Polarizing Cheap Talk

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Abstract

In this paper we extend the standard cheap talk model in a minimal way to capture the possible incentives of polarization. For this, we add one receiver and one binary payoff irrelevant state variable to the standard model. In contrast to payoff relevant variable, receivers have different beliefs about payoff irrelevant one. Besides the equilibria of standard model, our extension admits continuum of new equilibria and expands the set of sender's equilibrium payoffs. We characterize a set of polar equilibria and prove that this set of equilibria spans the sender's payoff space. We also show that in these equilibria, receivers are polarized. Namely, their posterior beliefs are ordered with first-order stochastic dominance. Moreover, when ex-ante receivers disagreement rises, then there is more scope for manipulation by the sender. Specifically, the set of aggregate actions and sender payoffs that can be supported in equilibrium expands as the receivers' posterior beliefs diverge. We show that even slight disagreement about payoff irrelevant state is enough for influential equilibrium to exist for any bias level. When disagreement increases, the equilibria set expands, and the bias threshold above which there are no informative equilibria also increases.

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1 Introduction

It is well-documented that public beliefs have polarized on many dimensions in recent years. (e.g. Alesina, Miano, and Stantcheva (2020)).[1] Explanations include the emergence of information bubbles(Allcott, Braghieri, Eichmeyer, and Gentzkow (2020)),[2] and tribal affiliations (Gerber and Green (1999) and Autor, Dorn, Hanson, and Majlesi (2020)). While these sources of polarization are likely important drivers of increasing polarization, this paper explores the incentives of privately informed agents to *induce* public polarization for their own gain. In order to discipline the exercise, we assume that the audience being polarized is rational and processes all new information using Bayes' rule. We do believe that cognitive bias' are not an important source of polarization. Rather, we believe that rational Bayesian benchmark models of polarization are useful for understanding the roots of polarization.

To fix ideas, assume we are interested in beliefs about the extent to which human activity has contributed to climate change. We administer a survey to assess prior beliefs in a group of subjects. Subjects then get exactly the same set of scientific studies to read. We then administer the same survey and discover that beliefs have polarized. Specifically, individuals who believed that climate change was "less than 50% caused by human activity" in the first survey decrease their estimate of the impact of humans on the climate in the second survey. While individuals who believed that climate change was "more than 50% caused by human activity" in the first survey increase their estimate of the impact of humans on the climate in the second survey.

Such polarization can be explained by invoking confirmatory bias: subjects concentrate on studies that support their existing beliefs, but do not account for this selection effect when they revise their beliefs. Alternatively, we could invoke affiliation theory: subjects interact with each other and seek to align their beliefs more closely with other people with similar prior beliefs. But neither of these explanations is required to explain polarization. Two individuals could read exactly the same studies, process any new information using Bayes' rule, and end up becoming more polarized. As Benoit and Dubra (2019) show, the key to understanding such rational polarization is to recognize that we live in a complex multidimensional world, but measure beliefs on a smaller (typically one-dimensional) subset. For an intuition, assume the individuals in the study have dif-

[1] Also see Boxell, Gentzkow, and Shapiro (2020), Pew Research Center (2014), and Gentzkow, Shapiro, and Taddy (2019)

[2] See also Törnberg (2018), Flaxman, Goel, and Rao (2016), Bail, Argyle, Brown, Bumpus, Chen, Hunzaker, Lee, Mann, Merhout, and Volfovsky (2018), and Melki and Sekeris (2019)

ferent views about biases in the scientific literature. If individual A believes that climate scientists are biased toward ascribing increases in global temperatures to human activity and individual B believes that climate scientists are biased against such findings, then it can be perfectly rational for these individuals to read the same studies and have A become more convinced that global warming is not caused by human activity and B become more convinced that it is.

We embed the above idea in the pure cheap talk communication game of Crawford and Sobel (1982). We add a second receiver so that belief polarization is meaningful. In order to distinguish the current model from information bubble explanations of polarization, we do not allow the sender to communicate separately with each receiver. Specifically, the sender must choose one costless, but unverifiable message, *publicly* observed by both receivers. After observing this common signal the receiver's update their beliefs according to Bayes' rule, and then each takes a separate action. The sender's payoff depends on the average action taken, and the state contingent preferences of the sender and each receiver are not fully aligned. In contrast the receiver's have identical state contingent preferences.

In order to allow for *rational* belief polarization, we assume that the state space is two dimensional.[3] The *payoff relevant* dimension is a scalar that directly enters the payoff function for all agents. The *payoff irrelevant* dimension is a binary variable that does not directly enter any payoff functions, but nonetheless may affect posterior beliefs. Receivers agree on the prior distribution of the payoff relevant variable, but their beliefs about the payoff irrelevant variable are different.

Receivers in our model are rational, but have different beliefs. To avoid contradicting the agreement theorem in Aumann (1976) we need to either assume that knowledge about beliefs is not common, or that the receivers "agree to disagree" about their initial prior on the payoff irrelevant state: we assume they agree to disagree. Formally receiver 1 does not update his belief based on the prior of receiver 2, and vice versa. We assume that receivers rationally process all *new* information, despite the initial disagreement.[4]

Suppose the sender employs a *one-dimensional partition*; namely, she either fully reveals the payoff irrelevant state or does not reveal any additional information about

[3]Similarly to Benoit and Dubra (2019), Jern, Chang, and Kemp (2014), Andreoni and Mylovanov (2012), Loh and Phelan (2019) show that rational polarization on one dimension is possible with a two dimensional state space, although the signal generating process in these papers is assumed to be exogenous, rather than controlled by a motivated agent like our sender.

