is motivated by evidence of a preference for commitment. This literature has typically put forth models for preferences over menus of lotteries that satisfy the Independence axiom. Independence requires that the ranking of two menus is not affected if each is mixed (probabilistically) with a common third menu. In particular, the preference for commitment is invariant under Independence. We argue that intuitive behavior may require that the preference for commitment be affected by such mixing, and hence be *mixture-dependent*. To capture such behavior, we generalize Gul and Pesendorfer (2001) by replacing their Independence axiom with a suitably adapted version of the Mixture-Betweenness axiom of Chew (1989)-Dekel (1986). Axiomatizing the model involves a novel extension of the Mixture Space Theorem to preferences that satisfy Mixture-Betweenness.

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1 Introduction

1.1 Overview

A key motivation for the literature on temptation and self-control problems comes from evidence of a preference for commitment (Bryan et al. (2010), Gul and Pesendorfer (2007) and Laibson (1997)). For instance a dieter might strictly prefer to eat at a salad bar rather than at a restaurant that offers both salad (s) and burgers (b). Identifying a restaurant with the set of alternatives it offers, this agent therefore exhibits

$$\{s\} \succ \{s, b\}.$$

In a seminal paper, Gul and Pesendorfer (2001) (henceforth GP) provide an axiomatic model of temptation and self-control that characterizes a preference over menus (of lotteries) in a manner that permits such preference for commitment. A distinct feature of GP is that it maintains the Independence axiom (appropriately adapted to the domain). We observe that Independence implies that preference for commitment must be *mixture independent* in the following sense:

$$\{s\} \succ \{s, b\} \implies \{\alpha s + (1-\alpha)i\} \succ \{\alpha s + (1-\alpha)i, \alpha b + (1-\alpha)i\},$$

where $\alpha s + (1-\alpha)i$ and $\alpha b + (1-\alpha)i$ are lotteries that yield s and b with probability α respectively and dish i with probability $(1-\alpha)$.

The main motivation of this paper is the idea that preference for commitment may in fact be mixture-dependent. To illustrate, consider a frugal vacationer who is planning a trip and needs to choose a hotel room. During her trip, she expects to use the room only to sleep. Hence, she strictly prefers a conventional room (c) to a fancy room (f). The vacationer can either choose to reserve the room in advance and commit to staying in the room she reserved, or choose the room once she arrives at the hotel. She believes that if she waits until she arrives, she will not feel tempted to choose f; after all, she is very careful with her spending and will only be using the room to sleep. Hence,

$$\{c\} \sim \{c, f\}.$$

However, being a preferred member of the hotel, she receives a promotion: her name will be added to a raffle where the prize is a free night in the fancy room, which we will denote by f', with full reimbursements if necessary. If she waits until she arrives at the hotel to choose a room, then she will face a choice between two lotteries: $\alpha c + (1-\alpha)f'$ and $\alpha f + (1-\alpha)f'$ where $(1-\alpha)$ is the probability she wins the raffle. Since there is a possibility that she might be able to stay in the fancy room for free, she will be dreaming about it for the rest of the day. Hence, if she waits to choose the room until she arrives, by then she will find the fancy room tempting. Thus, in order to avoid temptation and stick to her budget, she would rather book the conventional room in advance:

$$\{\alpha c + (1-\alpha)f'\} \succ \{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}.$$

Notice that the traveler's preference for commitment changed once the alternatives in $\{c, f\}$ were mixed with f'. Therefore, her preference for commitment is mixture-dependent.

Building on GP, we provide a novel axiomatic model of temptation and self-control. Like GP, our model characterizes preference over menus of lotteries. The key difference is that our model can accommodate mixture-dependent preference for commitment; it does do so by weakening Independence to a property we refer to as Mixture-Betweenness which adapts the Mixture-Betwenness axiom of Chew (1983) and Dekel (1986).

More specifically, let $\Delta(X)$ be the set of all lotteries with payoffs in X. GP consider a preference $\succeq$ over the set of all menus of lotteries (subsets of $\Delta(X)$). The interpretation is that at an unmodeled second stage, a lottery is selected from the menu chosen ex-ante according to $\succeq$. GP axiomatize the following utility function for $\succeq$,

$$V(x) = \max_{p \in x}\{u(p) + v(p) - \max_{q \in x} v(q)\},$$

for all menus x, where u and v are vNM utility functions over lotteries. For singleton menus, $V(\{p\}) = u(p)$ and thus u describes preference under commitment, which we interpret as describing the agent's normative view. The function v describes the agent's urges at the second stage. In the absence of commitment, there is a temptation to maximize v and hence to deviate from the choices that would be prescribed by u. Temptation can be resisted but at a cost described by $v(p) - \max_{q \in x} v(q)$. A balance between the normative preference and the cost of self-control is achieved by choosing a lottery ex-post that maximizes $u + v$. Since u and v are vNM utility functions, they satisfy the standard Independence axiom. This is imposed for reasons of analytical convenience rather than descriptive validity in view of the widely documented descriptive violations of the Independence axiom.[1]

The fact that in GP's model the normative and temptation utilities are linear precludes their model from accommodating mixture-dependent preference for commitment. In fact this limitation exists also in subsequent generalizations of GP in the literature (Dekel et al. (2009), Chatterjee and Krishna (2009), Noor and Takeoka (2010, 2015), Stovall (2010) and Kopylov (2012)). In order to accommodate mixture-dependent preference for commitment we need to allow for non-linear u and v.

This motivates us to consider the Chew (1983)-Dekel (1986) model for preference over risk. Their model is motivated by the descriptive failure of the Independence axiom and generalizes vNM utility theory. In particular, it is an implicit utility model in which the utility γ of a lottery p is the unique solution of

$$\gamma = u(p, \gamma),$$

where $u(., \gamma)$ is a vNM utility function over lotteries for all γ. The main ingredient in the characterization of the model is the Mixture-Betweenness axiom which is a weakening of Independence that is compatible with behavior such as the Allais paradox.

Our model combines GP and Chew-Dekel. The result is an implicit utility model in which the utility of a menu x is defined as the unique γ that solves

$$\gamma = \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\},$$

where $u(., \gamma)$ and $v(., \gamma)$ are vNM utility functions over lotteries for all γ. Notice that our model is inherently non-linear. Hence, standard tools cannot be used to axiomatize it because they rely on the Mixture Space Theorem (Herstein and Milnor (1953)) at a fundamental level. Thus, we are forced to take a different approach. In particular, we develop a novel extension of the Mixture Space Theorem. Since the Mixture Space Theorem is central to decision theory, our extension is potentially useful for addressing issues in economics other than temptation. Hence, we view it as a separate contribution of the paper.

[1]See the surveys by Camerer (1995) and Starmer (2000).

The paper proceeds as follows: The introduction concludes with a review of the relevant literature. Axioms and the implied representation of utility are described in Sections 2 and 3 respectively. Section 4 contains a discussion of mixture-dependent preference for commitment as well as axiomatic foundations for two special cases of the model. Section 5 concludes with our version of the Mixture Space Theorem and a discussion of its potential applications. The proof of the new Mixture Space Theorem is provided in Appendix A. The remaining proofs are collected in Appendix B.

1.2 Literature Review

This paper contributes to the axiomatic literature on temptation and self-control (Gul and Pesendorfer (2001), Dekel et al. (2001), and for a survey of the subsequent literature see Lipman and Pesendorfer (2013)). The closest papers to ours are Noor and Takeoka (2010, 2015) and Liang et al. (2019). They also generalize GP by weakening Independence. Noor and Takeoka (2010) extends GP to a model with a convex self-control cost, a feature shared by the non-axiomatic model of Fudenberg and Levine (2006). Liang et al. (2019) enrich GP by endowing the agent with a stock of willpower that cannot be exceeded by the cost of self-control in any menu.[2] In both models the agent's normative and temptation utilities satisfy the Independence axiom thereby ruling out mixture-dependent preference for commitment.

Dillenberger and Sadowski (2012) provide a model of shame for preferences over menus of monetary divisions between two agents, a dictator and a recipient. Their model adapts and extends GP, and can accommodate the following experimental finding: when subjects are presented with a menu of monetary divisions, they tend to behave altruistically whenever the recipient can observe the menu from which they are making the choice. However, at an ex-ante stage that is not observed by the recipient, some subjects are willing to give up part of their payoff in exchange for the removal of the altruistic monetary divisions so that in the second stage they can choose "unfair" divisions without feeling any shame. Since the objects of choice in their set up are menus of deterministic alternatives, their analysis is silent about mixture-dependence.

Following Dillenberger and Sadowski (2012) and GP, Saito (2015) devel-

[2] A closely related paper, Masatlioglu et al. (2020) deals with menus of abstract alternatives rather than menus of lotteries.

ops a model of impure shame and impure altruism (that is, shame and altruism driven by temptation) for preferences over menus of lotteries. His model satisfies the Independence axiom and thus, cannot accommodate mixture-dependent preference for commitment.

Dekel et al. (2009), Chatterjee and Krishna (2009), Stovall (2010) and Kopylov (2012) also generalize GP. Their models satisfy the Independence axiom. We believe that our Mixture Space Theorem and the arguments used in the proof of Theorem 3.1 can be used to generalize these models in the same way that we generalize GP.

Finally, outside the temptation literature but within the menus of lotteries literature, Ergin and Sarver (2010) derive a utility representation of costly contemplation. The model assumes that the agent chooses from a menu with imperfect knowledge of her preference over lotteries. In particular, the agent considers a set of possible preferences over lotteries where each of them satisfies the standard Independence axiom. Their key axiom, referred to as Aversion to Contingent Planning, is a weakening of our adaptation of the Mixture-Betweenness axiom.

2 Axioms

Let X be a finite set of cardinality n. A lottery is a probability measure over X. The set of all lotteries is denoted $\Delta(X)$ and $\mathcal{X}$ denotes the set of all of its non-empty closed subsets. We endow $\mathcal{X}$ with the topology generated by the Hausdorff metric.[3] A menu is an element of $\mathcal{X}$. Generic menus will be denoted by x, y and z and generic lotteries will be denoted by p, q and r. Each lottery p can be identified with the singleton menu $\{p\} \in \mathcal{X}$. Thus, where it does not cause confusion we will abuse notation and write p instead of $\{p\}$ and $\Delta(X)$ instead of $\{\{p\}|p \in \Delta(X)\}$.

Our primitive is a preference $\succeq$ over $\mathcal{X}$. We impose four axioms on $\succeq$ of which the first three are from GP.

Weak Order $\succeq$ is complete and transitive.

Hausdorff Continuity $\{y|x \succeq y\}$ and $\{y|y \succeq x\}$ are closed for all $x \in \mathcal{X}$.

[3]Let d be any metric on $\Delta(X)$. For any $x, y \in \mathcal{X}$ and $p, q \in \Delta(X)$, define $d(p, y) \equiv \inf_{q \in y} d(p, y)$ and $d_h(x, y) = \max\{\sup_{p \in x} d(p, y), \sup_{q \in y} d(q, x)\}$. The topology generated by d_h is the Hausdorff metric topology.

Set-Betweenness $x \succeq y$ implies $x \succeq x \cup y \succeq y$.

Set-Betweenness admits an interpretation in terms of temptation and self-control. To illustrate, consider the ranking $\{p\} \succ \{p, q\} \succ \{q\}$. The ranking $\{p\} \succ \{p, q\}$ is referred to as *preference for commitment*, it suggests that the agent expects to be temped by q if she faces $\{p, q\}$. Thus, $\{p, q\} \succ \{q\}$ implies that the agent expects to be able to resist temptation if she faces $\{p, q\}$, but it will require costly self-control. Similarly, $\{p\} \succ \{p, q\} \sim \{q\}$ suggests that the agent expects to be overwhelmed by temptation if she faces $\{p, q\}$. Finally, the lack of preference for commitment in $\{p\} \sim \{p, q\} \succ \{q\}$ suggests that the agent does not expect to be tempted by q if she faces $\{p, q\}$.

Whenever $x \subset y$ and $x \succ y$ we say $\succeq$ *has preference for commitment at* y. Under the temptation and self-control interpretation, preference for commitment at y reveals that there is some element in y that the agent expects to be tempted by and thus, would like to remove from the feasible set she will face in the second stage.

For any two menus x, y and $\alpha \in [0, 1]$, define the mixture $\alpha x + (1 - \alpha)y$ as the menu generated by the point-wise mixtures:

$$\alpha x + (1 - \alpha)y = \{r \in \Delta(X) | r = \alpha p + (1 - \alpha)q, p \in x, q \in y\}.$$

GP's fourth axiom formulates the standard vNM Independence axiom in the menus of lotteries setting.

Independence $x \succeq y$ implies $\alpha x + (1 - \alpha)z \succeq \alpha y + (1 - \alpha)z$ for all $\alpha \in [0, 1]$ and $z \in \mathcal{X}$.

GP, Dekel et al. (2001) and the literature that followed them adopt this axiom because of its normative appeal and the analytical convenience it offers. To understand their motivation consider an extension of the preference to the set of lotteries over $\mathcal{X}$, the interpretation being that randomization over menus is resolved before the second stage. Suppose that this extended preference satisfies the standard vNM Independence axiom: the preference between a lottery that yields with probability α a menu x and with probability $1 - \alpha$ a menu z (denoted by $\alpha \circ x + (1 - \alpha) \circ z$) and $\alpha \circ y + (1 - \alpha) \circ z$ is the same as the preference between x and y. If the agent is indifferent between uncertainty being resolved before the second stage or after the second stage, then she satisfies *Reduction*:

$$\alpha \circ x + (1-\alpha) \circ y \sim \alpha x + (1-\alpha) y \text{ for all } x, y \in \mathcal{X} \text{ and } \alpha \in [0,1].$$

Observe that vNM Independence and Reduction imply Independence. However, we claim that temptation may lead to violations of vNM Independence. Recall the rankings in our example:

$$\begin{gathered} \{c\} \sim \{c, f\} \\ \{\alpha c + (1-\alpha) f'\} \succ \{\alpha c + (1-\alpha) f', \alpha f + (1-\alpha) f'\}. \end{gathered} \tag{1}$$

The intuition behind these rankings implies that the agent would strictly prefer $\alpha \circ \{c\} + (1-\alpha) \circ \{f'\}$ to $\alpha \circ \{c, f\} + (1-\alpha) \circ \{f'\}$ because once there is a possibility she might be able to stay in the fancy room for free, she will think about it extensively. Hence, if the lottery $\alpha \circ \{c, f\} + (1-\alpha) \circ \{f'\}$ yields $\{c, f\}$, she will feel tempted to choose f.

More generally, Independence implies that preference for commitment is mixture independent: if $\succeq$ has preference for commitment at y, then $\succeq$ also has preference for commitment at $\alpha y + (1-\alpha) z$ for all $\alpha \in (0,1]$ and $z \in \mathcal{X}$. This follows from the fact that under Independence,

$$\begin{gathered} x \succ y \implies \alpha x + (1-\alpha) z \succ \alpha y + (1-\alpha) z \\ \text{and} \\ x \subset y \implies \alpha x + (1-\alpha) z \subset \alpha y + (1-\alpha) z. \end{gathered}$$

Hence, Independence needs to be weakened. We weaken it to Mixture-Betweenness.

Mixture-Betweenness

$x \succ y$ implies $x \succ \alpha x + (1-\alpha) y \succ y$ for all $\alpha \in (0,1)$, and
$x \sim y$ implies $x \sim \alpha x + (1-\alpha) y \sim y$ for all $\alpha \in (0,1)$.

Mixture-Betweenness requires that if an agent prefers x to y, then the mixture between x and y has to be between these two menus in terms of preference. In particular, if an agent is indifferent between two menus, then any mixture of these two is equally good. Hence, it implies that the indifference sets are convex. Next, we consider what Mixture-Betweenness permits and what it rules out.

Mixture-dependent preference for commitment is permitted by Mixture-Betweenness. For instance, the rankings in (1) are consistent with Mixture-Betweenness because the latter is silent about behavior that involves more than two menus. However, it does rule out a specific class of mixture-dependent preference for commitment. To illustrate, consider a dieter who prefers to go to a salad bar rather than a restaurant that offers salads (s) and burgers (b):

$$\{s\} \succ \{s, b\} \succeq \{b\}. \tag{2}$$

Suppose now that there is a small new restaurant that also offers salads and burgers, but its burgers are so popular that it sometimes runs out of them. Therefore, its menu offers salads and a lottery between burgers and salads. Mixture-Betweenness would then require that the dieter would also expect to feel temptation if she goes to the new restaurant because its menu is equal to a mixture between the salad bar and the restaurant. In particular, it is equal to

$$\{s, \alpha s + (1 - \alpha)b\},$$

where α is the probability that the restaurant runs out of burgers. Hence,

$$\{s\} \succ \{s, b\} \implies \{s\} \succ \{s, \alpha s + (1 - \alpha)b\}.$$

In general, Mixture-Betweenness requires that if $x \subset y$ and $x \succ y$, then $x \succ \alpha x + (1 - \alpha)y$. Thus, it implies that if the agent has preference for commitment at y, then the agent's preference for commitment is mixture independent whenever the mixture is with any subset of y that she would prefer to commit to.

Mixture-Betweenness also imposes some structure on the agent's self-control. To illustrate, consider again the dieting agent. Suppose now that there are two restaurants that belong to the same chain and assume that one is smaller than the other. The smaller restaurant does not have a fixed menu. Rather, its menu its picked randomly by the chef (chef's pick). It can either be a salad or a burger. The larger restaurant not only offers the chef's pick, but in addition it has a fixed burger selection. Consistent with the preference for commitment she exhibited in (2), she prefers to go to the smaller restaurant. Hence,

$$\{\alpha s + (1 - \alpha)b\} \succ \{\alpha s + (1 - \alpha)b, b\}$$

where α is the probability that the chef's pick is a salad. If the dieter exerted costly self-control in $\{s, b\}$ (that is, if $\{s, b\} \succ \{b\}$ in (2)), then since the menu at the larger restaurant is a mixture between $\{s, b\}$ and $\{b\}$, Mixture-Betweenness implies that

$$\{\alpha s + (1-\alpha)b, b\} \succ \{b\}.$$

Thus, if the dieter expects to resist temptation when facing $\{s, b\}$, then she must also expect to resist temptation when facing $\{\alpha s + (1-\alpha)b, b\}$. Hence, whenever her preference for commitment is preserved in a mixture with the tempting alternative, so is her ability to resist temptation.

As a matter of fact, Mixture-Betweenness also requires the lack of self-control to be mixture independent for certain mixtures. For instance, suppose that the dieter thinks she will be overwhelmed by temptation if she goes to the old restaurant that offers salads and burgers (that is, if $\{s, b\} \sim \{b\}$ in (2)). Suppose now that she is also considering the large chain restaurant. Since the menu at the large chain restaurant is a mixture between $\{s, b\}$ and $\{b\}$, Mixture-Betweenness requires that

$$\{\alpha s + (1-\alpha)b, b\} \sim \{b\}.$$

Thus, if the dieter expects to be overwhelmed by temptation when facing $\{s, b\}$, then she must also expect to be overwhelmed by temptation when facing $\{\alpha s+(1-\alpha)b, b\}$. Hence, her lack of self-control is mixture independent when mixing with the overwhelming alternative.