[4]Of course, people suffer both from biases in information processing in addition to having different beliefs about the information generating process. Ours is a benchmark model in which polarization is driven purely by differences in beliefs about the information generating process.

this binary variable. The set of equilibria with one-dimensional partitions in the current model is payoff equivalent to the set of equilibria in the standard cheap talk model (Proposition 0). However, allowing for the payoff irrelevant dimension, introduces a continuum of new equilibria, and expands the set of equilibrium payoffs.

In order to fix ideas, consider an application in which receivers decide how much to contribute to a project. The receivers each believe that the optimal contribution is uniformly distributed on $[0, 100]$, and wish to minimize the squared deviation between their actual contribution and the ideal contribution. Without additional information, they each would contribute 50. Now, imagine that all agents know the sender has commissioned two reports from two separate consulting companies, and publicly announces message 1: *"if consulting company #1 is more reliable, then the optimal contribution* > 80*, whereas if consulting company #2 is more reliable, then the optimal contribution is* < 80*"*, how would receivers process this information? This depends on the receivers' prior beliefs about the reliability of each consulting company (the payoff irrelevant dimension). For simplicity, assume that the first receiver is certain that consulting company #2 is more reliable, while the second receiver is certain that consulting company #1 is more reliable. Given message 1, receiver 1 believes that the optimal contribution is uniformly distributed on $[0, 80]$ and chooses to contribute 40, while the second receiver will contribute 90 as he believes that optimal contribution is now uniformly distributed on $[80, 100]$; and so, the total contribution is $90 + 40 = 130$. But why should receivers trust the sender's initial announcement? Assume instead that the alternative equilibrium announcement was message 2: "if consulting company #1 is more reliable, then optimal contribution < 80, and if consulting company #2 is more reliable, then optimal contribution > 80," this time the first receiver will contribute 40 and the second receiver will contribute 90, again totalling 130. Altogether, these two messages constitute an equilibrium: the two messages partition the two-dimensional state space, yield the same aggregate action (and thus the same payoff for the sender) when receivers best respond to their beliefs.

Notice the structure of the two messages considered in the consulting company example. The payoff relevant state space was partitioned into a low interval and a high interval. For each message the identity of the reliable consulting firm was associated with one of the intervals, but the mapping from the identity of the reliable firm to the payoff relevant interval was swapped between the signals. This is an example of a *diagonal partition* of $[0, 1]$. In a *polar partition* the payoff relevant dimension is partitioned into a finite number of intervals, and then each of these intervals is diagonally partitioned. It turns out that polar partitions are sufficient from the sender's point of view. Specifically, for any equilibrium, there exists an equilibrium with a polar partition that delivers the same aggregate actions, and thus the same payoff for the sender in every payoff relevant state (Proposition 1). Given this payoff equivalence, we primarily focus on polar partitions.

Recall the posterior beliefs in the consulting example: If message 1 is sent, receiver 1's posterior falls relative to his prior (in first-order stochastic dominance sense), while receiver 2's beliefs first order increase. If message 2 is sent then the opposite occurs: receiver 1's posterior belief first order rises and receiver 2's posterior belief first order falls. An equilibrium is *polarizing* if every message induces a first order increase in one receiver's beliefs and a first order decrease in the other receivers beliefs, and *strictly polarizing partition* if these first order changes are strict for at least one message. Clearly one-dimensional equilibria are not polarizing. However, *every* other equilibrium is aggregate action (and sender payoff) equivalent to a strictly polarizing equilibrium (Proposition 2). Furthermore, the equilibrium with the highest ex-ante utility for the sender is strictly polarizing (Corollary 2).

We show that if ex ante receiver disagreement rises, then there is more scope for manipulation by the sender. Specifically, the set of aggregate actions (and thus sender payoffs) that can be supported in equilibrium expands as the posterior beliefs of the receivers diverge (Proposition 3).

An equilibrium is *informative* if receiver aggregate actions differ across at least two messages. A standard result in the cheap talk literature is that informative equilibria exist, *iff* the sender's payoff bias is below some threshold. Intuitively, the sender cannot send a credible message if the receivers know that the sender has sufficiently extreme payoff bias. It turns out that it is easier to support informative equilibria with receiver ex ante belief disagreement. In fact, the sender bias threshold is monotonically increasing in the receiver ex ante belief disagreement (Proposition 5).

An equilibrium is *influential* if there exists at least one message for which the aggregate action differs from what the aggregate action would have been had the sender sent no message (i.e. with actions based only on priors). In the standard cheap talk model, an equilibrium is informative *iff* it is influential. But this equivalence no longer holds in the current model. In fact, when receiver's have different priors on the payoff irrelevant variable, *influential* equilibria always exist for any bounded sender bias.

There is vast descriptive literature on the polarization of attitudes on various issues and research in experimental economics and psychology, presenting experiments in which participants polarize upon observing the same information. For example, Lord and Ross (1979) show that attitudes toward the deterrent effect of the death penalty became more extreme after participants were exposed to the same information. Alesina, Miano, and Stantcheva (2020) documents increased polarization in subjects observing the same evidence on a number of debatable political topics. Some theoretical models capture polarization by assuming that people are biased in their processing of information. For example in FryerJr, Harms, and Jackson (2019), polarization owes to confirmation bias.

This is not the first paper to consider rational polarization. Benoit and Dubra (2019), Jern, Chang, and Kemp (2014), Loh and Phelan (2019), Andreoni and Mylovanov (2012) assume that agents optimally process new information, but allow for polarization by assuming that people have different models of how information is generated. Indeed, the current model is Crawford and Sobel (1982) with the two-dimensional state space and receiver disagreement in Benoit and Dubra (2019). Ex ante disagreement about priors is also found in Van den Steen (2010), den Steen (2010), Morris (1994), Morris (1995),Alonso and Câmara (2016) and Anderson and Pkhakadze (2020). The last one is a communication game with two receivers, but not cheap talk. Specifically, their sender commits to an experiment rather than choosing a message like in Kamenica and Gentzkow (2011).

Most of the cheap talk models are some type of generalization of the model described in Crawford and Sobel. Although our work is about cheap talk with multiple agents and a multidimensional state, our payoff functions and action spaces are most closely related to single receiver models, such as Crawford and Sobel (1982) and Chakraborty and Harbaugh (2010). Specifically, we adopt the quadratic loss specification for utility functions in Crawford and Sobel, which we refer to as the *standard model*.

Though the model in Chakraborty and Harbaugh (2010) has one receiver and sender with state-independent preferences, it provides one significant implication for our model. Their receiver is a consumer who cares about multiple attributes of the sender's product. Their sender partitions the space of all attributes so that the receivers' aggregate valuation of the good on each partition element implies the same utility level to the sender. Hence, the sender has no incentive to misrepresent in which partition element is the state. Similarly our sender has to expect the same payoff across messages if conditional on payoff relevant state, these messages differ only on payoff irrelevant state.

Other multidimensional but single receiver models are Chakraborty and Harbaugh (2007), Levy and Razin (2004), Levy and Razin (2007), and Sémirat (2019). Chakraborty and Harbaugh (2007) show how strategies which provide information on the ranking of realized states can guarantee credible information transmission when it would not be possible if the sender were using standard cheap talk across the state dimensions. Levy and Razin (2007) analyze the effects of informational spillovers, which might reduce information transmission. They show that in contrast to Chakraborty and Harbaugh (2007), two separate games, one for each dimension, can transmit more information than a multidimensional cheap talk game.

There is a literature that considers polarization of voters in the presence of strategic cheap talk from politician. But in contrast to current work, these models induce polarization by assuming receivers differ in their state contingent preferences, not in their prior beliefs. For example, Jeong (2019) provides a model in which, depending on her own preferences, using simple, cheap talk strategy sender can polarize or unify the voters. For polarization, the sender uses a comparative strategy similar to one in Chakraborty and Harbaugh (2007).

The rest of the paper is organized as follows. The next section defines the model. The third section gives an illustrative numerical example to build intuition for a results to follow. Section 4 characterizes equilibrium partitions as a form of two-stage communication. In the fifth section, we introduce polar partitions and show that these span the set of sender payoff functions. Section 6 contains the polarizing results. Section 7 describes how the set of equilibria changes when receivers disagree more about the payoff irrelevant state. Section 8 concludes. All omitted proofs can be found in the appendix.

2 Model

This section describes the public communication game between a privately informed sender and two receivers. The sender knows the two dimensional state $(t, \omega) \in \Theta = [0,1] \times \{0,1\}$. Both receivers share the common prior that t is distributed according to CDF F with density $f(t) > 0$ supported on $[0,1]$. But they have disparate priors on ω; namely, $q_i = Pr(\omega = 1) = q_i = 1 - Pr(\omega = 0)$ with $q_1 < q_2$.

While they differ in beliefs, receivers share the same the von Neumann-Morgenstern utility function conditional on t: $\mathbb{U}^1(a,t) = \mathbb{U}^2(a,t) = -(a-t)^2$, where $a \in [0,1]$ is the receiver's own action. The Sender's von Neumann-Morgenstern utility depends on t and the average action taken by the receivers $\bar{a}$:

$$\mathbb{U}^s(\bar{a}(a_1,a_2),t) = -\big(\bar{a} - (t+B)\big)^2 = -\Big(\frac{a_1+a_2}{2} - (t+B)\Big)^2$$

where $B \in [-1,1]$ is sender's bias. Since ω does not directly enter the utility function for any player, we will refer to ω as the *payoff irrelevant* state and t as the *payoff relevant* state, henceforth.

The timing is standard: The sender chooses a partition $M = \{m_1, ...m_k\}$ of the state space $\Theta = [0,1] \times \{0,1\}$. The sender after learning the state $(t, \omega) \in \Theta$, she sends the costless message (element of her partition) m that contains (t, ω). This message is commonly observed by both receivers, but not verifiable. After observing the message, the receivers update their beliefs on t and then choose the action that maximizes their expected utility given their posterior. Note that $a^*(\Phi) \equiv E(t|\Phi)$ is the unique maximizer of $E(\mathbb{U}(a,t)|\Phi)$ if $t \sim \Phi$.