We see therefore that the agent's self-control is mixture independent in a limited sense:

If $\{s\} \succ \{s, b\}$, then $\{s, b\} \succ \{b\}$ if and only if $\{\alpha s + (1-\alpha)b, b\} \succ \{b\}$.

This suggests that Mixture-Betweenness imposes a form of linearity of self control costs. In contrast, the model of convex self-control costs of Noor and Takeoka (2010) allows for mixture dependent self-control. In particular, their model can accommodate behavior of the following type:

$$\begin{gathered} \{s\} \succ \{s, b\} \sim \{b\} \\ \{\alpha s + (1-\alpha)b, b\} \succ \{b\}. \end{gathered} \tag{3}$$

The reason is that in their model, the marginal cost of self-control is increasing in the exertion of self-control. Hence, achieving "small" deviations from the tempting alternative is easier than large deviations. In terms of the rankings in (3), their model permits the dieter to believe that she will be able to choose $\alpha s + (1-\alpha)b$ in the presence of b even if she expects to choose b from $\{s, b\}$ for a small enough α. Therefore Mixture-Betweenness rules out convex self-control costs. Similarly, Mixture-Betweenness also rules out behavior related to concave self-control costs:

$$\{s\} \sim \{s, b\} \succ \{b\}$$
$$\{s\} \succ \{s, \alpha s + (1-\alpha)b\}.$$

3 Results

3.1 Representation Theorem

Say that $\succeq$ *is non-trivial* if there exists $x, y \in \mathcal{X}$ such that $x \succ y$.

The central result of the paper is the following axiomatization of utility over menus.

Theorem 3.1. *A non-trivial preference $\succeq$ satisfies Weak Order, Hausdorff Continuity, Set-Betweenness and Mixture-Betweenness if and only if there exist $u, v : \Delta(X) \times [0,1] \to \mathbb{R}$ such that:*

1. *$u(., \gamma)$ and $v(., \gamma)$ are vNM expected utility functions for all $\gamma \in [0,1]$.*
2. *u is continuous in its second argument on the interval $(0,1)$.*
3. *$u(\overline{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = 0$ for all $\gamma \in [0,1]$ for some $\overline{p}, \underline{p} \in \Delta(X)$.*
4. *$\succeq$ can be represented by a continuous utility function $V : \mathcal{X} \to [0,1]$ where, for each $x \in \mathcal{X}$, $V(x)$ is the unique $\gamma \in [0,1]$ that solves*

$$\gamma = \max_{p \in x}\{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\}.$$

One feature of our model that is not shared by any other model in the menus literature is that it does not reduce to expected utility over lotteries, that is, for menus that offer commitment: $x = \{p\}$ for some $p \in \Delta(X)$. Rather, it reduces to the Chew-Dekel model for preference under risk:

$$V(\{p\}) = u(\{p\}, V(\{p\})).$$

Say that (u, v) *represents* $\succeq$ if it satisfies the conditions of Theorem 3.1. Note that it is not the case that any (u, v) that satisfies conditions 1-3 in Theorem 3.1 implicitly defines a utility function V as in condition 4. In Appendix E we provide a set of sufficient conditions for (u, v) which guarantees the existence of an implicit utility representation.

3.2 Uniqueness

To state the uniqueness properties of our model we require some additional terminology. Given any pair of functions $f, g : \Delta(X) \to \mathbb{R}$, we say that f is a *positive affine transformation of* g if there exist $a, b \in \mathbb{R}$ such that $a > 0$ and $f = ag + b$. Similarly, f is a *negative affine transformation of* g if there exist $a, b \in \mathbb{R}$ such that $a < 0$ and $f = ag + b$.

Theorem 3.2. *Let (u, v) and (u', v') be such that $u(\overline{p}, \gamma) = u'(\overline{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = u'(\underline{p}, \gamma) = 0$ for all $\gamma \in [0, 1]$ for some $\overline{p}, \underline{p} \in \Delta(X)$. Then both (u, v) and (u', v') represent $\succeq$ if and only if for all $\gamma \in (0, 1)$, $u(., \gamma) = u'(., \gamma)$ and:*

1. *If $v(., \gamma)$ is a positive affine transformation of $u(., \gamma)$ or a constant, then $v'(., \gamma)$ is a positive affine transformation of $v(., \gamma)$ or a constant.*
2. *If $v(., \gamma) = -a_\gamma u(., \gamma) + b_\gamma$ for some $a_\gamma \geq 1$ and $b_\gamma \in \mathbb{R}$, then $v'(., \gamma) = a'_\gamma v(., \gamma) + b'_\gamma$ for some $a'_\gamma \geq \frac{1}{a_\gamma}$ and $b'_\gamma \in \mathbb{R}$.*
3. *If $v(., \gamma)$ is not a constant or a positive affine transformation of $u(., \gamma)$ and the condition in 2 does not hold, then $v'(., \gamma) = v(., \gamma) + b_\gamma$ for some $b_\gamma \in \mathbb{R}$.*

The uniqueness properties of u are completely characterized by the restriction of $\succeq$ to $\Delta(X)$. In particular, for every $p \in \Delta(X)$, $V(p)$ is the unique $\gamma \in [0, 1]$ that solves

$$\gamma = u(p, \gamma),$$

where $u(\overline{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = 0$ for all $\gamma \in [0, 1]$. Dekel (1986) shows that such representations are unique.

Because u is completely characterized by the restriction of $\succeq$ to menus that offer commitment, we interpret the utility function implicitly defined by u as the agent's commitment preferences.

To aid intuition for the uniqueness properties of v we describe why the conditions in Proposition 3.2 are sufficient. Assume (u, v) represents $\succeq$ and fix $\gamma \in (0,1)$. If $v(.,\gamma)$ is a constant or a positive affine transformation of $u(.,\gamma)$, then for all $x \in \mathcal{X}$,

$$\arg\max_{p\in x} v(p,\gamma) = \arg\max_{p\in x}\{u(p,\gamma) + v(p,\gamma)\},$$

and

$$\max_{p\in x}\{u(p,\gamma) + v(p,\gamma) - \max_{p\in x} v(p,\gamma)\} = \max_{p\in x} u(p,\gamma).$$

Thus, replacing $v(.,\gamma)$ with a constant or one of its positive affine transformations does not affect the representation. If $v(.,\gamma) = -a_\gamma u(.,\gamma) + b_\gamma$ for some $a_\gamma \geq 1$, then for all $x \in \mathcal{X}$,

$$\arg\max_{p\in x}\{v(p,\gamma)\} = \arg\min_{p\in x}\{u(p,\gamma)\}$$

and

$$\arg\min_{p\in x}\{u(p,\gamma)\} \subseteq \arg\max_{p\in x}\{u(p,\gamma) + v(p,\gamma)\}.$$

To see this, note that for any $a_\gamma \geq 1$ and $p, q \in \Delta(X)$ such that $u(p,\gamma) \geq u(q,\gamma)$,

$$\begin{aligned} u(p,\gamma) - u(q,\gamma) &\leq a_\gamma(u(p,\gamma) - u(q,\gamma)) \\ u(p,\gamma) - a_\gamma u(p,\gamma) + b_\gamma &\leq u(q,\gamma) - a_\gamma u(q,\gamma) + b_\gamma \\ u(p,\gamma) + v(p,\gamma) &\leq u(q,\gamma) + v(q,\gamma). \end{aligned}$$

Hence,

$$q \in \arg\min_{p\in x}\{u(p,\gamma)\} \implies q \in \arg\max_{p\in x}\{u(p,\gamma) + v(p,\gamma)\},$$

and

$$\max_{p\in x}\{u(p,\gamma) + v(p,\gamma) - \max_{p\in x} v(p,\gamma)\} = \min_{p\in x}\{u(p,\gamma)\},$$

for all $x \in \mathcal{X}$. Thus, if we replace $v(.,\gamma)$ with $v(.,\gamma) = a'_\gamma v(.,\gamma) + b'_\gamma$ for some $a'_\gamma \geq \frac{1}{a_\gamma}$, then $v'(.,\gamma)$ is also a negative affine transformation of $u(.,\gamma)$ in which the coefficient multiplying $-u(.,\gamma)$ is greater than or equal to one. Thus, the representation is not affected. Finally, note that if $v(.,\gamma)$ is not a constant or a positive affine transformation of $u(.,\gamma)$ and the condition in 2 does not hold, then for any $b_\gamma \in \mathbb{R}$,

$$\max_{p\in x}\{u(p,\gamma)+v(p,\gamma)+b_\gamma-\max_{p\in x}\{v(p,\gamma)+b_\gamma\}\} = \max_{p\in x}\{u(p,\gamma)+v(p,\gamma) - \max_{p\in x} v(p,\gamma)\}.$$

Hence, replacing $v(.,\gamma)$ with $v(.,\gamma)+b_\gamma$ does not affect the representation.

3.3 Illustration

We conclude this section by illustrating our model in the context of our motivating example. Details concerning the calculations below can be found in Appendix D. Recall the rankings in our motivating example:

$$\{c\} \sim \{c,f\} \succ \{f\}$$
$$\{\alpha c+(1-\alpha)f'\} \succ \{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}.$$

Consider the special case of our model in which the normative utility is linear and independent of the overall level of utility. Let

$$u(f')=1, \qquad v(f',\gamma)=1$$
$$u(c)=\frac{1}{2}, \qquad v(c,\gamma)=\frac{1-\gamma}{2}$$
$$u(f)=0, \qquad v(f,\gamma)=\frac{\gamma}{2}$$

for all $\gamma\in[0,1]$. For any lottery $p\in\Delta(\{c,f,f'\})$, we will abuse notation and write $u(p)$ and $v(p,\gamma)$ instead of $\sum_{a\in\{c,f,f'\}} p(a)u(a)$ and $\sum_{a\in\{c,f,f'\}} p(a)v(a,\gamma)$.

This special case of the model can accommodate the intuition behind the rankings in the example. The reason is that for $x=\{c,f\}$ and $y=\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}$,

$$V(x)=u(p)$$
$$V(y)=u(p')+v(p',V(y))-v(q,V(y)),$$

where $p=c$, $p'=\alpha c+(1-\alpha)f'$ and $q=\alpha f+(1-\alpha)f'$. Hence, the model predicts that the agent does not expect to feel any temptation if she faces $\{c,f\}$ in the second stage. However, if she faces $\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}$, then it suggests that she expects to choose $\alpha c+(1-\alpha)f$ and be tempted by $\alpha f+(1-\alpha)f'$. Further,

$$V(\{f'\}) > V(\{c\}) = V(\{c,f\}) > V(\{f\})$$
$$V(\{\alpha c+(1-\alpha)f'\}) > V(\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}),$$

where

$$V(\{f'\}) = 1, V(\{c\}) = \frac{1}{2}, V(\{c,f\}) = \frac{1}{2}, V(\{f\}) = 0$$
$$V(\{\alpha c+(1-\alpha)f'\}) = 1-\frac{\alpha}{2}, V(\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}) = \frac{1}{1+\alpha}.$$

Hence, the model accommodates our motivating example.

3.4 Specialization

To the extent that a systematic study of temptation calls for us to attribute any non-standard behavior to temptation, it is natural to attribute Independence violations to temptation and self-control rather than to normative preferences. Accordingly, we study a specialization of our model that retains linearity of normative preference and attributes non-standard effects of randomization to temptation preferences. Consequently, we consider:

Commitment Independence
$\{p\} \succeq \{q\}$ implies $\alpha\{p\}+(1-\alpha)\{r\} \succeq \alpha\{q\}+(1-\alpha)\{r\}$
for all $\alpha \in [0,1]$ and $r \in \Delta(X)$.

Commitment Independence requires that the agent's commitment preference satisfy the Independence axiom. Hence, a decision maker who views Independence over lotteries as an appealing normative property will satisfy it. Further, it does not restrict behavior over non-singleton menus and allows for violations of Independence.

Proposition 3.1. *Suppose $\succeq$ satisfies the axioms of Theorem 3.1. Then $\succeq$ also satisfies Commitment Independence if and only if it admits a representation as in Theorem 3.1 in which $u(.,\gamma) = u(.,\gamma')$ for all $\gamma,\gamma' \in [0,1]$.*

Apart from being appealing from a modeling perspective, the illustration in the previous section shows that it is also analytically convenient.

4 Commitment and Mixtures

4.1 General Self-Control Models

In the introduction we claimed that GP cannot accommodate mixture-dependent preference for commitment because of the linearity of the normative and temptation preferences. Here we show that in fact, any temptation model that maintains linearity of the normative and temptation preferences, and that satisfies Set-Betweenness cannot accommodate mixture-dependent preference for commitment.

Recall that $\succeq$ has preference for commitment at y if there exists $x \subset y$ such that $x \succ y$. Noor and Takeoka (2010, 2015) show that any temptation model that satisfies Set-Betweenness and maintains linearity of the normative and temptation preference can be written in the following way:

$$V(x) = \max_{p \in x}\{u(p) - c(p, \max_{q \in x} v(q))\},$$

for all menus x, where u and v are vNM utility functions over lotteries and c satisfies the following three properties:

1. c is weakly increasing in its second argument and continuous in both arguments.

2. $c(p, v(q)) > 0$ implies $v(q) > v(p)$.

3. $u(p) > u(q)$ and $v(p) < v(q)$ implies $c(p, v(q)) > 0$.

Intuitively, properties 1, 2 and 3 are the minimal properties c must possess in order to be interpretable as a self-control cost function. In particular, property 1 says that the higher the temptation, (weakly) higher the self-control is needed to resist it. Property 2 requires that if p is costly to choose in the presence of q, then it must be that q offers higher temptation utility. Property 3 provides a converse in that if q provides more temptation utility than p and there is conflict with the normative utility, then the cost of choosing p must be strictly positive.

Noor and Takeoka (2010) refer to this class of models as *general self-control* models, and identify them with the tuple (u, v, c).

Proposition 4.1. *Let (u, v, c) be a general self-control model and $\succeq$ the preference it represents. Then for all $x, y \in \mathcal{X}$:*

1. *If $\succeq$ has preference for commitment at x, then $\succeq$ has preference for commitment at $\alpha x + (1-\alpha)y$.*

2. *Suppose there is no preference for commitment at y. If $\succeq$ does not have a preference for commitment at x, then $\succeq$ does not have preference for commitment at $\alpha x + (1-\alpha)y$.*

To illustrate why mixture-dependent preference for commitment necessitates a violation of linearity of at least v, consider the following possible rankings involving binary menus:

$$(*)\;\; \{p\} \succ \{p, q\} \succeq \{q\} \text{ and } \{\alpha p + (1-\alpha)r\} \sim \{\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r\}$$
$$(**)\;\; \{p\} \sim \{p, q\} \succ \{q\} \text{ and } \{\alpha p + (1-\alpha)r\} \succ \{\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r\}.$$

Let (u, v, c) be a general self-control model and $\succeq$ the preference over menus it represents. Noor and Takeoka (2015) show that

$$\{p\} \succ \{p, q\} \succeq \{q\} \implies v(q) > v(p)$$
$$\{p'\} \sim \{p', q'\} \succ \{q'\} \implies v(p') \geq v(q').$$

Hence, the rankings in $(*)$ imply

$$v(q) > v(p)$$
$$v(\alpha p + (1-\alpha)r) \geq v(\alpha q + (1-\alpha)r).$$

Thus, by linearity of v, $v(q) > v(p)$ and $v(q) \leq v(p)$, an impossibility. Similarly, the rankings in $(**)$ imply

$$v(p) \geq v(q)$$
$$v(\alpha q + (1-\alpha)r) \geq v(\alpha p + (1-\alpha)r).$$

Hence, by linearity of v, $v(q) \geq v(p)$ and $v(q) < v(p)$, another impossibility.

4.2 Mixture Monotone Preference for Commitment

Here we specialize the model by focusing on the preference for commitment. In particular, we characterize two monotone patterns it can take under Commitment Independence and an additional assumption. In Appendix B we provide the corresponding results for the general model.

Mixture-Increasing Preference for Commitment

For all p and x such that $\succeq$ has preference for commitment at $\alpha x + (1-\alpha)\{p\}$ for some $\alpha \in [0,1]$:

I If $\{p\} \succ x$, then $\succeq$ has preference for commitment at $\beta x + (1-\beta)\{p\}$ for all $0 < \beta < \alpha$.

II If $x \succ \{p\}$, then $\succeq$ has preference for commitment at $\beta x + (1-\beta)\{p\}$ for all $\beta > \alpha$.

Mixture-Increasing Preference for commitment restricts how preference for commitment can vary along mixtures. Indeed, Part I implies that if $\succeq$ has preference for commitment at x, then $\succeq$ has preference for commitment at any mixture between x and a superior menu that offers commitment $\{p\}$. On the other hand, Part II requires that if the agent has preference for commitment at a mixture between a menu that offers commitment $\{p\}$ and a superior menu x, then increasing the weight on the superior menu does not affect preference for commitment.

The special case of our model described in Section 3.3 satisfies this axiom (see Appendix D for a proof). Hence, it is compatible with our motivating example. Recall that in the example, the agent had preference for commitment at a mixture between $\{c, f\}$ and the superior menu $\{f'\}$ but not at $\{c, f\}$:

$$\{\alpha c + (1-\alpha)f'\} \succ \{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}$$
$$\{c\} \sim \{c, f\}.$$

Thus, the axiom only requires that the agent has preference for commitment at $\{\beta c + (1-\beta)f', \beta f + (1-\beta)f'\}$ for all $0 < \beta < \alpha$ which is consistent with the behavior prescribed by the model in Section 3.3.

To state the next result, we require some additional notation. Let p^*, p_* denote a fixed pair of lotteries such that $\{p^*\} \succeq x \succeq \{p_*\}$ for all $x \in \mathcal{X}$. Given our axioms such lotteries always exist.[4]

[4]Weak Order and Continuity imply that the ordering over singleton menus has a best and worst menu. Thus, by Set-Betweenness, all finite menus are between these two menus in terms of preference. Hence, by Continuity the same holds for any menu.

Theorem 4.1. *Assume $\succeq$ satisfies the axioms of Proposition 3.1 and $\{p^*\} \sim \{p^*, p_*\}$. Then $\succeq$ satisfies Mixture-Increasing Preference for Commitment if and only if there exists (u, v) that represents $\succeq$ such that $0 < \gamma' < \gamma < 1$ implies that there exists $b_{\gamma,\gamma'} \in \mathbb{R}$ such that $v(., \gamma') + b_{\gamma,\gamma'}$ is a convex combination of u and $v(., \gamma)$.*

The assumption that $\{p^*\} \sim \{p^*, p_*\}$ implies that the agent does not expect to be tempted by p_* if she faces $\{p^*, p_*\}$ in the second stage. This assumption guarantees that for each $\gamma \in [0, 1]$, $v(., \gamma)$ is not a negative affine transformation of u. Theorem 4.1 shows that Mixture-Increasing Preference for Commitment forces the following concrete relationship between u and v across different levels of utility: as utility decreases, u and $v(., \gamma)$ get "closer together".