The sender knows all beliefs and utility functions. The receivers use Bayes' rule to update their beliefs based on their prior (F, q_i) and the public message m sent by the sender. However, the receivers "agree to disagree" about ω: receiver i does not update his belief given q_j. We are agnostic about the source of disagreement on the distribution of ω.[5]

Definition 1 (Equilibrium)**.** *Equilibrium consists of a partition* $M = \{m_1, ...m_k\}$ *of the state space* $\Theta = [0,1] \times \{0,1\}$ *and pair of receiver optimal actions* $a_1^*(m)$ *and* $a_2^*(m)$*, s.t.*

$$a_i^*(m) = \arg\max_a E\big(\mathbb{U}^i(a,t)\big|F, q_i, (t,y) \in m\big) \quad for\ i = 1,2$$

$$(t,\omega) \in m_i \Rightarrow \ m_i \in \arg\max_{m \in M} \mathbb{U}^s(\bar{a}(a_1^*(m), a_2^*(m)), t)$$

In equilibrium the sender does not have an incentive to misrepresent which part of the partition the realized state belongs to. Receivers assume the sender's message is truthful and best respond to their posterior beliefs. An element m_i of partition $M = \{m_1, ...m_k\}$, m_i is *one-dimensional* iff:

$$\forall i \ \exists b_i \subseteq [0,1], \ \exists s_i \subseteq \{0,1\} \ s.t. \ m_i = b_i \times s_i$$

Otherwise we say that m_i is *two-dimensional.*

We call a sender's partition *one dimensional* or equivalently we say *sender does not use the second dimension* if all elements of the partition are one-dimensional, otherwise we call a sender's a partition *two-dimensional* or we say *sender uses the second dimension.*

Now note that measure zero changes to a partition do not alter receiver optimal

[5] Similarly to Alonso and Camara (2016), Van den Steen (2010), and Morris (1994, 1995) and Anderson and Pkhakadze (2020)

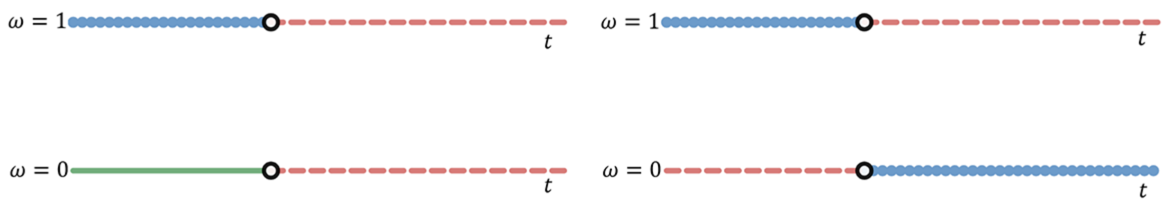

Figure 1: on the left picture, partition $\{red, green, blue\}$ ($\{dashed, solid, dotted\}$) is one-dimensional, because blue(dotted) set is $[0, a] \times \{1\}$, green(solid) set is $[0, a] \times \{0\}$ and red(dashed) set is $[a, 1] \times \{0, 1\}$ and the partition on the right picture $\{red, blue\}$ ($\{dashed,\ doted\}$) is two-dimensional

actions; and thus, do not change payoffs. But two partitions can be payoff equivalent even if they differ on a set of positive measure. Every partition uniquely defines a mapping from the payoff relevant state t to payoffs for each player. We say that two equilibrium partitions are *equivalent from the player's point of view* if they generate the same payoff mapping for that player. Two sets of equilibrium payoffs are equivalent from a player's point of view if they generate the same set of payoff mappings for that player.

Proposition 0 (Crawford and Sobel 1982)**.** *The set of one-dimensional equilibrium partitions in the current model is equivalent from all players point of view to the set of equilibrium partitions in the Crawford and Sobel cheap talk model.*[6]

The goal for the rest of the paper is to characterize two-dimensional equilibria and understand how the extra dimension changes the set of equilibrium sender values. We start by exploring a parameterized example that illustrates the core results to come.

3 An Illustrative Example

Assume the receivers believe the payoff irrelevant state is $\omega = 1$ with chances $q_1 = 0.3$ and $q_2 = 0.8$, and they have a common uniform prior on the payoff relevant state t. We claim that in this case the following two-dimensional partition is an equilibrium (illustrated in Figure 2):

$$m_1 \cup m_2 = \Big([0, 0.8) \times \{1\} \cup [0.7, 1] \times \{0\}\Big) \cup \Big([0, 0.7) \times \{0\} \cup [0.8, 1] \times \{1\}\Big)$$

[6]More precisely, the cartesian product of equilibrium partitions of standard cheap talk model and any subset of {0,1}.

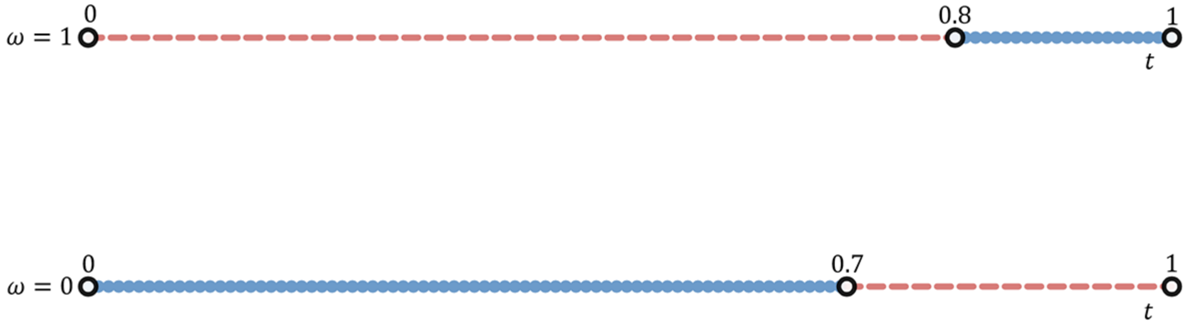

Figure 2: Partition of $\Theta = [0,1] \times \{0,1\}$. On the figure Θ is union of two lines $[0,1] \times \{1\}$ (upper line) and $[0,1] \times \{0\}$ (lower line). And partition elements are red dashed (m_1) and blue doted lines (m_2).

Given this partition, Figure 3 illustrates the posterior CDF on t for each receiver given each message, assuming the receivers believe that the message is truthful.[7]
Beliefs have polarized: Given m_1 the posterior CDF for receiver 1 first order dominates the posterior of receiver 2, and this dominance relation is reversed given message m_2.

Since each receiver's payoff is the squared difference between their action and the true state, each will set their action equal to the expectation of t given their posterior, i.e. $a_j^i = \int_0^1 t dF_j^i(t)$ is receiver $i's$ optimal action when message is m_j, which implies:

$$a_1^1 \approx 0.41, \quad a_1^2 \approx 0.64 \quad a_2^1 \approx 0.61 \quad a_2^2 \approx 0.44$$

and thus, the aggregate actions for m_1 and m_2 will be:

$$\bar{a}_1 = \frac{a_1^1 + a_1^2}{2} = \frac{1.05}{2} = 0.525 \quad and \quad \bar{a}_2 = \frac{a_2^1 + a_2^2}{2} = \frac{1.05}{2} = 0.525$$

Aggregate actions are the same for each signal, and since the sender's payoff only depends on the aggregate action and the state, no sender type can gain by announcing m_1 when the true state is in m_2 (or vice versa). Altogether, this partition and the receiver's best responses a_j^i constitute an equilibrium. And this is a *babbling equilibrium* from the point of view of the sender: her payoff is the same regardless of message sent.

But notice that the equilibrium aggregate action differs from the aggregate action that receivers would have taken had no communication taken place. That is, if the receivers had taken actions based on their prior alone; and thus, each taken action 0.5.

[7]For precise derivation of posteriors check *proof of lemma 3*

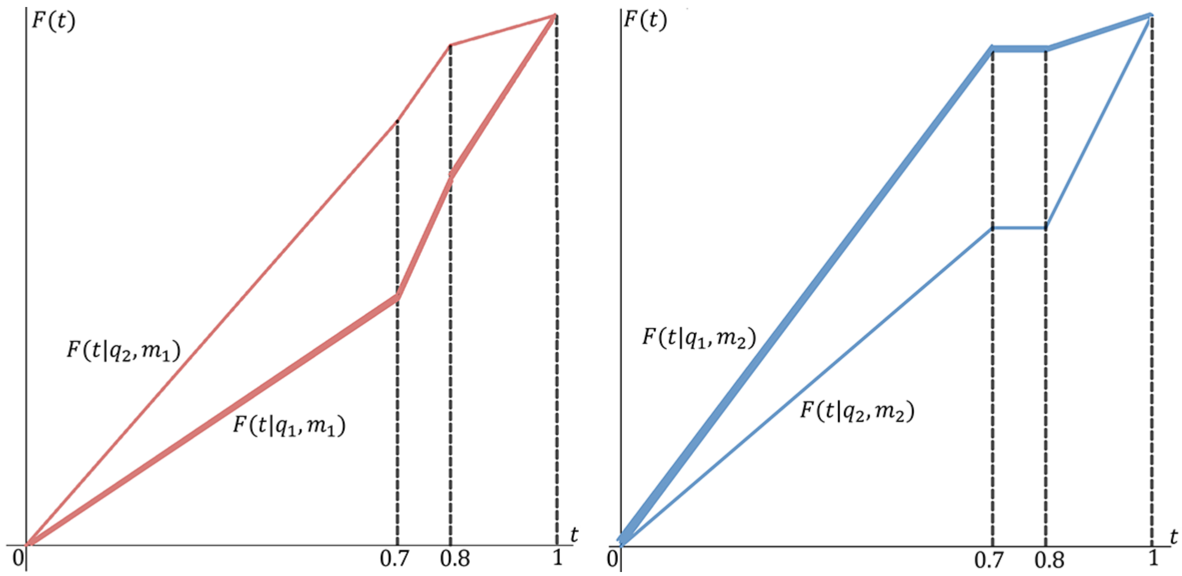

Figure 3: Receivers' posterior CDF-s when they know that the state is in m_1 (left figure with red graphs) and when they know that the state is in m_2 (right figure with blue graphs). Receiver 1's CDF is given by thick lines.

Recall that an equilibrium is *influential* if there exists at least one message in which the aggregate action differs from what the aggregate action would have been had the receivers based their action on their prior alone. In an *informative equilibrium* the receiver aggregate actions must differ across at least two messages. In the standard model; equivalently with a one-dimensional partition, an equilibrium is influential *iff* it is informative. But as we see in this example, this equivalence no longer holds, once we allow for two-dimensional partitions (trivially, informative implies influential for all types of equilibria).