The following axiom characterizes the opposite case: as utility decreases, u and $v(., \gamma)$ move "further apart".

Mixture-Decreasing Preference for Commitment

For all p and x such that $\succeq$ has preference for commitment at $\alpha x + (1 - \alpha)\{p\}$ for some $\alpha \in [0, 1]$:

I If $x \succ \{p\}$, then $\succeq$ has preference for commitment at $\beta x+(1-\beta)\{p\}$ for all $0 < \beta < \alpha$

II If $\{p\} \succ x$, then $\succeq$ has preference for commitment at $\beta x+(1-\beta)\{p\}$ for all $\beta > \alpha$.

An agent modeled after this axiom would have mixture independent preference for commitment when mixing with inferior menus that offer commitment. Its interpretation is analogous to the interpretation of the previous axiom.

Part II of this axiom is inconsistent with the rankings in our motivating example. In particular, an agent that satisfies II that has preference for commitment at $\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}$ would have preference for commitment at $\{c, f\}$. However, it can be motivated by similar examples. To illustrate, consider again the traveler but assume now that she wishes to stay at the fancy room but thinks that if she waits to choose the room until she arrives at the hotel, then she will be tempted to choose the conventional room to save money. Hence,

$$\{f\} \succ \{c, f\}.$$

Assume she receives the same promotion as in the motivating example: her name will be added to a raffle in which the prize is a free night in the fancy room. The possibility of staying for free in the fancy room makes her dream about it for the rest of the day. Hence, if she waits until she arrives to choose the room, by then she will not find the cheap room tempting. Hence,

$$\{\alpha f + (1-\alpha)f'\} \sim \{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}.$$

This type of behavior is permitted by Mixture-Decreasing Preference for Commitment. The reason is that this axiom is silent about behavior that involves a mixture between a menu in which the agent has preference for commitment ($\{c, f\}$) and a superior menu that offers commitment ($\{f'\}$). However, it is inconsistent with Part I of Mixture-Increasing Preference for Commitment. In particular, an agent that satisfies Part I of that axiom that has preference for commitment at $\{c, f\}$ would have preference for commitment at $\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}$.

Theorem 4.2. *Assume $\succeq$ satisfies the axioms of Proposition 3.1 and $\{p^*\} \sim \{p^*, p_*\}$. Then $\succeq$ satisfies Mixture-Decreasing Preference for Commitment if and only if there exists (u, v) that represents $\succeq$ such that $0 < \gamma' < \gamma < 1$ implies that there exists $b_{\gamma,\gamma'} \in \mathbb{R}$ such that $v(., \gamma) + b_{\gamma,\gamma'}$ is a convex combination of u and $v(., \gamma')$.*

5 Betweenness Mixture Space Theorem

In the introduction we briefly discussed that one of the contributions of the paper is to provide a novel extension of the main result of Herstein and Milnor (1953), commonly known as the Mixture Space Theorem. Here we provide our extension.

A mixture space is a non-empty set $\mathcal{M}$ which is endowed with an operation π,

$$\begin{aligned} \pi : [0,1] \times \mathcal{M} \times \mathcal{M} &\to \mathcal{M} \\ (a, x, y) &\to \pi_a(x, y), \end{aligned}$$

where π satisfies the following three properties:

(A1) $\pi_1(x, y) = x$.

(A2) $\pi_a(x, y) = \pi_{1-a}(y, x)$ for all $a \in [0, 1]$.

(A3) $\pi_a(\pi_b(x, y), y) = \pi_{ab}(x, y)$ for all $a, b \in [0, 1]$.

Basically, a mixture space is an abstract version of a convex set.

Let $\succeq$ be a binary relation over a mixture space $(\mathcal{M}, \pi)$. Consider the following axioms:

Weak Order $\succeq$ is complete and transitive.

Continuity $x \succ y \succ z$ implies that there exist $a, b \in (0, 1)$ such that $\pi_a(x, z) \succ y \succ \pi_b(x, z)$.

Independence $x \succeq y$ implies $\pi_a(x, z) \succeq \pi_a(y, z)$ for all $z \in \mathcal{M}$ and $a \in [0, 1]$.

Theorem 5.1. *(Mixture Space Theorem) Let $\succeq$ be a binary relation on a mixture space $(\mathcal{M}, \pi)$. Then the following two statements are equivalent:*

a) $\succeq$ satisfies Weak Order, Continuity and Independence.

b) There exists a utility function $\Phi : \mathcal{M} \to \mathbb{R}$ that represents $\succeq$ such that

$$\forall x, y \in \mathcal{M} \text{ and } a \in [0, 1],\ \Phi(\pi_a(x, y)) = a\Phi(x) + (1 - a)\Phi(y). \quad (4)$$

Further, Φ is unique up to a positive affine transformation.

Say that a function $\Phi : \mathcal{M} \to \mathbb{R}$ is *mixture-linear* if it satisfies (4). The Mixture Space Theorem provides necessary and sufficient conditions for a preference to have a mixture-linear utility representation. Our theorem characterizes a representation where Φ instead conforms to an analogue of the Chew-Dekel utility representation. However, it is less general than the Mixture Space Theorem in the sense that it only applies to mixture spaces that satisfy the following additional property:

(A4) $\pi_a(\pi_b(x, y), z) = \pi_{ab}(x, \pi_{\frac{a(1-b)}{1-ab}}(y, z))$ $\forall a, b \in [0, 1]$ such that $ab \neq 1$.

A4 requires that the order of the mixture does not matter. Hence, it is an associative property. Thus we refer to any mixture space that satisfies A4 as an associative mixture space. Not all mixture spaces are associative (see Mongin (2001) for an example). However, any space that is isomorphic to a convex subset of a linear space is an associative mixture space (examples include the frameworks employed in Anscombe and Aumann (1963), Dekel et al. (2001) and the probability simplex).[5] Not all associative mixture spaces have this feature. In particular, Stone (1949) and Mongin (2001) show that the missing ingredient is the following property.

(A5) For any $a \in (0,1)$ and $x \in \mathcal{M}$, $y \mapsto \pi_a(x,y)$ is injective.

Theorem 5.2. *(Stone-Mongin) Let $(\mathcal{M}, \pi)$ be an associative mixture space. Then the following two statements are equivalent:*

a) $(\mathcal{M}, \pi)$ satisfies A5.

b) $(\mathcal{M}, \pi)$ is isomorphic to a convex subset of a linear space.

Thus, even though our result is not as general as the Mixture Space Theorem, it applies to settings that are more general than convex subsets of linear spaces.

In our version of the Mixture Space Theorem, we replace the Independence axiom with the following two axioms.

Mixture-Betweenness

$x \succ y$ implies $x \succ \pi_a(x,y) \succ y$ for all $a \in (0,1)$, and
$x \sim y$ implies $x \sim \pi_a(x,y) \sim y$ for all $a \in (0,1)$.

Strict Best and Worst There exist $\overline{x}, \underline{x}$ such that $\overline{x} \succ x \succ \underline{x}$ for all $x \in \mathcal{M} \backslash \{\overline{x}, \underline{x}\}$.

To state our result we require some additional terminology. Say that a function $V : \mathcal{M} \to \mathbb{R}$ is *mixture-continuous* if for all $x, y \in \mathcal{M}$, $V(\pi_a(x,y))$ is continuous as a function of a.

[5] To the best of our knowledge, every mixture space employed in the Decision Theory literature is associative.

Theorem 5.3. *Let $\succeq$ be a binary relation on an associative mixture space $(\mathcal{M}, \pi)$. Then the following two statements are equivalent:*

a) $\succeq$ satisfies Weak Order, Continuity, Mixture-Betweenness and Strict Best and Worst.

b) There exists $\Phi : \mathcal{M} \times (0,1) \to \mathbb{R}$ such that

1.- Φ is continuous in its second argument on the interval $(0,1)$.

2.- Φ is mixture linear in its first argument for all $\gamma \in (0,1)$.

3.- $\Phi(\overline{x}, \gamma) = 1, \Phi(\underline{x}, \gamma) = 0$ for all $\gamma \in (0,1)$.

4.- $\Phi(x, \gamma) = \gamma$ has a unique solution for all $x \in \mathcal{M} \backslash \{\overline{x}, \underline{x}\}$.

5.- $\succeq$ can be represented by a mixture-continuous function $V : \mathcal{M} \to [0,1]$ such that

$$V(x) = \begin{cases} 1 & x = \overline{x}, \\ \Phi(x, V(x)) & x \in \mathcal{X} \backslash \{\overline{x}, \underline{x}\}, \\ 0 & x = \underline{x}. \end{cases}$$

Further, $\Phi(., \gamma)$ is unique for all $\gamma \in (0,1)$.

Mixture-Betweenness is obviously weaker than Independence. However, Strict Best and Worst is unrelated to Independence and limits the generality of our theorem. In the supplemental appendix (Payro (2020)) we show that it can be deleted if the associative mixture space is a compact and convex subset of a linear space and Continuity is replaced with the following axiom.

Strong Continuity The sets $\{y | y \succeq x\}$ and $\{y | x \succeq y\}$ are closed for all $x \in \mathcal{M}$.

The proof of Theorem 5.3 is similar *in spirit* to the proofs in Chew (1989), Dekel (1986) and Conlon (1995). However, they exploit properties of the probability simplex that have no evident analog in our framework. More specifically, Chew (1989) and Dekel (1986) use its geometric properties and Conlon (1995) uses its topological properties. Hence, to prove the theorem we were forced to develop novel arguments.

Due to its generality, Theorem 5.3 has several potential applications. In particular, it can be used to derive a natural counterpart of Dekel (1986) for the Anscombe-Aumann domain (Anscombe and Aumann (1963)). In this setting with uncertainty as opposed to risk, the result is a non-standard model of beliefs. More specifically, let Ω be a finite state space. Consider a preference $\succeq$ over $\mathcal{F} = \{f : \Omega \to \Delta(X)\}$, set of all AA acts. In ongoing work, we use Theorem 5.3 as a stepping stone in the axiomatization of a representation for $\succeq$ in which the utility of an AA act f is the unique γ that solves

$$\gamma = \int_{\Omega} u(f(\omega))\mu(d\omega, \gamma), \tag{5}$$

where u is a vNM utility function over lotteries and $\mu(., \gamma)$ is a probability measure over Ω for all γ.

In a separate project, we use our result to generalize Dekel et al. (2001) and therefore also Kreps (1979). In particular, we derive a counterpart of (5) in which the state space is subjective. To be precise, we axiomatize a utility representation for preferences over menus of lotteries in which the utility of a menu x is the unique γ that solves

$$\gamma = \int_{S} \sup_{p \in x} u(p, s)\mu(ds, \gamma),$$

where S is a non-empty set (subjective state space), $u(., s)$ is a vNM utility function over lotteries for all $s \in S$ and $\mu(., \gamma)$ is a probability measure over S for all γ.

Appendix A: Mixture Space Theorem

Preliminaries

Lemma 5.1. *Let $\succeq$ be a binary relation over a Mixture Space $(\mathcal{M}, \pi)$ that satisfies Weak Order, Continuity and Mixture-Betweenness. Then:*

1. *$x \succ y$ and $0 \leq a < b \leq 1$ implies $\pi_b(x, y) \succ \pi_a(x, y)$.*

2. *For all x, y and z the following sets are closed:*

$$\{a | \pi_a(x, y) \succeq z\} \text{ and } \{a | z \succeq \pi_a(x, y)\}.$$

3. *If $x \succ z \succ y$, then there exists a unique $a \in (0, 1)$ such that $z \sim \pi_a(x, y)$.*

The proof of this lemma is provided in Appendix C.

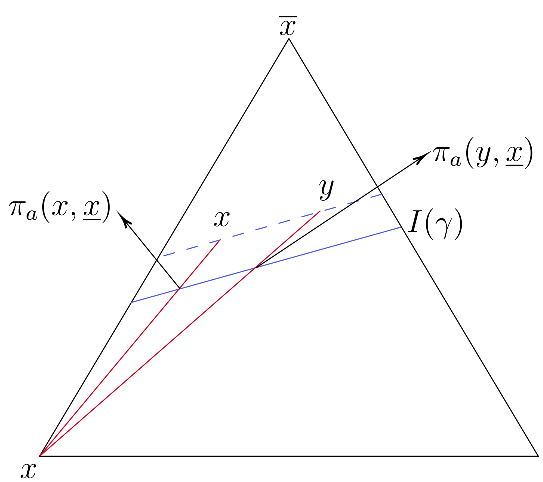
Sufficiency

Lemma 5.1 shows that for each $x \in \mathcal{M}$ there exists a unique $\gamma(x) \in [0,1]$ such that $x \sim \pi_{\gamma(x)}(\overline{x}, \underline{x})$. Define

$$V(x) \equiv \gamma(x).$$

Then, V is mixture-continuous and represents $\succeq$. Now we proceed with the construction of Φ.

For each $\gamma \in (0,1)$, let x_γ denote $\pi_\gamma(\bar{x}, \underline{x})$ and $I(\gamma) = \{x | x \sim x_\gamma\}$. Our goal is to construct a mixture-linear $\Phi(., \gamma)$ such that it represents an artificial preference that has indifference curves that are "parallel" to $I(\gamma)$. Informally, x, y are in a "higher" artificial indifference curve parallel to $I(\gamma)$ if $\pi_a(x, \underline{x}) \sim x_\gamma$ and $\pi_a(y, \underline{x}) \sim x_\gamma$. The following figure illustrates this idea.



Similarly, x, y are in a "lower" artificial indifference curve parallel to $I(\gamma)$ if $\pi_a(x, \overline{x}) \sim x_\gamma$ and $\pi_a(y, \overline{x}) \sim x_\gamma$.

Consider the mapping $\lambda : \mathcal{M} \times (0,1) \to [0,1]$ given by

$$\lambda(x, \gamma) = \begin{cases} a | \pi_a(x, \underline{x}) \sim x_\gamma & V(x) > \gamma \\ 1 & V(x) = \gamma \\ b | \pi_b(x, \bar{x}) \sim x_\gamma & V(x) < \gamma. \end{cases}$$

By Lemma 5.1, λ is well defined. λ can be used to define an artificial preference that has indifference curves parallel to $I(\gamma)$: if either $x, y \succ x_\gamma$ or $x_\gamma \succ x, y$, then x, y are in the same artificial indifference curve if and only if $\lambda(x,\gamma) = \lambda(y,\gamma)$. $\Phi(.,\gamma)$ would represent such artificial preference. Hence, mixture linearity of $\Phi(.,\gamma)$ and the definition of λ would imply that if $x \succ x_\gamma$,

$$\begin{aligned} \Phi(\pi_{\lambda(x,\gamma)}(x,\underline{x}),\gamma) &= \Phi(x_\gamma,\gamma) \\ \lambda(x,\gamma)\Phi(x,\gamma) &= \gamma \\ \frac{\gamma}{\lambda(x,\gamma)} &= \Phi(x,\gamma). \end{aligned}$$

A similar argument shows that if $x_\gamma \succ x$, then

$$\Phi(x,\gamma) = 1 - \frac{1-\gamma}{\lambda(x,\gamma)}.$$

Hence, $\Phi(.,\gamma)$ most have the following form:

$$\Phi(x,\gamma) \equiv \begin{cases} \frac{\gamma}{\lambda(x,\gamma)} & V(x) > \gamma \\ \gamma & V(x) = \gamma \\ 1 - \frac{1-\gamma}{\lambda(x,\gamma)} & V(x) < \gamma. \end{cases}$$

Notice that $\Phi(x,\gamma) = \gamma$ if and only if $V(x) = \gamma$. Further,

$$\begin{aligned} V(x), V(y) \geq \gamma &\implies \Phi(x,\gamma) \geq \Phi(y,\gamma) \iff \lambda(x,\gamma) \leq \lambda(y,\gamma) \\ V(x) \geq \gamma \geq V(y) &\implies \Phi(x,\gamma) \geq \Phi(y,\gamma) \\ V(x), V(y) \leq \gamma &\implies \Phi(x,\gamma) \geq \Phi(y,\gamma) \iff \lambda(x,\gamma) \geq \lambda(y,\gamma). \end{aligned}$$

We will show that $\Phi(.,\gamma)$ is mixture linear and continuous in its second argument. In our proof we employ the following lemma that describes several properties of λ.

Lemma 5.2. *If $\succeq$ satisfies the axioms of Theorem 5.3, then λ satisfies the following properties:*

1. $V(x), V(y) \geq \gamma \implies \lambda(\pi_a(x,y),\gamma) = \frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$ *for all* $a \in [0,1]$.

2. $V(x), V(y) \leq \gamma \implies \lambda(\pi_a(x,y),\gamma) = \frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$ *for all* $a \in [0,1]$.

3. $V(x), V(y) > \gamma > V(z)$ *and* $\lambda(x,\gamma) < \lambda(y,\gamma)$ $\implies \lambda(\pi_a(x,z),\gamma) = \lambda(y,\gamma)$ *for a unique* $a \in (0,1)$.

4. $V(x), V(y) < \gamma < V(z)$ *and* $\lambda(x,\gamma) < \lambda(y,\gamma)$ $\implies \lambda(\pi_a(x,z),\gamma) = \lambda(y,\gamma)$ *for a unique* $a \in (0,1)$.

5. $V(x), V(y) > \gamma > V(z)$, $\lambda(x,\gamma) = \lambda(y,\gamma)$ *and* $\pi_a(x,z) \sim x_\gamma$ $\implies \pi_a(y,z) \sim x_\gamma$.

6. $V(x), V(y) < \gamma < V(z)$, $\lambda(x,\gamma) = \lambda(y,\gamma)$ *and* $\pi_a(x,z) \sim x_\gamma$ $\implies \pi_a(y,z) \sim x_\gamma$.

The proof is provided in Appendix C. Here we provide two figures that illustrate properties 3 (Figure 1) and 5 (Figure 2).

Notice that properties 1 and 2 imply that if x, y are both above or both below $I(\gamma)$ in terms of preferences, then

$$\lambda(\pi_a(x,y),\gamma) = (a\lambda(x,\gamma)^{-1} + (1-a)\lambda(y,\gamma)^{-1})^{-1}.$$

Hence, $\lambda(\pi_a(x,y),\gamma)$ is the harmonic weighted mean of $\lambda(x,\gamma)$ and $\lambda(y,\gamma)$.