Next we construct an informative two-dimensional equilibrium. To verify this, we now construct an informative equilibrium with four messages. Assume $q_1 = q = 1/3$ and $q_2 = 1 - q = 2/3$, and sender bias $B = 0.1$. For these values in standard model there is an equilibrium with two partition elements: message $t \in [0, 0.3)$ and message $t \in [0.3, 1]$, which would imply respective aggregate actions 0.15 and 0.65. Now consider instead the following partition:

$$
\begin{aligned}
m_1 &= [0, 0.23) \times \{1\} \cup [0.23, 0.315) \times \{0\} \quad (\textit{Blue solid line on figure}) \\
m_2 &= [0, 0.23) \times \{0\} \cup [0.23, 0.315) \times \{1\} \quad (\textit{Blue dotted line on figure}) \\
m_3 &= [0.315, 0.81) \times \{1\} \cup [0.81, 1) \times \{0\} \quad (\textit{Red solid line on figure}) \\
m_4 &= [0.315, 0.81) \times \{0\} \cup [0.81, 1) \times \{1\} \quad (\textit{Red dotted line on figure})
\end{aligned}
$$

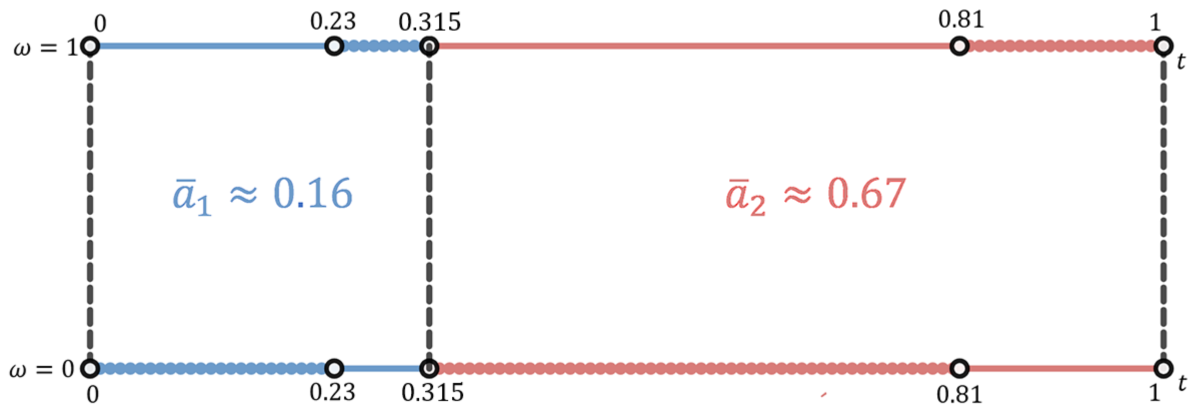

Figure 4: This equilibrium consists of 4 partition elements, blue solid, blue dotted, red solid and red dotted. Action for same partition elements is the same, but note that blue and red areas are divided with 0.31 and not 0.3 as in standard game and aggregate actions are different too

If the receivers assume these messages are truthful, we calculate that the aggregate action is constant on the blue partition elements m_1 and m_2 (i.e. $t < 0.31$) and equal to $\bar{a}_1 \approx .16$, while for the red partition elements m_3 and m_4 (i.e. $t \geq 0.31$) the aggregate action is $\bar{a}_2 \approx 0.67$. Altogether, the equilibrium is informative across signal groups $\{m_1, m_2\}$ and $\{m_3, m_4\}$, and babbling within each group.

Figure 5 compares the sender's interim payoff when $B = 0.1$ as a function of t across the four equilibria: babbling with one or two-dimensional partitions on the left and informative with one or two-dimensional partitions on the right. In each case the sender's ex-ante payoff rises when using the second dimension. In particular, in the standard one-dimensional "Babbling" equilibrium sender's average (average with F) utility is -0.093 and in standard informative with two partition elements sender's average utility is -0.041. The same utilities if she uses two dimensional signals are -0.089 and -0.037 respectively.

The partitions and aggregate actions in these examples share a common structure.

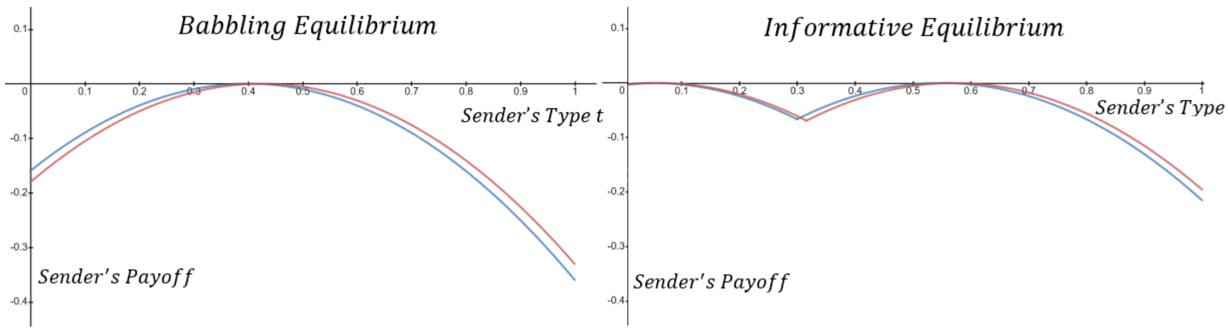

Figure 5: Sender's utility as a function of t For "babbling" (left) and informative (right) equilibrium. Red graph is sender's utility as a function of t if she uses two dimensional signal, blue is the graph of sender's utility if she uses only standard signals

In the first example even though number of partition elements is two, there is unique aggregate action, while in the second example, the partition size was four, but there were just two different aggregate actions. These examples could be interpreted as two stage message. In the first stage, the sender specifies a one dimensional partition element of type $[\alpha, \beta] \times \{0, 1\}$ similarly to the standard cheap talk model and in the second stage, the sender partitions each one dimensional element from the first stage into two dimensional partition elements so that the aggregate action on each partition element is the same. The next section establishes that all equilibria in our model have this "two stage" structure.

4 Equilibrium As a Two Stage Communication

In the previous section, we interpreted the equilibrium partition as two stage communication. This section defines the class of two stage communication protocols and shows that all equilibria are in this class.

Let us denote by $\mathbb{G}$ the operator which projects subset of state space on payoff relevant dimension. More formally,

$$\mathbb{G}(X) = \mathbb{G}\Big(\big(B_0 \times \{0\}\big) \cup \big(B_1 \times \{1\}\big)\Big) = B_0 \cup B_1$$

Where $X = \Big(\big(B_0 \times \{0\}\big) \cup \big(B_1 \times \{1\}\big)\Big) \subseteq \Theta$ is arbitrary subset of state space.

We define two stage communication using this projection operator. Recall that the second example in the previous section had 4 partition elements but only two different aggregate actions. We now consider partitions by aggregate action. Specifically, let $\{\bar{a}_1, ..., \bar{a}_\ell\}$ be the set of aggregate actions induced by partition $M = \{m_1, m_2, ..., m_k\}$ and $\forall i \in \{1, ..., \ell\}$ denote the aggregate action partition elements G_i as follows:

$$G_i = \mathbb{G}\Big(\bigcup_{j\in K_i} m^j\Big)$$

Where $\forall j \in K_i$, aggregate action induced on m_j is equal to $\bar{a}_i$.

We will say that *partition* $M = \{m_1, m_2, ..., m_k\}$ *is a two stage communication protocol* if $\forall i \in \{1, ..., \ell\}$, $int(G_i) = (x_i, x_{i+1})$ where $0 = x_1 < x_2 < ... < x_{\ell+1} = 1$. For example, in numerical example in the previous section $K_1 = \{1, 2\}$, $K_2 = \{3, 4\}$ and $\Big\{G_1, G_2\Big\} = \Big\{[0, 0.315); [0.315, 1]\Big\}$

Now we are ready to state main result of this section.

Lemma 1. *If M is equilibrium partition then M is two stage communication protocol.*

Sketch of the proof: In the first step of the proof, we show that if different partition elements induce posteriors, which give positive probability to some subset of $[0, 1]$, then the sender must be indifferent between sending these signals. We then construct the action partition elements as defined above. Finally, we show each element of the action partition is an interval on the payoff relevant dimension, i.e. $[x_i, x_{i+1}] \times \{0, 1\}$. This final step critically relies on Crawford and Sobel (1982) **Lemma 1**.

The idea for the proof is inspired by Chakraborty and Harbaugh (2010). In particular, since the sender's utility does not depend on payoff irrelevant state, her signaling strategy will be credible iff for all values of payoff relevant state t, points of state space $(t, 0)$ and $(t, 1)$ are either in the same partition element or if they are in different partition elements, Sender is indifferent between sending these two messages. Otherwise, she would always choose one with a higher payoff.

Figure 5 illustrates a two-stage communication. The first stage partitions $[0, 1]$ into $\{G_1, G_2, G_3\}$, where

$$G_1 = [0, a), \qquad G_2 = [a, b) \qquad and \quad G_3 = [b, 1]$$

Why would receivers believe that Sender is not lying? The scalars a, b, x_i, z_i are chosen so that, conditional on the receivers assuming that messages are true, the aggregate actions induced by *green dotted* and *green solid* sets are the same. Similarly, the aggregate action is the same for all red also for all blue messages. Thus, the Sender has

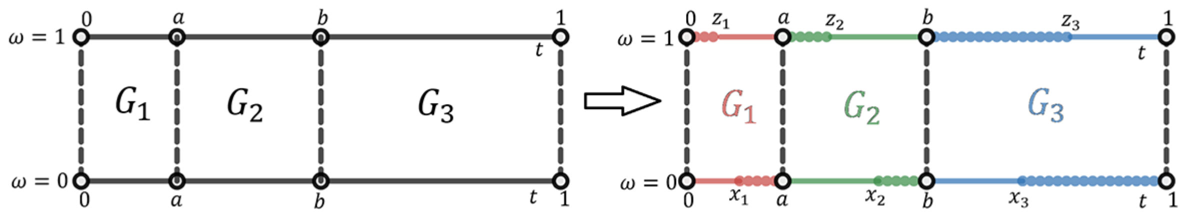

Figure 6: Final partition consists of 6 elements (red solid and dotted, green solid and dotted, blue solid and doted). Despite of 6 element in the partition, set of aggregate actions in equilibrium, consist of only 3 elements. $[0,1]$ is partitioned into 3 intervals, $red = G_1 = [0,a)$, $green = G_2 = [a,b)$ *and* $blue = G_3 = [b,1]$. Sender gets private information about $t \in [0,1]$ and $\omega \in \{0,1\}$. Just for sake of illustration assume $(t,y) \in$ *green solid*, sender will report that (t,ω) is in *green solid* element of the partition. Or it can be interpreted as follows, sender informs receivers that payoff relevant state is between a and b and if payoff irrelevant state $\omega = 1$, then t is above z_2 and if payoff irrelevant state is $\omega = 0$, then t is below x_2.

no incentive to misrepresent the state within any color. What about misrepresentations across colors? Here the idea is the same as in the canonical cheap talk model. For all payoff relevant states in the green area (for example), the Sender prefers the aggregate action induced by given truthful messages.[8] Sender will be indifferent between red and green if $t = a$ and indifferent between green and blue if $t = b$.