Figure 1

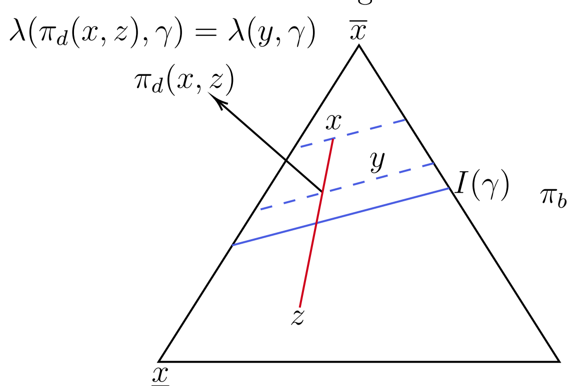
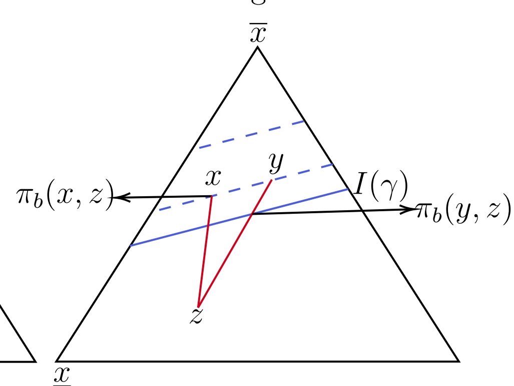


Figure 2



$\Phi(.,\gamma)$ **is Mixture Linear**

Fix $x, y \in \mathcal{X}, \gamma \in (0,1)$ and $a \in (0,1)$. Assume WLOG that $V(x) \geq V(y)$. Then, by Mixture-Betweenness, $V(x) \geq V(\pi_a(x,y)) \geq V(y)$. We will show that

$$\Phi(\pi_a(x,y),\gamma) = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma).$$

There are four possible cases to consider:
(i) $V(x) \geq V(y) \geq \gamma$. *(ii)* $\gamma \geq V(x) \geq V(y)$.
(iii) $V(x) \geq V(\pi_a(x,y)) \geq \gamma \geq V(y)$. *(iv)* $V(x) \geq \gamma \geq V(\pi_a(x,y)) \geq V(y)$.
Since the proofs of *(i)* and *(ii)* are analogous, and the proofs of *(iii)* and *(iv)* are analogous we only consider *(i)* (Lemma 5.3) and *(iii)* (Lemma 5.4).

Lemma 5.3. $V(x), V(y) \geq \gamma \implies \Phi(\pi_a(x,y),\gamma) = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma)$.

Proof. First note that if $V(x) = V(y) = \gamma$, then there is nothing to prove. Assume $V(x) > V(y) \geq \gamma$. Then, by Mixture-Betweenness, $V(\pi_a(x,y)) > \gamma$. Thus,

$$\begin{aligned}
\Phi(\pi_a(x,y),\gamma) &= \frac{\gamma}{\lambda(\pi_a(x,y),\gamma)} \\
&= \frac{\gamma}{\frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}} \\
&= \frac{\gamma}{\lambda(x,\gamma)\lambda(y,\gamma)}(a\lambda(y,\gamma) + (1-a)\lambda(x,\gamma)) \\
&= a\frac{\gamma}{\lambda(x,\gamma)} + (1-a)\frac{\gamma}{\lambda(y,\gamma)} \\
&= a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma).
\end{aligned}$$

where the second equality follows from Part 1 of Lemma 5.2. □

Lemma 5.4. *$V(x) > \gamma > V(y)$ and $V(\pi_a(x,y)) \geq \gamma \implies$ $\Phi(\pi_a(x,y),\gamma) = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma)$.*

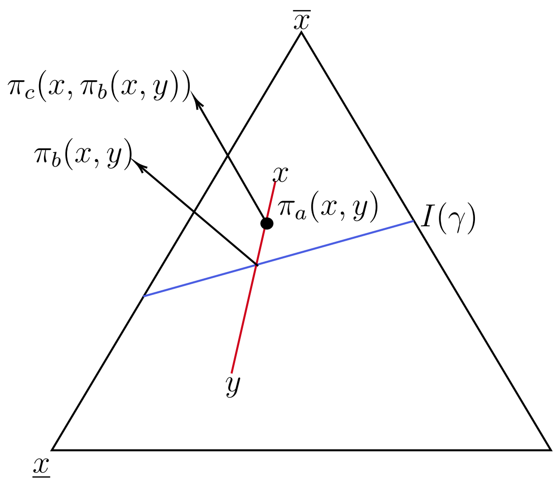
Proof. Let $b \in (0,1)$ be such that

$$\pi_b(x,y) \sim x_\gamma.$$

The proof of this lemma is done in two steps.

Step 1: Calculate $\Phi(\pi_a(x,y),\gamma)$

The following figure illustrates the argument for this step.



$\pi_a(x,y) \succeq x_\gamma$ implies $a \geq b$. Let $c = \frac{a-b}{1-b} \leq 1$. We claim that $\pi_a(x,y) = \pi_c(x,\pi_b(x,y))$. To prove this claim note that

$$\begin{aligned}
\pi_c(x,\pi_b(x,y)) &= \pi_{1-c}(\pi_{1-b}(y,x),x) \\
&= \pi_{(1-c)(1-b)}(y,x) \\
&= \pi_{1-(1-c)(1-b)}(x,y).
\end{aligned}$$

Further,

$$\begin{aligned}
1-(1-c)(1-b) &= 1-(1-\frac{a-b}{1-b})(1-b) \\
&= 1-(1-b-(a-b)) \\
&= a.
\end{aligned}$$

Hence, $\pi_a(x,y) = \pi_c(x,\pi_b(x,y))$. Since $V(x), V(\pi_b(x,y)) \geq \gamma$, by Part 1 of Lemma 5.2,

$$\begin{aligned}
\lambda(\pi_c(x,\pi_b(x,y)),\gamma) &= \frac{\lambda(x,\gamma)}{c+(1-c)\lambda(x,\gamma)} \\
&= \frac{\lambda(x,\gamma)}{\frac{a-b}{1-b}+(1-\frac{a-b}{1-b})\lambda(x,\gamma)} \\
&= (1-b)\frac{\lambda(x,\gamma)}{(a-b)+(1-a)\lambda(x,\gamma)}.
\end{aligned}$$

Hence,

$$\begin{aligned}\Phi(\pi_a(x,y),\gamma) &= \Phi(\pi_c(x,\pi_b(x,y)),\gamma) \\ &= \gamma\frac{1}{\lambda(\pi_c(x,\pi_b(x,y)),\gamma)} \\ &= \gamma\frac{1}{(1-b)\frac{\lambda(x,\gamma)}{(a-b)+(1-a)\lambda(x,\gamma)}} \\ &= \gamma\frac{a-b+(1-a)\lambda(x,\gamma)}{(1-b)\lambda(x,\gamma)}.\end{aligned}$$

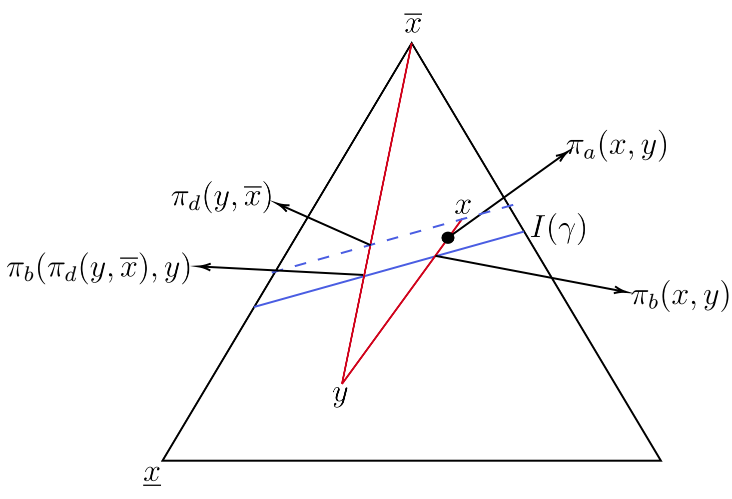
Step 2: Calculate $a\Phi(x,\gamma)+(1-a)\Phi(y,\gamma)$ and show it is equal to $\Phi(\pi_a(x,y),\gamma)$

To calculate $a\Phi(x,\gamma)+(1-a)\Phi(y,\gamma)$ we first need to derive the relation between $\lambda(y,\gamma),\lambda(x,\gamma),\gamma$ and b. In particular, we need to show that

$$\lambda(y,\gamma)=\frac{(1-b)\lambda(x,\gamma)(\gamma-1)}{\lambda(x,\gamma)(\gamma-1)-b(\gamma-\lambda(x,\gamma))}. \tag{6}$$

To prove (6) we need to distinguish between two cases: (i) $\lambda(x,\gamma)>\lambda(\overline{x},\gamma)$ and (ii) $\lambda(x,\gamma)\leq\lambda(\overline{x},\gamma)$. The proof of both cases is analogous. Thus, we only consider (i).

Assume $\lambda(x,\gamma)>\lambda(\overline{x},\gamma)$. By Part 3 of Lemma 5.2 there exists a unique $d\in(0,1)$ such that $\lambda(\pi_d(y,\overline{x}),\gamma)=\lambda(x,\gamma)$. First we show that $\lambda(y,\gamma)=1-b(1-d)$. The following figure illustrates why this is true.



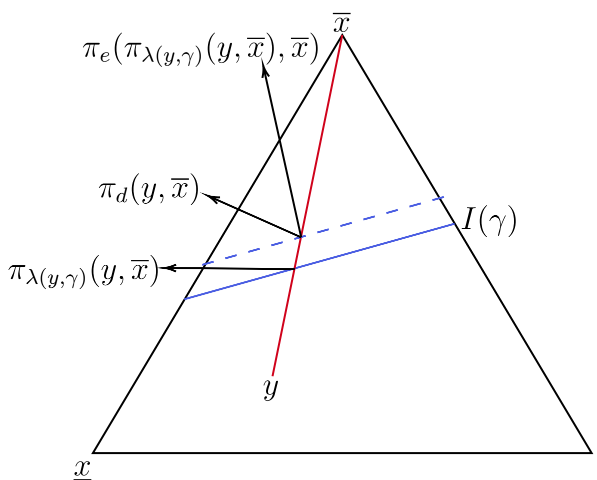
By Part 5 of Lemma 5.2, $\pi_b(\pi_d(y,\overline{x}),y) \sim x_\gamma$. Hence $\pi_{\lambda(y,\gamma)}(y,\overline{x}) = \pi_b(\pi_d(y,\overline{x}),y)$. Further,

$$\begin{aligned}\pi_b(\pi_d(y,\overline{x}),y) &= \pi_b(\pi_{1-d}(\overline{x},y),y)\\ &= \pi_{b(1-d)}(\overline{x},y)\\ &= \pi_{1-b(1-d)}(y,\overline{x}).\end{aligned}$$

Thus,

$$\lambda(y,\gamma) = 1 - b(1-d).$$

Next, we need to replace d in $\lambda(y,\gamma) = 1 - b(1-d)$ with $e\lambda(y,\gamma)$ where $e = \frac{\gamma-\lambda(x,\gamma)}{\lambda(x,\gamma)(\gamma-1)}$. The following figure illustrates how we do this.



First we show that $e \in (0,1)$ and $\pi_d(y,\overline{x}) = \pi_e(\pi_{\lambda(y,\gamma)}(y,\overline{x}),\overline{x})$. To prove this, first note that $\gamma = \lambda(\overline{x},\gamma)$ so $\lambda(x,\gamma) > \lambda(\overline{x},\gamma)$ and $\gamma \in (0,1)$ imply $e > 0$. Further, $\lambda(x,\gamma) \in (0,1)$. Thus,

$$\begin{aligned}\gamma &> \lambda(x,\gamma)\gamma\\ \gamma - \lambda(x,\gamma) &> \lambda(x,\gamma)\gamma - \lambda(x,\gamma)\\ \frac{\gamma-\lambda(x,\gamma)}{\lambda(x,\gamma)(\gamma-1)} &< 1.\end{aligned}$$

Next, note that

$$\pi_e(\pi_{\lambda(y,\gamma)}(y,\overline{x}),\overline{x}) = \pi_{e\lambda(y,\gamma)}(y,\overline{x}).$$

Thus, by Part 1 of Lemma 5.2,

$$\lambda(\pi_e(\pi_{\lambda(y,\gamma)}(y,\overline{x}),\overline{x}),\gamma) = \frac{\lambda(\overline{x})}{e\lambda(\overline{x},\gamma)+(1-e)}.$$

Notice that if $e = \frac{\gamma-\lambda(x,\gamma)}{\lambda(x,\gamma)(\gamma-1)}$, then

$$\begin{aligned}\pi_e(\pi_{\lambda(y,\gamma)}(y,\overline{x}),\overline{x}) &= \frac{\lambda(\overline{x},\gamma)}{(\frac{\gamma-\lambda(x,\gamma)}{\lambda(x,\gamma)(\gamma-1)})\lambda(\overline{x},\gamma)+\frac{\gamma(\lambda(x,\gamma)-1)}{\lambda(x,\gamma)(\gamma-1)}}\\ &= \frac{\gamma\lambda(x,\gamma)(\gamma-1)}{\gamma(\gamma-\lambda(x,\gamma))+\gamma(\lambda(x,\gamma)-1)}\\ &= \frac{\lambda(x,\gamma)(\gamma-1)}{(\gamma-\lambda(x,\gamma))+(\lambda(x,\gamma)-1)}\\ &= \lambda(x,\gamma).\end{aligned}$$

Hence by the uniqueness in Part 3 of Lemma 5.2, $d = e\lambda(y,\gamma)$. Thus,

$$\begin{aligned}\lambda(y,\gamma) &= 1-b(1-d)\\ &= 1-b(1-e\lambda(y,\gamma))\\ &= \frac{1-b}{1-be}\\ &= \frac{(1-b)\lambda(x,\gamma)(\gamma-1)}{\lambda(x,\gamma)(\gamma-1)-b(\gamma-\lambda(x,\gamma))}.\end{aligned}$$

Hence, (6) holds.

Finally, we finish the proof of this step by showing $a\Phi(x,\gamma)+(1-a)\Phi(y,\gamma) = \Phi(\pi_a(x,y))$.

$$\begin{aligned}a\Phi(x,\gamma)+(1-a)\Phi(y,\gamma) &= a\frac{\gamma}{\lambda(x,\gamma)}+(1-a)(1-\frac{1-\gamma}{\lambda(y,\gamma)})\\ &= a\frac{\gamma}{\lambda(x,\gamma)}\\ &\quad+(1-a)(1-(1-\gamma)\frac{\lambda(x,\gamma)(\gamma-1)-b(\gamma-\lambda(x,\gamma))}{(1-b)\lambda(x,\gamma)(\gamma-1)})\\ &= a\frac{\gamma}{\lambda(x,\gamma)}+(1-a)(1-(1-\gamma)\frac{b(\gamma-\lambda(x,\gamma))-\lambda(x,\gamma)(\gamma-1)}{(1-b)\lambda(x,\gamma)(1-\gamma)})\end{aligned}$$

$$= a\frac{\gamma}{\lambda(x,\gamma)}$$
$$+ (1-a)(\frac{(1-b)\lambda(x,\gamma) - (b(\gamma - \lambda(x,\gamma)) - \lambda(x,\gamma)(\gamma - 1))}{(1-b)\lambda(x,\gamma)})$$
$$= a\frac{\gamma}{\lambda(x,\gamma)} + \frac{1-a}{(1-b)\lambda(x,\gamma)}(\lambda(x,\gamma)\gamma - b\gamma)$$

$$= \frac{\gamma}{\lambda(x)(1-b)}(a - ab + \lambda(x,\gamma) - b - a\lambda(x,\gamma) + ab)$$
$$= \gamma\frac{(a-b) + \lambda(x,\gamma)(1-a)}{\lambda(x,\gamma)(1-b)}$$
$$= \Phi(\pi_a(x,y),\gamma).$$

□

Φ is continuous in its second argument

To show that Φ is continuous in its second argument it is enough to show that λ is continuous in its second argument.

Lemma 5.5. *λ is continuous in its second argument.*

Proof. Fix $x \in \mathcal{M}$, $\gamma \in (0,1)$ and γ_n such that $\gamma_n \to \gamma$. There are three possible cases: (i) $V(x) > \gamma$, (ii) $V(x) < \gamma$ and (iii) $V(x) = \gamma$. The proof of the three cases is analogous. Hence, we only consider (i).

Assume $V(x) < \gamma$. By standard arguments it is without loss to assume $\gamma_n > V(x)$ for all n. Let $\lambda_n = \lambda(x,\gamma_n)$ and assume towards a contradiction that $\lambda_n \not\to \lambda(x,\gamma)$. Then, there exists a neighborhood $B(\lambda(x,\gamma))$ such that $\lambda_n \notin B(\lambda(x,\gamma))$. Let λ_m denote the corresponding subsequence. Since λ_m is a subsequence in $[0,1]$, there exists a convergent subsequence λ_l with limit $\lambda^* \neq \lambda(x,\gamma)$. Then, either $\pi_{\lambda^*}(x,\underline{x}) \succ \pi_{\lambda(x,\gamma)}(x,\underline{x})$ or $\pi_{\lambda(x,\gamma)}(x,\underline{x}) \succ \pi_{\lambda^*}(x,\underline{x})$. The proof that either case leads to a contradiction is analogous. Hence, we only consider the case in which $\pi_{\lambda^*}(x,\underline{x}) \succ \pi_{\lambda(x,\gamma)}(x,\underline{x})$.

By Continuity there exists z such that $\pi_{\lambda^*}(x,\underline{x}) \succ z \succ \pi_{\lambda(x,\gamma)}(x,\underline{x})$. Further, by Lemma 5.1, there exists N such that $l > N$ implies $\pi_{\lambda_l}(x,\underline{x}) \succ z$. Since $\pi_{\lambda_l}(x,\underline{x}) \sim \pi_{\gamma_l}(\overline{x},\underline{x})$ for all l and $\pi_{\lambda(x,\gamma)}(x,\underline{x}) \sim x_\gamma$, then by Lemma 5.1, $\pi_{\gamma_l}(\overline{x},\underline{x}) \succ z$ for all $l > N$ and $\gamma_l \to \gamma$ implies $\pi_\gamma(\overline{x},\underline{x}) \succeq z$ a contradiction. □

Necessity: Mixture-Betweenness

Say that Φ *represents* $\succeq$ if it satisfies the conditions of Theorem 5.3.

Preliminaries

Lemma 5.6. *Assume Φ represents $\succeq$. Then for any x such that $V(x) \in (0,1)$, the following two properties hold:*

1. *$\Phi(x,\gamma) < \gamma$ implies $V(x) < \gamma$.*
2. *$\Phi(x,\gamma) > \gamma$ implies $V(x) > \gamma$.*

Proof. Assume towards a contradiction that there exists γ and x such that $V(x) \geq \gamma$ and $\Phi(x,\gamma) < \gamma$. If $V(x) = \gamma$, then by the unique solution property $\Phi(x,\gamma) = \gamma$, a contradiction. Thus, $V(x) > \gamma$. Since $\Phi(x,\gamma) < \gamma < 1$, there exists $a \in (0,1)$ such that $a\Phi(x,\gamma) + (1-a) = \gamma$. Hence, $V(\pi_a(x,\overline{x})) = \gamma < V(x) < 1$. Since V is mixture-continuous, there exists $b \in (0,1)$ such that $V(\pi_b(x,\overline{x})) = V(x)$, Hence, $V(x) = b\Phi(x,V(x)) + (1-b) = bV(x) + (1-b)$, a contradiction.

Property 2 Follows from an analogous argument, the only difference is that one needs to use $\underline{x}$ instead of $\overline{x}$ to derive a contradiction. □

Next, we complete the proof. Showing that $V(x) = V(y)$ implies $V(x) = V(\pi_a(x,y)) = V(y)$ for all $a \in (0,1)$ is straight forward. Assume $V(x) > V(y)$ and let $\gamma = V(\pi_a(x,y))$. Then

$$\gamma = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma).$$

Notice that if $\gamma = V(x)$ or $\gamma = V(y)$, then the unique solution property implies a contradiction of $V(x) > V(y)$. Further, if $\gamma > V(x)$, then by Lemma 5.6, $\gamma > \Phi(x,\gamma), \Phi(y,\gamma)$. Thus, $\gamma = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma)$ does not have a solution. Similarly, if $\gamma < V(y)$, then by Lemma 5.6 $\gamma < \Phi(x,\gamma), \Phi(y,\gamma)$ and $\gamma = a\Phi(x,\gamma) + (1-a)\Phi(y,\gamma)$ does not have a solution. Hence, $\gamma \in (V(y), V(x))$.

Uniqueness

Preliminaries

The proof will employ the following lemma which states that if two preferences that satisfy Weak Order, Continuity and Independence share an indifference curve, then they are equal.

Lemma 5.7. *Let $\succeq$ and $\succeq'$ be two binary relations on a Mixture Space $(\mathcal{M}, \pi)$ that satisfy Weak Order, Continuity and Independence. Assume*

1. $\{x|x \sim \pi_a(\overline{x}, \underline{x})\} = \{x|x \sim' \pi_a(\overline{x}, \underline{x})\}$ *for some* $a \in (0, 1)$.
2. $\overline{x} \succ \underline{x}$ *and* $\overline{x} \succ' \underline{x}$.

Then $\succeq = \succeq'$.

Proof. Assume towards a contradiction that there exist x, y such that $x \succ y$ and $y \succeq' x$. Since $\succeq$ is complete, then there are two possible cases: $y \succeq \pi_a(\overline{x}, \underline{x})$ and $\pi_a(\overline{x}, \underline{x}) \succeq y$. The proof that either case leads to a contradiction is analogous. Hence, we only consider the case in which $y \succeq \pi_a(\overline{x}, \underline{x})$.

Assume $y \succeq \pi_a(\overline{x}, \underline{x})$. Then there exists b such that $\pi_b(y, \underline{x}) \sim \pi_a(\overline{x}, \underline{x})$. Hence, $\pi_b(y, \underline{x}) \sim' \pi_a(\overline{x}, \underline{x})$. Since $x \succ y$ and $y \succeq' x$, by Independence,

$$\pi_b(x, \underline{x}) \succ \pi_b(y, \underline{x}) \sim \pi_a(\overline{x}, \underline{x}) \text{ and } \pi_a(\overline{x}, \underline{x}) \sim \pi_b(y, \underline{x}) \succeq' \pi_b(x, \underline{x})$$

Thus, there exists $c < b$ such that $\pi_c(x, \underline{x}) \sim \pi_a(\overline{x}, \underline{x})$ and $\pi_a(\overline{x}, \underline{x}) \succ' \pi_c(x, \underline{x})$ a contradiction. □

Next, we complete the proof. First we show that if Φ and Φ' represent $\succeq$, then $V = V'$. To see this, note that

$$V(\pi_\gamma(\overline{x}, \underline{x})) = \gamma$$
$$V'(\pi_\gamma(\overline{x}, \underline{x})) = \gamma$$

for all $\gamma \in (0, 1)$. Hence, $V(x)$ and $V'(x)$ are the unique γ such that $x \sim \pi_\gamma(\overline{x}, \underline{x})$. Thus, $V = V'$.

To see that $\Phi = \Phi'$ let $\succeq_\gamma$ and $\succeq_{\gamma'}$ be the preference represented by $\Phi(., \gamma)$ and $\Phi'(., \gamma)$ respectively. Then,

$$\{x|x \sim \pi_\gamma(\overline{x}, \underline{x})\} = \{x|x \sim' \pi_\gamma(\overline{x}, \underline{x})\}.$$

Thus, by Lemma 5.7, $\succeq_\gamma = \succeq'_\gamma$. By the Mixture Space Theorem, $\Phi(., \gamma)$ is the unique mixture-linear representation of $\succeq_\gamma$ in which $\Phi(\overline{x}, \gamma) = 1$ and $\Phi(\underline{x}, \gamma) = 0$. Hence, $\Phi = \Phi'$.

Appendix B

Proof of Theorem 3.1

Preliminaries

$\Delta(X)$ compact implies $\mathcal{X}$ is compact (Aliprantis and Border 1999, Theorem 3.71(3)). Let $\mathcal{K}(\Delta(X))$ denote the set of all closed and convex menus of lotteries. By the Blaschke Selection Theorem, $\mathcal{K}(\Delta(X))$ is compact.

Lemma 5.8. *$\mathcal{K}(\Delta(X))$ is an associative mixture space.*

Proof. By Lemmas S.2 and S.3 in Dekel et al. (2007), there exists a bijection from $\mathcal{K}(\Delta(X))$ to a convex subset of a linear space. Further, the bijection is mixture preserving. Hence, by Theorem 5.2, $\mathcal{K}(\Delta(X))$ is an Associative Mixture Space. □

We conclude the preliminaries by introducing some notation. Let $\overline{p}, \underline{p} \in \Delta(X)$ denote a fixed pair of lotteries such that $\{\overline{p}\} \succeq x \succeq \{\underline{p}\}$ for all $x \in \mathcal{X}$. Given our axioms, such lotteries always exist (see footnote 4). Further, for each $\gamma \in (0, 1)$, let $\{p_\gamma\}$ denote $\gamma\{\overline{p}\} + (1-\gamma)\{\underline{p}\}$.

Sufficiency

Define V and Φ as in the proof of Theorem 5.3 using $\{\overline{p}\}$ and $\{\underline{p}\}$ as the best and worst menus. Then, $\Phi(., \gamma)$ is mixture linear for all $\gamma \in (0, 1)$ and, $\Phi(\{\overline{p}\}, \gamma) = 1$ and $\Phi(\{\underline{p}\}, \gamma) = 0$ for all $\gamma \in [0, 1]$. Further, for all x such that $\{\overline{p}\} \succ x \succ \{\underline{p}\}$, $V(x) = \gamma$ is the unique solution of

$$\gamma = \Phi(x, \gamma).$$

An identical argument to the one in the proof of Theorem 1.1 in the supplemental material (Payro (2020)) shows that $\Phi(., \gamma)$ and V are continuous for all $\gamma \in (0,1)$. In what follows we will use $\Phi(.,\gamma)$ to construct the representation in Theorem 3.1 for the case in which $\gamma \in (0,1)$. Afterwards we construct the representation for the case in which $\gamma = 1$ and $\gamma = 0$.

Step 1: Extend V and $\Phi(.,\gamma)$ to $\mathcal{X}$.

For each menu $x \in \mathcal{X}$, let $ch(x)$ denote its convex hull. Extend V by letting $V(x) = V(ch(x))$ for all $x \in \mathcal{X} \backslash \mathcal{K}(\Delta(X))$. We claim that V represents $\succeq$. To prove this, it is enough to show that our axioms imply $x \sim ch(x)$ for all $x \in \mathcal{X}$.

Lemma 5.9. *Let $\succeq$ be a binary relation over $\mathcal{X}$ that satisfies Weak Order, Continuity and Mixture-Betweenness. Then $x \sim ch(x)$ for all $x \in \mathcal{X}$.*

Proof. Let $K = |X|$ and assume, by way of contradiction, that there exists x such that $x \not\sim ch(x)$. Then, by Mixture-Betweenness, $\alpha x + (1-\alpha)ch(x) \not\sim ch(x)$ for all $\alpha \in (0,1)$. This is a contradiction. In particular, Lemma S.6 in the supplemental appendix of Dekel et al. (2007) shows that $\alpha x + (1-\alpha)ch(x) = ch(x)$ for all $\alpha \in [0, \frac{1}{K}]$. □

Extend Φ by letting $\Phi(x,\gamma) = \Phi(ch(x),\gamma)$ for all $x \in \mathcal{X} \backslash \mathcal{K}(\Delta(X))$. Then,

$$\begin{aligned}\Phi(\alpha x + (1-\alpha)y, \gamma) &= \Phi(ch(\alpha x + (1-\alpha)y), \gamma) \\ &= \Phi(\alpha ch(x) + (1-\alpha)ch(y), \gamma) \\ &= \alpha \Phi(ch(x),\gamma) + (1-\alpha)\Phi(ch(y),\gamma) \\ &= \alpha \Phi(x,\gamma) + (1-\alpha)\Phi(y,\gamma)\end{aligned}$$

for all $x, y \in \mathcal{X}$ and $\alpha \in [0,1]$. Thus, the extension of Φ is also mixture linear in its first argument and continuous in its second.

Step 2: Show that $\Phi(.,\gamma)$ satisfies Set-Betwenness.

Fix $x, y \in \mathcal{X}$ and $\gamma \in (0,1)$. Assume WLOG that $\Phi(x,\gamma) \geq \Phi(y,\gamma)$. There are three possible cases:
(i) $\Phi(x,\gamma) \geq \Phi(y,\gamma) \geq \gamma$. (ii) $\Phi(x,\gamma) \geq \gamma \geq \Phi(y,\gamma)$.
(iii) $\gamma \geq \Phi(x,\gamma) \geq \Phi(y,\gamma)$.

(i) $\Phi(x,\gamma) \geq \Phi(y,\gamma) \geq \gamma$.

By Lemma 5.6, $V(x), V(y) \geq \gamma$. Thus, by Set-Betweenness, $V(x \cup y) \geq \gamma$.
By construction, $\Phi(x,\gamma) \geq \Phi(y,\gamma)$ if and only if $\lambda(x,\gamma) \leq \lambda(y,\gamma)$ so

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \sim \{p_\gamma\} \succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\}.$$

Hence, by Set-Betweenness,

$$\begin{aligned}\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} &\succeq \lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\} \\ &\succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\}.\end{aligned}$$

However, note that

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\} \sim \lambda(x,\gamma)(x \cup y) + (1-\lambda(x,\gamma))\{\underline{p}\}$$

because their convex hulls are equal. Thus,

$$\{p_\gamma\} \succeq \lambda(x,\gamma)(x \cup y) + (1-\lambda(x,\gamma))\{\underline{p}\}.$$

Hence, $\lambda(x \cup y,\gamma) \geq \lambda(x,\gamma)$ so $\Phi(x,\gamma) \geq \Phi(x \cup y,\gamma)$. To see that
$\Phi(x \cup y,\gamma) \geq \Phi(y,\gamma)$, note that

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\underline{p}\} \succeq \{p_\gamma\} \sim \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\underline{p}\}.$$

Thus, by Set-Betweenness,

$$\begin{aligned}\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\underline{p}\} &\succeq \lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\underline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\underline{p}\} \\ &\succeq \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\underline{p}\}.\end{aligned}$$

However,

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\underline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\underline{p}\} \sim \lambda(y,\gamma)(x \cup y) + (1-\lambda(y,\gamma))\{\underline{p}\}.$$

Thus, $\lambda(y,\gamma)(x \cup y) + (1-\lambda(y,\gamma))\{\underline{p}\} \succeq \{p_\gamma\}$ and $\lambda(x \cup y,\gamma) \leq \lambda(y,\gamma)$ so
$\Phi(x \cup y,\gamma) \geq \Phi(y,\gamma)$.

(ii) $\Phi(x,\gamma) \geq \gamma \geq \Phi(y,\gamma)$.

By Lemma 5.6, $V(x) \geq \gamma \geq V(y)$. Hence, by Set-Betweenness, $V(x) \geq V(x \cup y) \geq V(y)$. There are two cases: $V(x \cup y) \geq \gamma$ or $\gamma \geq V(x \cup y)$.

If $V(x \cup y) \geq \gamma$, then, by Lemma 5.6, $\Phi(x \cup y, \gamma) \geq \Phi(y,\gamma)$. Hence, we only need to show $\Phi(x,\gamma) \geq \Phi(x \cup y, \gamma)$. By Mixture-Betweenness,

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \sim \{p_\gamma\} \succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\}.$$

Thus, by Set-Betweenness,

$$\begin{aligned}\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} &\succeq \lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\} \\ &\succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\}.\end{aligned}$$

However,

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\} \sim \lambda(x,\gamma)x \cup y + (1-\lambda(x,\gamma))\{\underline{p}\}.$$

Therefore,

$$\begin{aligned}\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\underline{p}\} &\succeq \lambda(x,\gamma)x \cup y + (1-\lambda(x,\gamma))\{\underline{p}\} \\ &\succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\underline{p}\}.\end{aligned}$$

Hence, $\lambda(x \cup y, \gamma) \geq \lambda(x,\gamma)$ so $\Phi(x,\gamma) \geq \Phi(x \cup y, \gamma)$.

Next, assume $\gamma \geq V(x \cup y)$. Then, by Lemma 5.6, $\Phi(x,\gamma) \geq \Phi(x \cup y, \gamma)$. By Mixture-Betweenness,

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \succeq \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\} \sim \{p_\gamma\}.$$

Hence, by Set-Betweenness,

$$\begin{aligned}\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} &\succeq \lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\} \\ &\succeq \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\}.\end{aligned}$$

However,

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\} \sim \lambda(y,\gamma)x \cup y + (1-\lambda(y,\gamma))\{\overline{p}\}.$$

Therefore,

$$\begin{aligned}\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} &\succeq \lambda(y,\gamma)x \cup y + (1-\lambda(y,\gamma))\{\overline{p}\} \\ &\succeq \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\}.\end{aligned}$$

Hence, $\lambda(x \cup y, \gamma) \geq \lambda(y,\gamma)$ and $\Phi(x \cup y, \gamma) \geq \Phi(y,\gamma)$.

(iii) $\gamma \geq \Phi(x,\gamma) \geq \Phi(y,\gamma)$.

By Lemma 5.6, $\gamma \geq V(x), V(y)$. Thus, by Set-Betweenness, $\gamma \geq V(x \cup y)$. $\Phi(x,\gamma) \geq \Phi(y,\gamma)$ implies $\lambda(x,\gamma) \geq \lambda(y,\gamma)$ so

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \succeq \{p_\gamma\} \sim \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\}.$$

Thus, by Set-Betweenness,

$$\begin{aligned}\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} &\succeq \lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\} \\ &\succeq \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\}.\end{aligned}$$

However,

$$\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} \cup \lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\} \sim \lambda(y,\gamma)(x \cup y) + (1-\lambda(y,\gamma))\{\overline{p}\}.$$

Hence,

$$\begin{aligned}\lambda(y,\gamma)x + (1-\lambda(y,\gamma))\{\overline{p}\} &\succeq \lambda(y,\gamma)(x \cup y) + (1-\lambda(y,\gamma))\{\overline{p}\} \\ &\succeq \lambda(y,\gamma)y + (1-\lambda(,\gamma))\{\overline{p}\}.\end{aligned}$$

Therefore, $\lambda(x \cup y, \gamma) \geq \lambda(y,\gamma)$ and $\Phi(x \cup y, \gamma) \geq \Phi(y,\gamma)$. To see that $\Phi(x,\gamma) \geq \Phi(x \cup y, \gamma)$ note that

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\overline{p}\} \sim \{p_\gamma\} \succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\overline{p}\}.$$

Further, by Set-Betweenness,

$$\begin{aligned}\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\overline{p}\} &\succeq \lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\overline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\overline{p}\} \\ &\succeq \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\overline{p}\}.\end{aligned}$$

However,

$$\lambda(x,\gamma)x + (1-\lambda(x,\gamma))\{\overline{p}\} \cup \lambda(x,\gamma)y + (1-\lambda(x,\gamma))\{\overline{p}\} \sim \lambda(x,\gamma)(x \cup y) + (1-\lambda(x,\gamma))\{\overline{p}\}.$$

Hence, $\lambda(x,\gamma) \geq \lambda(x \cup y,\gamma)$ and $\Phi(x,\gamma) \geq \Phi(x \cup y,\gamma)$.

Step 3: Show that there exists $u(.,\gamma)$ and $v(.,\gamma)$ such that for all $\gamma \in (0,1)$ and $x \in \mathcal{X}$,

$$\Phi(x,\gamma) = \max_{p\in x}\{u(p,\gamma) + v(p,\gamma) - \max_{q\in x} v(q,\gamma)\}.$$

Restrict $\succeq$ to $\Delta(X)$, then by Proposition 1 in Dekel (1986) there exists a unique $u : \Delta(X) \times (0,1) \to \mathbb{R}$ such that $u(.,\gamma)$ is a vNM utility function for all $\gamma \in [0,1]$, u is continuous on its second argument on the open interval $(0,1)$, $u(\overline{p},\gamma) = 1$, $u(\underline{p},\gamma) = 0$ and $V(\{p\})$ is the unique $\gamma \in [0,1]$ that solves

$$\gamma = u(p,\gamma).$$

Hence, $\Phi(\{p\},\gamma) = u(p,\gamma)$ for all $p \in \Delta(X)$ and $\gamma \in (0,1)$.

By Lemmas 2, 4 and 5 in GP, there exists a vNM function $v(.,\gamma)$ such that

$$\Phi(x,\gamma) = \max_{p\in x}\{u(p,\gamma) + v(p,\gamma) - \max_{q\in x} v(q,\gamma)\}$$

for all $x \in \mathcal{X}$.

Step 4: Construct $v(.,1)$ and $v(.,0)$.

The construction of $v(.,0)$ and $v(.,1)$ are analogous. Thus, we only show the latter and specify how to adapt the argument for the former. Our construction of $v(.,1)$ is similar to the construction in the proof of Theorem 2 in Noor and Takeoka (2015).

If $\{\overline{p}\} \succ \{p,q\}$ for all $p,q \in \Delta(X)$ such that $q \neq \overline{p}$, let $v(.,1) = -u(.,1)$. Otherwise, let $\mathcal{P} = \{p|\{p\} \succeq \{q\}$ for all $q\}$. If $\{p,q\} \sim \{p\}$ for all $p \in \mathcal{P}$ and $q \notin \mathcal{P}$, let $u(.,1) = 0$. Otherwise, define $\succeq^T$ over $\Delta(X)$ as follows

$$p \succeq^T q \text{ if and only if } \{p\} \sim \{p,q\} \succ \{q\} \text{ and } p \in \mathcal{P}$$
$$q \succ^T p \text{ if and only if } \{p\} \succ \{p,q\} \text{ and } p \in \mathcal{P}.$$

We will show that there exists a vNM utility function $v(.,1)$ such that $p \succeq^T q$ implies $v(p,1) \geq v(q,1)$ and $q \succ^T p$ implies $v(q,1) > v(p,1)$. We claim that if such function exists, then

$$1 = \max_{p\in x}\{u(p,1) + v(p,1) - \max_{q\in x} v(q,1)\} \tag{7}$$

if and only if $V(x) = 1$.

Since $\succeq$ satisfies Set-Betweenness, it is enough to prove the claim for binary menus. Let $x = \{p, q\}$.

$V(x) = 1$: If $\{p\}, \{q\} \sim \{\overline{p}\}$, then $u(p, 1) = u(q, 1) = 1$. Thus the RHS of (7) must be equal to 1 because it satisfies Set-Betweenness. Assume WLOG $\{\overline{p}\} \sim \{p\} \succ \{q\}$. Then, $\{p, q\} \succ \{q\}$ so by construction $v(p, 1) \geq v(p, 1)$. Hence, the RHS is equal to $u(p, 1)$ which is equal to 1. Thus, (7) holds. Finally, note that if $\{\overline{p}\} \succ \{p\}, \{q\}$, then by Set-Betweenness $V(x) < 1$. Thus, we have shown that $V(x) = 1$ implies (7) holds.

$V(x) < 1$: If $\{\overline{p}\} \succ \{p\}, \{q\}$, then $u(p, 1), u(q, 1) < 1$. Thus, since the RHS of (7) satisfies Set-Betweenness, then it is strictly less than 1. Thus, (7) does not hold. Assume WLOG $\{\overline{p}\} \sim \{p\} \succ \{q\}$. Then, $\{p\} \succ \{p, q\}$ so by construction, $v(q, 1) > v(p, 1)$. Thus, the RHS of (7) is equal to either

$$u(p, 1) + v(p, 1) - v(q, 1) \text{ or } u(q, 1),$$

either way (7) is strictly less than 1 and (7) does not hold. Finally, note that if $\{p\}, \{q\} \sim \{\overline{p}\}$, then $V(x) = 1$. Thus, we have shown that (7) holds implies $V(x) = 1$.

Next, we show that there exists a vNM utility function that represents $\succeq^T$. The proof of the existence of such function is done in four steps.

Let $\mathcal{T} = cl(ch(\{\lambda(p - q) | \lambda > 0, p \succ q \text{ or } p \succeq^T q\})$. Then, $\mathcal{T}$ is the closed convex cone generated by $\{(p - q) | p \succeq^T q \text{ or } p \succ^T q\}$. Let $\overline{0}$ denote the zero vector.

Step 4.1: Show that if $\{p_t\} \sim \{p_t, q_t\}$ for $t = 1, ..., n$ and $p_t \in \mathcal{P}$ for all t, then $\{\sum_t^n \lambda_t p_t\} \sim \{\sum_t^n \lambda_t p_t, \sum_t^n \lambda_t q_t\}$ for all $\lambda \in \Delta(\{1, ..., n\})$.

The proof is by induction.

Base case: Fix $p_1, p_2 \in \mathcal{P}$ such that $\{p_1\} \sim \{p_1, q_1\}$ and $\{p_2\} \sim \{p_2, q_2\}$. By Mixture-Betweenness, $\{\lambda p_1 + (1 - \lambda) p_2\} \succeq x$ for all $x \in \mathcal{X}$. Assume towards a contradiction that $\{\lambda p_1 + (1 - \lambda) p_2\} \succ \{\lambda p_1 + (1 - \lambda) p_2, \lambda q_1 + (1 - \lambda) q_2\}$.

By Mixture-Betweenness,

$$\begin{aligned}\{p_2, q_2\} &\succ \{\lambda q_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \\ \{p_1, q_1\} &\succ \{\lambda p_1 + (1-\lambda)q_2, \lambda q_1 + (1-\lambda)q_2\}.\end{aligned}$$

Thus, by Set-Betweenness,

$$\{p_2, q_2\} \succ \{\lambda q_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)q_2\}.$$

Assume WLOG that

$$\{\lambda q_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)q_2\} \succeq \{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\}.$$

Thus, By Set-Betweenness,

$$\{p_2, q_2\} \succ \{\lambda q_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)p_2\}.$$

However,

$$\{\lambda q_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)q_2, \lambda p_1 + (1-\lambda)p_2\} = \lambda\{p_1, q_1\} + (1-\lambda)\{p_2, q_2\}$$

a contradiction.

Induction step: Suppose the result is true for n. We will now show it holds for $n+1$.

Fix $p_t \in \mathcal{P}$ such that $p_t \sim \{p_t, q_t\}$ for all $t = 1, ..., n+1$. By the induction hypothesis $\{\sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} p_t\} \sim \{\sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} p_t, \sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} q_t\}$. Hence, by the base case, $\{\sum_{t=1}^{n+1} \lambda_t p_t\} \sim \{\sum_{t=1}^{n+1} \lambda_t p_t, \sum_t \lambda_t q_t\}$

Step 4.2: Show that if $\{p_t\} \succ \{p_t, q_t\}$ $t = 1, ..., n$ and $p_t \in \mathcal{P}$ for all t, then $\{\sum_{t=1}^{n} \lambda_t p_t\} \succ \{\sum_{t=1}^{n} \lambda_t p_t, \sum_t \lambda_t q_t\}$.

The proof is by induction.

Base case: Fix $p_1, p_2 \in \mathcal{P}$ such that $\{p_1\} \succ \{p_1, q_1\}$ and $\{p_2\} \succ \{p_2, q_2\}$. By Mixture-Betweenness, $\{\lambda p_1 + (1-\lambda)p_2\} \succeq \{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\}$. Assume towards a contradiction that $\{\lambda p_1 + (1-\lambda)p_2\} \sim \{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\}$. First note that

$$\begin{aligned}\lambda \to 1 &\implies \{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \to \{p_1, q_1\} \\ \lambda \to 0 &\implies \{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \to \{p_2, q_2\}.\end{aligned}$$

Thus, there exist $\alpha > \lambda > \beta$ such that

$$\{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \succ \{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2\}$$
$$\{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \succ \{\beta p_1 + (1-\beta)p_2, \beta q_1 + (1-\beta)q_2\}$$

Let η be such that

$$\eta(\alpha p_1 + (1-\alpha)p_2) + (1-\eta)(\beta p_1 + (1-\beta)p_2) = \lambda p_1 + (1-\lambda)p_2$$

Thus, by Mixture-Betweenness,

$$\{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\} \succ$$
$$\eta\{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2\} + (1-\eta)\{\beta p_1 + (1-\beta)p_2, \beta q_1 + (1-\beta)q_2\}.$$

However,

$$\eta\{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2\} + (1-\eta)\{\beta p_1 + (1-\beta)p_2, \beta q_1 + (1-\beta)q_2\} =$$
$$\{\lambda p_1 + (1-\lambda)p_2, \eta(\alpha p_1 + (1-\alpha)p_2) + (1-\eta)(\beta q_1 + (1-\beta)q_2), \lambda q_1 + (1-\lambda)q_2,$$
$$\eta(\alpha q_1 + (1-\alpha)q_2) + (1-\eta)(\beta p_1 + (1-\beta)p_2)\}.$$

Further, for $\nu = \frac{\lambda-(1-\eta)\beta}{\lambda}$

$$\{\lambda p_1 + (1-\lambda)p_2, \eta(\alpha p_1 + (1-\alpha)p_2) + (1-\eta)(\beta q_1 + (1-\beta)q_2)\} =$$
$$\nu\{\lambda p_1 + (1-\lambda)p_2\} + (1-\nu)\{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\}$$

and

$$\{\lambda p_1 + (1-\lambda)p_2, \eta(\alpha q_1 + (1-\alpha)q_2) + (1-\eta)(\beta p_1 + (1-\beta)p_2)\} =$$
$$(1-\nu)\{\lambda p_1 + (1-\lambda)p_2\}) + \nu\{\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2\}.$$

Hence, by Set-Betweenness and Mixture-Betweenness,

$$\{\lambda p_1 + (1-\lambda)p_2, \eta(\alpha p_1 + (1-\alpha)p_2) + (1-\eta)(\beta q_1 + (1-\beta)q_2), \lambda q_1 + (1-\lambda)q_2,$$
$$\eta(\alpha q_1 + (1-\alpha)q_2) + (1-\eta)(\beta p_1 + (1-\beta)p_2)\} \sim \{p_1\}$$

a contradiction.

Induction step: Suppose the result is true for n. We will show it is true for $n+1$.

Fix $\{p_t\} \in \mathcal{P}$ such that $\{p_t\} \succ \{p_t, q_t\}$ for $t = 1, ..., n+1$. By the induction hypothesis $\{\sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} p_t\} \succ \{\sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} p_t, \sum_{t=1}^{n} \frac{\lambda_t}{\sum_t^n \lambda_t} q_t\}$. Hence, by the base case, $\{\sum_t \lambda_t p_t\} \succ \{\sum_t \lambda_t p_t, \sum_t \lambda_t q_t\}$.

Step 4.3: show that $0 \notin \mathcal{T}$.

We only show that $0 \notin int(\mathcal{T})$ because Continuity implies that if $0 \notin int(\mathcal{T})$, then $0 \notin \mathcal{T}$.

Fix $\lambda(p-q) \in \mathcal{T}$, then

$$\lambda(p-q) = \sum_{t=1}^{n} \lambda_t \alpha_t p_t + \sum_{t=n+1}^{N} \lambda_t \alpha_t q'_t - \sum_{t=1}^{n} \lambda_t \alpha_t q_t - \sum_{t=n+1}^{N} \lambda_t \alpha_t p'_t$$

where $p_t \sim \{p_t, q_t\}$ for all $t = 1, ..., n$ and $p'_t \succ \{p'_t, q'_t\}$ for all $t = 1, ..., N$. Let

$$p = \sum_{t} \frac{\lambda_t \alpha_t}{\sum_{t=1}^{N} \lambda_t \alpha_t} p_t \text{ and } p' = \sum_{t=n+1}^{N} \frac{\lambda_t \alpha_t}{\sum_{t=1}^{N} \lambda_t \alpha_t} p'_t,$$
$$q = \sum_{t=1}^{n} \frac{\lambda_t \alpha_t}{\sum_{t=1}^{N} \lambda_t \alpha_t} q_t \text{ and } q' = \sum_{t=n+1}^{N} \frac{\lambda_t \alpha_t}{\sum_{t=1}^{N} \lambda_t \alpha_t} q'_t.$$

Assume $\lambda(p-q) = 0$ and let $\lambda^* = \sum_{t=1}^{n} \frac{\lambda_t \alpha_t}{\sum_{t=1}^{N} \lambda_t \alpha_t}$. Then, $\lambda^* p + (1-\lambda^*) q' = \lambda^* q + (1-\lambda_*) p'$.

Note that by steps 4.1 and 4.2, $\{p\} \sim \{p, q\}$ and $\{p'\} \succ \{p', q'\}$. Hence, by Mixture-Betweenness,

$$\begin{aligned}
\{\lambda^* p + (1-\lambda^*) p'\} &\succ \lambda^* \{p\} + (1-\lambda^*) \{p', q'\} \\
&= \{\lambda^* p + (1-\lambda^*) p', \lambda^* p + (1-\lambda^*) q'\} \\
&= \{\lambda^* p + (1-\lambda^*) p', \lambda^* q + (1-\lambda^*) p'\} \\
&= \lambda^* \{p, q\} + (1-\lambda^*) \{p'\} \sim \{\lambda^* p + (1-\lambda^*) p'\}
\end{aligned}$$

a contradiction.

Step 4.4: Finish the proof

By step 4.3 if $\{p\} \succ \{p,q\}$, then $p - q \notin \mathcal{T}$. Thus, $\bar{0} \notin \mathcal{T}$ so by the hyperplane separation theorem there exists $v(.,1)$ such that $q \succeq^T p$ implies $v(q,1) \geq v(p,1)$. Further, since $\mathcal{T}$ is a closed cone and v is linear, $v(p-q) \geq 0$ for all $p - q \in \mathcal{T}$. Next, we will that $q \succ_T p$ implies $v(q,1) > v(p,1)$.

Suppose $\{p\} \succ \{p,q\}$ and $v(p,1) = v(q,1)$. By construction, $v(.,1)$ is not a constant. Hence, there exists $p' \in \mathcal{P}, q' \notin \mathcal{P}$ such that $v(p',1) < v(q',1)$. Thus, by Continuity and steps 4.2-4.3, for a large enough α,

$$\{\alpha p + (1-\alpha)p'\} \succ \{\alpha p + (1-\alpha)p', \alpha q + (1-\alpha)q'\}.$$

Thus, $\alpha q + (1-\alpha)q' - \alpha p + (1-\alpha)p' \in \mathcal{T}$. Hence,

$$v(\alpha q + (1-\alpha)q' - \alpha p + (1-\alpha)p', 1) < 0,$$

a contradiction.

To conclude the proof we outline the construction of $v(.,0)$. If $\{p,q\} \succ \{\underline{p}\}$ for all $p,q \in \Delta(X)\backslash\{\underline{p}\}$, let $v(r,0) = 0$ for all $r \in \Delta(X)$. Otherwise, let $\mathcal{Q} = \{q \in \Delta(X) | \{q\} \sim \{\underline{p}\}\}$. If $\{p,q\} \sim \{q\}$ for all $q \in \mathcal{Q}$ and $p \in \Delta(X)$, then let $v(r,0) = -u(r,0)$. Otherwise, define $\succeq_T$ over $\Delta(X)$ as follows:

$$\begin{aligned} &p \succeq_T q \text{ if and only if } \{p\} \succ \{p,q\} \sim \{q\} \text{ and } q \in \mathcal{Q} \\ &q \succ_T p \text{ if and only if } \{p,q\} \succ \{q\} \text{ and } q \in \mathcal{Q}. \end{aligned}$$

An identical argument to the one in the construction of $v(.,1)$ shows there exists v' such that $p \succeq_T q$ implies $v'(p) \geq v'(q)$, $p \succ_T q$ implies $v'(p) > v(q)$. Let $v(.,0) = -u(.,0) + v'$, we will now show that

$$0 = \max_{p \in x}\{u(p,0) + v(p,0) - \max_{q \in x} v(q,0)\} \tag{8}$$

if and only if $V(x) = 0$.

As in the $v(.,1)$ case, it is WLOG to only consider binary menus $x = \{p,q\}$.

$V(x) > 0$: If $\{p\},\{q\} \succ \{\underline{p}\}$, then $u(p,0), u(q,0) > 0$. Thus the RHS of (8) must be positive because it satisfies Set-Betweenness. Assume WLOG $\{p\} \succ \{q\} \sim \{\underline{p}\}$. Then, $u(p,0) > 0$ and $v(p,0) > v(q,0)$. Thus, the RHS of (8) is equal to

$$v'(p) - v'(q) \text{ or } u(p),$$

either way, (8) does not hold. Hence, (8) holds implies $V(x) = 0$.

$V(x) = 0$: Then, either $\{p\} \sim \{\underline{p}\}$ or $\{q\} \sim \{\underline{p}\}$. If both hold, then $u(p, 0) = u(q, 0)$, thus by Set-Betweenness, (8) holds. Assume WLOG $\{p\} \succ \{q\} \sim \{\underline{q}\}$, then $V(x) = 0$ implies $\{p\} \succ \{p, q\} \sim \{q\}$, thus $v'(q) \geq v'(p)$. Then, the RHS of (8) is equal to

$$v'(q) - \max\{v'(q), v'(p) - u(p, 0)\}.$$

$\{p\} \succ \{\underline{p}\}$ implies $u(p, 0) > 0$. Thus, since $v'(q) \geq v(p)$, then $\max\{v'(q), v'(p) - u(p, 0)\} = v'(q)$. Hence, $V(x) = 0$ implies (8) holds.

Necessity: Set-Betweenness

Assume (u, v) represent $\succeq$, then for any menu $z \in \mathcal{X}$, $V(z) = V(\{p_z, q_z\})$ for any $p_z, q_z \in z$ such that

$$p_z \in \arg\max_{p \in z}\{u(p, V(z)) + v(p, V(z))\}$$
$$q_z \in \arg\max_{q \in z} v(q, V(z)).$$

To see this, note that

$$\begin{aligned} V(z) &= \max_{p \in z}\{u(p, V(z)) + v(p, V(z)) - \max_{q \in z}\{v(q, V(z))\}\} \\ &= \max_{p \in \{p_z, q_z\}}\{u(p, V(z)) + v(p, V(z)) - \max_{q \in \{p_z, q_z\}}\{v(q, V(z))\}\}.\text{‘} \end{aligned}$$

Hence, by the unique solution property, $V(z) = V(\{p_z, q_z\})$.

Fix x, y such that $V(x) \geq V(y)$ and

$$p_z \in \arg\max_{p \in z}\{u(p, V(z)) + v(p, V(z))\}$$
$$q_z \in \arg\max_{q \in z} v(q, V(z))$$

for $z \in \{x, y, x \cup y\}$. Then, $V(z) = V(\{p_z, q_z\})$ for all $z \in \{x, y, x \cup y\}$. We will show that $V(\{p_x, q_x\}) \geq V(\{p_{x \cup y}, q_{x \cup y}\}) \geq V(\{p_y, q_y\})$.

Note that for $z = x \cup y$, p_z and q_z can be either in x or in y. Hence, there are 4 possible cases: (i) $p_{x\cup y}, q_{x\cup y} \in x$. (ii) $p_{x\cup y}, q_{x\cup y} \in y$. (iii) $p_{x\cup y} \in x\backslash y, q_{x\cup y} \in y\backslash x$. (iv) $p_{x\cup y} \in y\backslash x, q_{x\cup y} \in x\backslash y$.

If $p_{x\cup y}, q_{x\cup y} \in x$, then $V(x) = V(x \cup y)$ and if $p_{x\cup y}, q_{x\cup y} \in y$, then $V(x \cup y) = V(y)$. Hence we only consider cases (iii) and (iv).

(iii) $p_{x\cup y} \in x\backslash y, q_{x\cup y} \in y\backslash x$.

First note that

$$q_{x\cup y} \in \arg\max_{q\in x\cup y} v(q, V(x \cup y)) \text{ and } q_{x\cup y} \in y$$
$$\implies v(q_x, V(x \cup y)), v(p_{x\cup y}, V(x \cup y)) \leq v(q_{x\cup y}, V(x \cup y))$$

Thus,

$$u(p_{x\cup y}, V(x \cup y)) + v(p_{x\cup y}, V(x \cup y)) - v(q_x, V(x \cup y)) \geq V(x \cup y).$$

Hence,

$$\max_{p\in\{p_{x\cup y},q_x\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q\in\{p_{x\cup y},q_x\}} v(q, V(x \cup y)) \geq V(x \cup y).$$

Thus, by Lemma 5.6, $V(\{p_{x\cup y}, q_x\}) \geq V(\{p_{x\cup y}, q_{x\cup y}\})$. Moreover,

$$u(p_x, V(x)) + v(p_x, V(x)) \geq u(p_{x\cup y}, V(x)) + v(p_{x\cup y}, V(x)).$$

Hence,

$$V(x) \geq u(p_{x\cup y}, V(x)) + v(p_{x\cup y}, V(x)) - v(q_x, V(x)).$$

Thus, by Lemma 5.6, $V(\{p_x, q_x\}) \geq V(\{p_{x\cup y}, q_x\})$. Hence, $V(\{p_x, q_x\}) \geq V(\{p_{x\cup y}, q_{x\cup y}\})$. To see that $V(\{p_{x\cup y}, q_{x\cup y}\}) \geq V(\{p_y, q_y\})$ note that $p_{x\cup y} \in x$ implies that

$$V(x \cup y) \geq u(p_y, V(x \cup y)) + v(p_y, V(x \cup y)) - v(q_{x\cup y}, V(x \cup y))$$
$$\text{and}$$
$$V(x \cup y) \geq u(q_{x\cup y}, V(x \cup y)) + v(q_{x\cup y}, V(x \cup y)) - v(q_{x\cup y}, V(x \cup y)).$$

Hence,

$$V(x \cup y) \geq \max_{p\in\{p_y,q_{x\cup y}\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q\in\{p_y,q_{x\cup y}\}} v(q, V(x \cup y)).$$

Thus, by Lemma 5.6, $V(\{p_{x\cup y}, q_{x\cup y}\}) \geq V(\{p_y, q_{x\cup y}\})$. Finally, since $q_{x\cup y} \in y$, then

$$u(p_y, V(y)) + v(p_y, V(y)) - v(q_{x\cup y}, V(y)) \geq V(y).$$

Hence, by Lemma 5.6, $V(\{p_y, q_{x\cup y}\}) \geq V(\{p_y, q_y\})$.

(iv) $p_{x\cup y} \in y \backslash x, q_{x\cup y} \in x \backslash y$.

First note that

$$\begin{aligned} &V(y) \geq u(p_{x\cup y}, V(y)) + v(p_{x\cup y}, V(y)) - v(q_y, V(y)) \\ &\qquad \text{and} \\ &V(y) \geq u(q_y, V(y)) + v(q_y, V(y)) - v(q_y, V(y)). \end{aligned}$$

Hence,

$$V(y) \geq \max_{p\in\{p_{x\cup y}, q_y\}} \{u(p, V(y)) + v(p, V(y))\} - \max_{q\in\{p_{x\cup y}, q_y\}} v(q, V(y)).$$

Thus, by Lemma 5.6, $V(p_y, q_y) \geq V(\{p_{x\cup y}, q_y\})$. Next, note that

$$\begin{aligned} &q_{x\cup y} \in \arg\max_{q\in x\cup y} \{v(q, V(x\cup y))\} \\ &\implies V(x\cup y) \leq u(p_{x\cup y}, V(x\cup y)) + v(p_{x\cup y}, V(x\cup y)) - v(q_y, V(x\cup y)) \end{aligned}$$

Hence,

$$V(x\cup y) \leq \max_{p\in\{p_{x\cup y}, q_y\}} \{u(p_{x\cup y}, V(x\cup y)) + v(p_{x\cup y}, V(x\cup y))\} - \max_{q\in\{p_{x\cup y}, q_y\}} v(p_{x\cup y}, V(x\cup y))$$

Thus, by Lemma 5.6, $V(\{p_{x\cup y}, q_y\}) \geq V(\{p_{x\cup y}, q_{x\cup y}\})$. However,

$$V(x\cup y) \geq u(p_x, V(x\cup y)) + v(p_x, V(x\cup y)) - v(q_{x\cup y}, V(x\cup y)).$$

Thus, by Lemma 5.6, $V(\{p_{x\cup y}, q_{x\cup y}\}) \geq V(\{p_x, q_{x\cup y}\})$. Moreover,

$$V(x) \leq u(p_x, \gamma + v(p_x, V(x)) - v(q_{x\cup y}, V(x)).$$

Hence,

$$V(x) \leq \max_{p\in\{p_x, q_{x\cup y}\}} \{u(p, V(x)) + v(p, V(x))\} - \max_{q\in\{p_x, q_{x\cup y}\}} v(q, V(x)).$$

Thus, by Lemma 5.6, $V(\{p_x, q_x\}) \leq V(\{p_x, q_{x\cup y}\})$. Hence,

$$\begin{aligned} V(\{p_x, q_x\}) &\geq V(\{p_y, q_y\}) &&\geq V(\{p_{x\cup y}, q_y\}) \\ &\geq V(\{p_{x\cup y}, q_{x\cup y}\}) &&\geq V(\{p_x, q_{x\cup y}\}) \\ &\geq V(\{p_x, q_x\}) \end{aligned}$$

Thus, $V(x) = V(y) = V(x \cup y)$.

Uniqueness

Sufficiency is given in the text. Here we prove necessity.

Assume (u, v) and (u', v') represent $\succeq$. Fix $\gamma \in (0, 1)$ and let

$$\Phi(x, \gamma) = \max_{p\in x}\{u(p, \gamma) + v(p, \gamma) - \max_{q\in x}\{v(q, \gamma)\}\}$$
$$\Phi'(x, \gamma) = \max_{p\in x}\{u(p, \gamma) + v'(p, \gamma) - \max_{q\in x}\{v'(q, \gamma)\}\}.$$

By an identical argument to the one in the uniqueness part of Theorem 5.3, $\Phi(x, \gamma) = \Phi'(x, \gamma)$ for all $x \in \mathcal{K}(\Delta(X))$ and $\gamma \in (0, 1)$. In particular, $\Phi(\{p\}, \gamma) = \Phi'(\{p\}, \gamma)$ for all $p \in \Delta(X)$ and $\gamma \in (0, 1)$. Hence, $u = u'$. Denote $\succeq_\gamma$ and $\succeq_{\gamma'}$ the preference over menus represented by $\Phi(., \gamma)$ and $\Phi'(., \gamma)$ respectively. Fix $\gamma \in (0, 1)$. If $v(., \gamma)$ is a constant or a positive affine transformation of $u(., \gamma)$. Then,

$$\Phi(x, \gamma) = \max_{p\in x}\{u(p, \gamma)\}$$

for all $x \in X$. Hence, for all x there exists $p \in x$ such that $x \sim_\gamma \{p\}$. Thus, by GP (p. 1414), either $v'(., \gamma)$ is a constant or a positive affine transformation of $u(., \gamma)$.

If $v(., \gamma)$ is not a constant or a positive affine transformation of $u(., \gamma)$. Then, by GP (p.1414), there are two cases: either there exist p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) > \Phi(\{q\}, \gamma)$ or $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) = \Phi(\{q\}, \gamma)$ for all p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{q\}, \gamma)$. If there exist p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) > \Phi(\{q\}, \gamma)$, then since $u(., \gamma)$ is unique, by GP's Theorem 4, $v'(., \gamma) = v(., \gamma) + b_\gamma$. If $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) = \Phi(\{q\}, \gamma)$ for all p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{q\}, \gamma)$, then by GP (p.1414), $v(., \gamma)$ is a negative affine transformation of $u(., \gamma)$. Hence, we only need to rule out the case in which $v(.,\gamma) = -a_\gamma u(.,\gamma) + b_\gamma$ and $a_\gamma \in (0,1)$. To see that this is impossible let p_γ be such that $\gamma = u(p_\gamma,\gamma)$. Then, $u(\overline{p},\gamma) = 1 > u(p_\gamma,\gamma)$. Hence, $\Phi(\{p_\gamma,\overline{p}\},\gamma) = \gamma$. However, if $a_\gamma < 1$, then

$$u(\overline{p},\gamma) - a_\gamma u(\overline{p},\gamma) = 1 - a_\gamma > u(p_\gamma,\gamma) - a_\gamma u(p_\gamma,\gamma) = \gamma - a_\gamma \gamma.$$

Hence,

$$\Phi(\{p_\gamma,\overline{p}\},\gamma) = 1 - a_\gamma + a_\gamma \gamma \neq \gamma.$$

Therefore, $a_\gamma \geq 1$.

Commitment and Mixtures

Proof of Proposition 4.1

Let (u, v, c) represent $\succeq$. Then, by Theorem 1 in Noor and Takeoka (2010), $\succeq$ satisfies Commitment Independence and Set-Beteenness. Since general self-control models are continuous, it is enough to show the result for finite menus. In what follows we will use the following property of c (Noor and Takeoka (2010 p.134)): $c(p, v(p)) = 0$ for all $p \in \Delta(X)$. For any menu x, let p_x and q_z denote an arbitrary element of $\arg\max_{p\in x} u(p)$ and $\arg\max_{p\in x} v(p)$ respectively.

Part 1: Suppose $\succeq$ has preference for commitment at x. Then, by Set-Betweenness (applied repeatedly), for all p_x and q_x it must be the case that

$$u(p_x) > u(q_x) \text{ and } c(p_x, v(q_x)) > 0,$$

so by Property 2

$$v(p_x) < v(q_x).$$

Further,

$$\alpha u(p_x) + (1-\alpha)u(p_z) \geq \alpha u(p) + (1-\alpha)u(p')$$

for all $p \in x$ and $p' \in z$. By linearity of u and v,

$$v(q_x) > v(p_x) \text{ and } v(q_z) \geq v(p_z) \implies \alpha v(q_x) + (1-\alpha)v(q_z) > \alpha v(p_x) + (1-\alpha)v(p_z)$$
$$u(p_x) > u(q_z) \text{ and } u(p_z) \geq u(q_z) \implies \alpha u(p_x) + (1-\alpha)u(p_z) > \alpha u(q_z) + (1-\alpha)u(q_z).$$

Hence, by Property 3, $0 < c(\alpha p_x + (1-\alpha)p_z, \alpha v(q_x) + (1-\alpha)v(q_z)) = c(\alpha p_x + (1-\alpha)p_z, \max_{q \in \alpha x + (1-\alpha)z} v(q))$. Finally, by linearity of u,

$$\arg \max_{p \in \alpha x + (1-\alpha)z} u(p) = \alpha \arg \max_{p' \in x} u(p') + (1-\alpha) \arg \max_{p'' \in z} u(p'').$$

Hence, $\succeq$ has preference for commitment at $\alpha x + (1-\alpha)z$.

Part 2: Assume $\succeq$ does not have preference for commitment at y and z but has preference for commitment at $\alpha y + (1-\alpha)z$. Fix q_y. Since $\succeq$ does not have preference for commitment at y, then there exists p_y such that $c(p_y, v(q_y)) = 0$ and $\{p_y\} \sim y$. Similarly, there exists p_z such that $c(p_z, v(q_z)) = 0$. Further, $\alpha u(p_y) + (1-\alpha)u(p_z) \geq \alpha u(p) + (1-\alpha)u(p')$ for all $p \in y, p' \in z$. Thus, since $\succeq$ has preference for commitment at $\alpha y + (1-\alpha)z$, $c(\alpha p_y + (1-\alpha)p_z, \alpha v(q_y) + (1-\alpha)v(q_z)) > 0$ so by Property 2, $\alpha v(q_y) + (1-\alpha)v(q_z) > \alpha v(p_y) + (1-\alpha)v(p_z)$.

There are two cases: *(i)* $v(q_y) > v(p_y)$ or *(ii)* $v(q_z) > v(p_z)$.

(i) $v(q_y) > v(p_y)$

Notice that if $u(q_y) = u(p_y)$, then we can replace p_y with q_y in the above argument and rule out this case. If $u(p_y) > u(q_y)$, then by Property 3 $c(p_y, q_y) > 0$ a contradiction.

(ii) $v(q_z) > v(p_z)$

Similarly, if $u(q_z) = u(p_z)$, then we can replace p_z with q_z in the above argument and rule out this case. If $u(p_z) > u(q_z)$, then by Property 3 $c(p_z, q_z) > 0$ a contradiction.

Notation

For the rest of Appendix B, for any $\gamma \in (0,1)$, define

$$\Phi(x, \gamma) \equiv \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\}$$

and let $\succeq_\gamma$ denote the preference over menus it represents.

Mixture Monotone Preference for Commitment without Commitment Independence

Here we provide the counterparts of Theorems 4.1 and 4.2 for the case in which the preference need not satisfy Commitment Independence. The proof for the case in which the preference satisfies Commitment Independence is almost identical; it requires one extra step which we provide at the end of the proof of the general case.

To state the theorems we require additional terminology. Let $f, g, f', g : \Delta(X) \to \mathbb{R}$. Say that (f, g) are a joint positive affine transformation of (f', g') if there exists $a \in \mathbb{R}_{++}, b_f, b_g \in \mathbb{R}$ such that $f = af' + b_f$ and $g = ag' + b_g$.

Theorem 5.4. *Assume $\succeq$ satisfies the axioms of Theorem 3.1 and $\{p^*\} \sim \{p^*\} \sim \{p^*, p_*\}$. Then $\succeq$ satisfies Mixture-Increasing Preference for Commitment if and only if there exists (u, v) that represents $\succeq$ such that $\gamma, \gamma' \in (0, 1)$ and $\gamma' < \gamma$ implies that there exists a joint possitive affine transformation of $(u(., \gamma), v(., \gamma))$ such that $u(., \gamma)$ is a convex combination of $u(., \gamma')$ and $v(., \gamma')$, and $v(., \gamma)$ is a convex combination of $u(., \gamma')$ and $v(., \gamma')$.*

Theorem 5.5. *Assume $\succeq$ satisfies the axioms of Proposition 3.1 and $\{p^*\} \sim \{p^*\} \sim \{p^*, p_*\}$. Then $\succeq$ satisfies Mixture-Decreasing Preference for Commitment if and only if there exists (u, v) that represents $\succeq$ such that $\gamma, \gamma' \in (0, 1)$ and $\gamma \in (0, 1)$ and $\gamma, \gamma' \in (0, 1)$, $\gamma' < \gamma$ implies that there exists a joint positive affine transformation of $(u(., \gamma'), v(., \gamma'))$ such that $u(., \gamma')$ is a convex combination of $u(., \gamma)$ and $v(., \gamma)$, and $v(., \gamma')$ is a convex combination of $u(., \gamma)$ and $v(., \gamma)$.*

The proof of Theorem 5.4 is analogous to the proof of Theorem 5.5. Thus, we only show the proof of Theorem 5.5.

Sufficiency

Let (u, v') be the representation of $\succeq$ constructed using $\{p^*\}$ and $\{p_*\}$ as the best and worst menus. Then, $\overline{p} = p^*$ and $\underline{p} = p_*$. By the uniqueness properties of (u, v') there exists v such that (u, v) represents $\succeq$ and if $v'(., \gamma)$ is a constant or a positive affine transformation of $u(., \gamma)$, then $v(., \gamma) = u(., \gamma)$. First we show that $v(., \gamma)$ is not a negative affine transformation of $u(., \gamma)$. By GP (p. 1414) it is enough to show that for each $\gamma \in (0, 1)$ either there exists y such that $\{p\} \succeq_\gamma y \succ_\gamma \{p'\}$ for some $p, p' \in y$.

Fix $\gamma \in (0,1)$, Note that since $\{\overline{p}, \underline{p}\} \sim \{\overline{p}\}$. Then, by Mixture-Betweenness $\{\alpha\overline{p} + (1-\alpha)\underline{p}, \underline{p}\} \succ \{\underline{p}\}$ for all $\alpha > 0$. Fix $\gamma \in (0,1)$, then by Lemma 5.1, there exists α such that

$$\{\alpha\overline{p} + (1-\alpha)\underline{p}, \underline{p}\} \sim \{\gamma\overline{p} + (1-\gamma)\underline{p}\}.$$

Further, by Set-Betweenness

$$\{\alpha\overline{p} + (1-\alpha)\underline{p}\} \succeq \{\alpha\overline{p} + (1-\alpha)\underline{p}, \underline{p}\} \succ \{\underline{p}\}.$$

Hence, by Lemma 5.6, $\Phi(\{\alpha\overline{p} + (1-\alpha)\underline{p}\}, \gamma) \geq \Phi(\{\alpha\overline{p} + (1-\alpha)\underline{p}, \underline{p}\}, \gamma) = \gamma > \Phi(\{\underline{p}\}, \gamma)$.

Next we prove the result. Fix $\gamma > \gamma'$ and $y \in \mathcal{X}$ such that $\succeq_\gamma$ has preference for commitment at y. There two possible cases: (i) $V(y) \geq \gamma$ or (ii) $V(y) \leq \gamma$.

(i) $V(y) \geq \gamma$

$\succeq_\gamma$ has preference for commitment at y implies $\succeq$ has preference for commitment at $\lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\underline{p}\}$. Thus by Part I, and $\succeq$ has preference for commitment at $\lambda(y,\gamma')y + (1-\lambda(y,\gamma'))\{\underline{p}\}$

(ii) $V(y) \leq \gamma$

$\succeq_\gamma$ has preference for commitment at y implies $\succeq$ has preference for commitment at $\lambda(y,\gamma)y + (1-\lambda(y,\gamma))\{\overline{p}\}$. Thus, by Part II $\succeq$ has preference for commitment at $\lambda(y,\gamma')y + (1-\lambda(y,\gamma'))\{\overline{p}\}$ if $\gamma' > V(y)$. If $\gamma' \leq V(y)$, then by Part II, $\succeq$ has preference for commitment at y, thus by Part I $\succeq$ has preference for commitment at $\lambda(y,\gamma')y + (1 - \lambda(y,\gamma'))\{\underline{p}\}$

Hence if $\succeq_\gamma$ has preference for commitment at y, $\succeq_{\gamma'}$ has preference for commitment at y. Now, there are two cases: $v(.,\gamma') = u(.,\gamma')$ or $v(.,\gamma') \neq u(.,\gamma')$. If $v(.,\gamma') = u(.,\gamma')$, then it must be the case that $v(.,\gamma) = u(.,\gamma')$, otherwise there would exists y such that $\succeq_\gamma$ has preference for commitment at y and thus $\succeq_{\gamma'}$ would also have preference for commitment at y so $v(.,\gamma') \neq u(.,\gamma')$, a contradiction. If $v(.,\gamma') \neq u(.,\gamma')$, then by GP Theorems 4 and 8 there exists a joint positive affine transformation of $(u(.,\gamma), v(.,\gamma))$ such that $u(.,\gamma)$

is a convex combination of $u(.,\gamma')$ and $v(.,\gamma')$, and $v(.,\gamma)$ is a convex combination of $u(.,\gamma')$ and $v(.,\gamma')$.[6]

Commitment Independence: If $\succeq$ satisfies Commitment Independence, then in the case in which $u \neq v(.,\gamma)$ the application of GP Theorems 4 and 8 implies that u is a convex combination of a joint positive affine transformation of u and $v(.,\gamma)$. However, if the coefficient multiplying u of the joint affine transformation of $(u, v(.,\gamma))$ is different from 1, then we can write v as an affine transformation of u which is an impossibility as $\{p^*\} \sim \{p^*, p_*\}$ implies $v(.,\gamma)$ is not a negative affine transformation of u and $v(.,\gamma)$ was chosen such that either $v(.,\gamma) = u$ or $v(.,\gamma)$ is not a positive affine transformation of u.

Necessity

Part I: Note that by GP Theorem 8 the conditions in the theorem imply that if $\gamma > \gamma' > 0$, then if $\succeq_\gamma$ has preference for commitment at y, then $\succeq_{\gamma'}$ also has preference for commitment at y. Assume $x \succ \{p\}$ and $\succeq$ has preference for commitment at $\alpha x + (1-\alpha)\{p\}$. Then there exists $y \subseteq \alpha x + (1-\alpha)\{p\}$ such that $y \succ \alpha x + (1-\alpha)\{p\}$. Thus, by Lemma 5.6,

$$\Phi(y, V(\alpha x + (1-\alpha)\{p\})) > \Phi(\alpha x + (1-\alpha)\{p\}, V(\alpha x + (1-\alpha)\{p\})).$$

Hence, $\succeq_{V(\alpha x+(1-\alpha)\{p\})}$ has preference for commitment at $\alpha x + (1-\alpha)\{p\}$. Fix $0 < \beta < \alpha$, then since $x \succ \{p\}$, by Lemma 5.1, $V(\beta x + (1-\beta)\{p\}) < V(\alpha x + (1-\alpha)\{p\})$. Hence, by the observation at the beginning of the proof, $\succeq_{V(\beta x+(1-\beta)\{p\})}$ has preference for commitment at $\alpha x + (1-\alpha)\{p\}$. Thus, there exists $p_x \in x$ such that $\Phi(\{\alpha p_x + (1-\alpha)p\}, V(\beta x + (1-\beta)\{p\})) > \Phi(\alpha x + (1-\alpha)\{p\}, V(\beta x + (1-\beta)\{p\}))$. Finally, since Φ is mixture linear in its first argument, then $\Phi(\{\beta p_x + (1-\beta)p\}, V(\beta x + (1-\beta)\{p\})) > \Phi(\beta x + (1-\beta)\{p\}, V(\beta x + (1-\beta)\{p\})) = V(\beta x + (1-\beta)\{p\})$. Hence, by Lemma 5.6, $\{\beta p_x + (1-\beta)p\} \succ \beta x + (1-\beta)\{p\}$ and $\succeq$ has preference for commitment at $\beta x + (1-\beta)\{p\}$.

Part II: Assume $\{p\} \succ x$ and $\succeq$ has preference for commitment at $\alpha x + (1-\alpha)\{p\}$. Then, there exists $y \subseteq \alpha x + (1-\alpha)\{p\}$ such that $y \succ \alpha x + (1-\alpha)\{p\}$. Hence, by Lemma 5.6, $\Phi(y, V(\alpha x + (1-\alpha)\{p\})) > \Phi(\alpha x + (1-\alpha)\{p\}, V(\alpha x + (1-\alpha)\{p\}))$. Hence, $\succeq_{V(\alpha x+(1-\alpha)\{p\})}$ has preference for

[6] Here we are citing the version of GP Theorem 8 in Gul and Pesendorfer (2004).

commitment at $\alpha x + (1-\alpha)\{p\}$. Since $\{p\} \succ x$, then for any $\beta > \alpha$, $V(\alpha x + (1-\alpha)\{p\}) > V(\beta x + (1-\beta)\{p\})$. Hence, by the observation at the beginning of the proof of Part I, $\succeq_{V(\beta x+(1-\beta)\{p\})}$ has preference for commitment at $\alpha x+(1-\alpha)\{p\}$. Thus, there exists $p_x \in x$ such that $\Phi(\{\alpha p_x + (1-\alpha)p\}, V(\beta x + (1-\beta)\{p\})) > \Phi(\alpha x + (1-\alpha)\{p\}, V(\beta x + (1-\beta)\{p\}))$. Finally, since Φ is mixture linear in its first argument, then $\Phi(\{\beta p_x + (1-\alpha)p\}, V(\beta x + (1-\beta)\{p\})) > \Phi(\beta x + (1-\beta)\{p\}, V(\beta x + (1-\beta)\{p\})) = V(\beta x+(1-\beta)\{p\})$. Hence, by Lemma 5.6, $\{\beta p_x+(1-\alpha)p\} \succ \beta x+(1-\beta)\{p\}$. Hence, $\succeq$ has preference for commitment at $\beta x + (1-\beta)\{p\}$.

Appendix C: Proofs of Lemma 5.1 and 5.2

Preliminaries

Lemma 5.10. *Assume $(\mathcal{M}, \pi)$ is an Associative Mixture Space. Then, for all $x, y, z \in \mathcal{M}$ and $a, b, c \in [0,1]$ such that $ca + (1-c)b \neq 0$,*

$$\pi_c(\pi_a(x,y), \pi_b(z,y)) = \pi_{ca+(1-c)b}(\pi_{\frac{ca}{ca+(1-c)b}}(x,z), y).$$

Proof. Fix $x, y, z \in \mathcal{M}$ and $a, b, c \in [0,1]$ such that $ca + (1-c)b \neq 0$. Then,

$$\begin{aligned}\pi_c(\pi_a(x,y), \pi_b(z,y)) &= \pi_{ca}(x, \pi_{\frac{c(1-a)}{1-ac}}(y, \pi_b(z,y)) \\ &= \pi_{ca}(x, \pi_{\frac{1-c}{1-ac}}(\pi_b(z,y), y) \\ &= \pi_{ca}(x, \pi_{\frac{b(1-c)}{1-ac}}(z,y) \\ &= \pi_{ca+(1-c)b}(\pi_{\frac{ca}{ca+(1-c)b}}(x,z), y)\end{aligned}$$

Where the first inequality follows from (A4), the second from (A2), the third from (A3) and the fourth from (A4). ☐

Proof of Lemma 5.1

Part 1: If $a = 0$ or $b = 1$, then Mixture-Betweenness implies $\pi_b(x,y) \succ \pi_a(x,y)$. Suppose $0 < a < b < 1$. Then, there exists $c \in (0,1)$ such that $cb = a$. By Mixture-Betweenness, $\pi_b(x,y) \succ y$. Further, Mixture-Betweenness also implies $\pi_b(x,y) \succ \pi_c(\pi_b(x,y), y) = \pi_a(x,y)$.

Part 2: Fix $x, y, z \in \mathcal{M}$ and let $A = \{a | \pi_a(x,y) \succeq z\}$. WLOG assume

$x \succeq y$. If $A = \emptyset$ or a singleton, then there is nothing to prove. Assume $A \neq \emptyset$ and $|A| > 1$. Define $a^* = \inf A$. If $a^* \in A$, then by Part 1 $A = [a^*, 1]$. Hence, A is closed. Assume towards a contradiction that $a^* \notin A$. Thus, $z \succ \pi_{a^*}(x, y)$. Since $|A| > 1$, then $x \succ z \succ \pi_{a^*}(x, y)$. Hence, by Continuity, there exists b such that

$$z \succ \pi_b(x, \pi_{a^*}(x, y)) = \pi_{1-b}(\pi_{(1-a^*)}(y, x), x) = \pi_{(1-b)(1-a^*)}(y, x) = \pi_{1-(1-b)(1-a^*)}(x, y).$$

Note that $1-(1-b)(1-a^*) = a^*+b(1-a^*) > a^*$. Hence, $z \succ \pi_{1-(1-b)(1-a^*)}(x, y)$ and $1-(1-b)(1-a^*) > a^*$, a contradiction of the fact that a^* is the infimum.

The proof that $\{a|z \succeq \pi_a(x, y)\}$ is closed is analogous, the only difference is that instead of using the infimum to derive a contradiction, one needs to use the supremum.

Part 3: Follows from Part 2.

Proof of Lemma 5.2

The proofs of Parts 2, 4 and 6 are analogous to the proofs of Parts 1, 3 and 5 respectively. Hence, we only prove Parts 1,3 and 5.

Part 1: Fix x, y such that $V(x), V(y) \geq \gamma$ and $a \in (0, 1)$. Then, by Mixture-Betweenness, $V(\pi_a(x, y)) \geq \gamma$. Further,

$$\pi_{\lambda(x,\gamma)}(x, \underline{x}) \sim x_\gamma \sim \pi_{\lambda(y,\gamma)}(y, \underline{x}).$$

Hence, by Mixture-Betweenness, $\pi_c(\pi_{\lambda(x,\gamma)}(x, \underline{x}), \pi_{\lambda(y,\gamma)}(y, \underline{x})) \sim x_\gamma$ for all $c \in (0, 1)$. Notice that if for some $c \in (0, 1)$

$$\pi_c(\pi_{\lambda(x,\gamma)}(x, \underline{x}), \pi_{\lambda(y,\gamma)}(y, \underline{x})) = \pi_{\frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}(\pi_a(x, y), \underline{x})$$
$$\implies \lambda(\pi_a(x, y), \gamma) = \frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1 - a)\lambda(x, \gamma)}.$$

Thus, it is enough to show that the first equality holds.

By Lemma 5.10, for all $c \in [0, 1]$

$$\pi_c(\pi_{\lambda(x,\gamma)}(x, \underline{x}), \pi_{\lambda(y,\gamma)}(y, \underline{x})) = \pi_{c\lambda(x,\gamma)+(1-c)\lambda(y,\gamma)}(\pi_{\frac{c\lambda(x,\gamma)}{c\lambda(x,\gamma)+(1-c)\lambda(y,\gamma)}}(x, y), \underline{x}).$$

In particular, if $c = \frac{a\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$, then

$$c\lambda(x,\gamma)+(1-c)\lambda(y,\gamma) = \frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)} + \frac{(1-a)\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$$
$$= \frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$$

and

$$\frac{c\lambda(x,\gamma)}{c\lambda(x,\gamma)+(1-c)\lambda(y,\gamma)} = \frac{\frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}{\frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)} + \frac{(1-a)\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}$$
$$= a.$$

Hence, $\pi_c(\pi_{\lambda(x,\gamma)}(x,\underline{x}),\pi_{\lambda(y,\gamma)}(y,\underline{x})) = \pi_{\frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}(\pi_a(x,y),\underline{x})$.

Part 3: Fix $x,y,z \in \mathcal{X}$ such that $V(x),V(y) > \gamma > V(z)$ and $\lambda(x,\gamma) < \lambda(y,\gamma)$. Let $b \in (0,1)$ be such that

$$\pi_b(x,z) \sim x_\gamma.$$

Then, by Part 1,

$$\lambda(\pi_a(x,\pi_b(x,z))) = \frac{\lambda(x,\gamma)}{a+(1-a)\lambda(x,\gamma)}.$$

Define $\phi(a) = \frac{\lambda(x,\gamma)}{a+(1-a)\lambda(x,\gamma)}$. Then, $\phi(0)=1, \phi(1)=\lambda(x)$ and $\phi'(a)<0$ for all $a\in[0,1]$. Hence, there exists a unique $a\in(0,1)$ such that $\phi(a)=\lambda(y,\gamma)$.

Part 5: Let a_1 and a_2 be such that

$$x_\gamma \sim \pi_{a_1}(x,z)$$
$$x_\gamma \sim \pi_{a_2}(y,z).$$

Assume towards a contradiction that $a_2 > a_1$. Then, $\pi_{a_2}(x,z) \succ x_\gamma$ and $\lambda(\pi_{a_2}(x,z),\gamma) < 1$. Since $\lambda(x,\gamma)=\lambda(y,\gamma)$, then by Part 1, for all $c\in(0,1)$

$$\lambda(\pi_c(x,\pi_{a_2}(y,z)),\gamma) = \lambda(\pi_c(y,\pi_{a_1}(x,z)),\gamma).$$

However,

$$\pi_c(x,\pi_{a_2}(y,z) = \pi_{1-c}(\pi_{a_2}(y,z),x)$$
$$= \pi_{(1-c)a_2}(y,\pi_{\frac{(1-c)(1-a_2)}{1-(1-c)a_2}}(z,x))$$
$$= \pi_{(1-c)a_2}(y,\pi_{\frac{c}{1-(1-c)a_2}}(x,z)).$$

Thus, for $c = \frac{a_2}{1+a_2}$,

$$\pi_c(x, \pi_{a_2}(y, z) = \pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_2}(x, z))$$

and

$$\pi_c(y, \pi_{a_1}(x, z)) = \pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_1}(x, z)).$$

Thus, $\lambda(\pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_2}(x, z)), \gamma) = \lambda(\pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_1}(x, z)), \gamma)$ implies

$$\frac{\lambda(y,\gamma)\lambda(\pi_{a_2}(x,z))}{c + (1-c)\lambda(y,\gamma)} = \frac{\lambda(y,\gamma)}{c + (1-c)\lambda(y,\gamma)}$$
$$\lambda(\pi_{a_2}(x,z)) = 1,$$

a contradiction.

Appendix D: Details for Section 3.3

First we show that the special case of the model considered in Section 3.3 satisfies Mixture-Increasing Preference for Commitment.

Note that for all $\gamma \in [0, 1]$,

$$v(., \gamma) = (1 - \gamma)u(.) + \gamma v(., 1).$$

Hence, for any $\gamma > \gamma'$, $v(., \gamma')$ is a convex combination of u and $v(., \gamma)$. Hence, by Theorem 4.1, the preference represented by (u, v) satisfies Mixture-Increasing Preference for Commitment.

Next, we calculate the utility of each menu involved in the motivating example. It is easy to see that $V(c) = \frac{1}{2}, V(f) = 0$ and $V(\{\alpha c + (1-\alpha)f\}) = 1 - \frac{\alpha}{2}$. Consider $V(\{c, f\})$. Notice that for all γ,

$$u(c) + v(c, \gamma) \geq u(f) + v(f, \gamma).$$

Hence,

$$2\gamma = \frac{1}{2} + 1 - \max_{q \in \{c,f\}} v(q, \gamma).$$

Further,

$$\max_{q\in\{c,f\}} v(q,\gamma) = \frac{1-\gamma}{2} \implies V(\{c,f\}) = \frac{1}{2}$$
$$\max_{q\in\{c,f\}} v(q,\gamma) = \frac{\gamma}{2} \implies V(\{c,f\}) = \frac{1}{2}.$$

Thus, $V(\{c,f\}) = \frac{1}{2}$ and

$$V(\{c\}) = V(\{c,f\}) > V(\{f\}).$$

Next, consider $V(\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\})$. Notice that for all γ,

$$u(\alpha c + (1-\alpha)f') + v(\alpha c + (1-\alpha)f', \gamma) \geq u(\alpha f + (1-\alpha)f') + v(\alpha f + (1-\alpha)f', \gamma).$$

Hence,

$$\gamma = 2 - \alpha - \frac{\alpha\gamma}{2} - \max_{q\in\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}} v(q,\gamma)$$

If $\gamma \geq \frac{1}{2}$, then

$$\max_{q\in\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}} v(q,\gamma) = \frac{\alpha\gamma}{2} + (1-\alpha)$$
$$\implies V(\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}) = \frac{1}{1+\alpha} \geq \frac{1}{2}.$$

If $\gamma < \frac{1}{2}$, then

$$\max_{q\in\{\alpha c+(1-\alpha)f', \alpha f+(1-\alpha)f'\}} v(q,\gamma) = \alpha\frac{(1-\gamma)}{2} + (1-\alpha)$$
$$\implies V(\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}) = 1 - \frac{\alpha}{2} > \frac{1}{2},$$

a contradiction. Hence, $V(\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}) = \frac{1}{1+\alpha}$.

Finally, $1 - \frac{\alpha}{2} > \frac{1}{1+\alpha}$ if and only if $\alpha > \alpha^2$. Hence,

$$V(\{\alpha c + (1-\alpha)f'\}) > V(\{\alpha c + (1-\alpha)f', \alpha f + (1-\alpha)f'\}).$$

Appendix E: Sufficient Conditions on (u, v)

The following proposition provides sufficient conditions on (u, v) such that an implicit representation as in Theorem 3.1 exists.

Proposition 5.1. *Let $u, v : \Delta(X) \times [0,1] \to \mathbb{R}$ be such that $u(., \gamma)$ and $v(., \gamma)$ are vNM utility functions for all $\gamma \in [0,1]$. Then, the following are sufficient for (u, v) to represent some preference $\succeq$.*

1. *u, v are continuous in their second argument on $[0,1]$*

2. $$1 = \max_{x \in \mathcal{X}} \max_{p \in x} \{u(p,1) + v(p,1) - \max_{q \in x} v(q,1)\}$$

3. $$0 = \min_{x \in \mathcal{X}} \max_{p \in x} \{u(p,0) + v(p,0) - \max_{q \in x} v(q,0)\}$$

4. *For all $x \in \mathcal{X}$*
$$\phi(\gamma) = \gamma - \max_{p \in x} \{u(p,\gamma) + v(p,\gamma) - \max_{q \in x} v(q,\gamma)\}$$
is strictly increasing in $[0,1]$

Proof. Notice that if for every x there exists a unique γ such that

$$\gamma = \max_{p \in x} \{u(p,\gamma) + v(p,\gamma) - \max_{q \in x} v(q,\gamma)\},$$

then 1 implies the utility function defined by the above equation is continuous. Hence, we only need to show that for any x, the solution exists and it is unique. Define

$$\Phi(x,\gamma) = \max_{p \in x} \{u(p,\gamma) + v(p,\gamma) - \max_{q \in x} v(q,\gamma)\}.$$

Then, condition (1) implies $\Phi(x, .)$ is continuous. Further, conditions 2 and 3 imply $\Phi(x, 0) \geq 0$ and $\Phi(x, \gamma) \leq 1$. Thus, if $\Phi(x, 0) > 0$ and $\Phi(x, \gamma) < 1$, continuity implies that a solution exists. Thus, to finish the proof we only need to show it is unique. To see this, note that if $\Phi(x, \gamma) = \gamma$, then $\phi(\gamma) = 0$. Thus, by condition 4 $\phi(\gamma') > 0$ for all $\gamma' > \gamma$ and $\phi(\gamma') < 0$ for all $\gamma'' < \gamma$. Hence, the solution is unique. □

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