Denote $\{G_1 \times \{0,1\}, ..., G_\ell \times \{0,1\}\}$ by Γ and note that Γ is finest one dimensional coarsening of M and M is a refinement of Γ. Based on this, for every equilibrium partition M we will call Γ *one-dimensional coarsening of* M or simply *coarsening* and for all i, we will call $M^i = \{m^j | j \in K_i\} \subseteq M$ *two dimensional equilibrium refinement of* $G_i \times \{0,1\}$ or simply *refinement* where

$$G_i \times \{0,1\} = \bigcup_{j \in K_i} m^j$$

For our figure, $\Gamma = \{red,\ green,\ blue\}$ is a coarsening and respective $\{dotted,\ solid\}$ sets are refinements.

So we can interpret equilibrium as a two stage communication. On the first stage,

[8]Here, the language we use is not entirely correct. Equilibrium does not require that receivers need to know what will be the aggregate action. If we have assumed that q_i is not common knowledge and receivers update q_i just based on the received signal(without thinking about Sender's intentions and thinking of other receiver's q), the given example would be still an equilibrium.

the sender signals to which coarsening element G the payoff relevant state belongs to and then on the second stage sender reveals which refinement element of $G \times \{0,1\}$ is the state (t, ω). Crawford and Sobel (1982) gives all tools to work with one-dimensional coarsening of M as soon as we know for each coarsening element what aggregate action can a refinement provide on it.

Lemma 1 has a simple corollary, which gives sufficient condition for two equilibria to be equivalent from the sender's point of view.

Corollary 1. *Let M and N be two equilibrium partitions. If M and N induce the same sets of aggregate actions, then M and N are payoff equivalent for all sender types t.*

Proof Let $\{a_1, ..., a_n\}$ be set of aggregate actions induced by equilibrium partitions M and N. Assume $a_i < a_j$ when $i < j$. Then by Lemma 1 a_i is induced by every two-dimensional refinement element m_j^i of one-dimensional coarsening element $G_M^i \times \{0,1\}$ in equilibrium M and at the same time a_i is induced by all two-dimensional refinement elements n_j^i of one-dimensional coarsening element G_N^i in equilibrium N. G_N^i and G_M^i are intervals for all i.
Note that if $x_i = \sup(G_M^{i-1}) = inf(G_M^i)$ and $y_i = \sup(G_N^{i-1}) = inf(G_N^i)$ then

$$\mathbb{U}^s(a_{i-1}, x_i, B) = \mathbb{U}^s(a_i, x_i, B)$$

Given sender's utility function $x_i + B$ must be middle point of a_{i-1} and a_i. Similarly are determined y_i-s. So we get that:

$$int(G_M^i) = (x_i, x_{i+1}) = [y_i, y_{i+1}] = int(G_N^i)$$

Coarsening elements and aggregate actions on this coarsening elements are the same (not necessarily true for refinement elements), meaning that for every t sender gets the same utility in both equilibria and M and N are equivalent from her point of view.

Q.E.D.

Our main goal in the next section will be to finalize all equilibria' characterization, and for this, we will need to evaluate what aggregate actions can two-dimensional refinements provide.

5 Diagonal Partitions

In the previous sections characterized the structure of equilibrium partitions. We now introduce an important class of two-dimensional refinements. The refinements in Section 3 share a similar structure, each was of type: $t > x$ *if* $\omega = i$ *and* $t < y$ *if* $\omega = 1 - i$ *where* $i \in \{0, 1\}$.

More generally, For any interval $[\alpha, \beta] \subseteq [0, 1]$ and any $x, y \in [\alpha, \beta]$, denote the partition of $[\alpha, \beta] \times \{0, 1\} = m \cup m'$ by $\mathbb{S}^x_y(\alpha, \beta)$ and call it *diagonal partition* where

$$m = [\alpha, y) \times \{0\} \cup [x, \beta] \times \{1\} \quad \textit{and} \quad m' = [y, \beta] \times \{0\} \cup [\alpha, x) \times \{1\}$$

We will drop the α, β arguments in $\mathbb{S}^x_y(\alpha, \beta)$ when they are clear from context. An equilibrium is a *polar equilibrium*, if for every one-dimensional coarsening element G, the two-dimensional equilibrium refinement is a diagonal partition or $G \times \{0, 1\}$. *polar equilibrium* is *strictly polar equilibrium* if it is not an equilibrium of standard model.

Now we are ready to state one of the main results of the paper.

Proposition 1. *For every equilibrium, there exists aggregate action equivalent polar equilibrium to it. Thus, polar equilibria span the set of sender equilibrium payoffs.*

Proof Sketch: For standard equilibria the proposition holds trivially. For other equilibria It is enough to show the following: If for any interval $[\alpha, \beta] \subseteq [0, 1]$ with aggregate action $\bar{a}$ on every element of refinement of $[\alpha, \beta] \times \{0, 1\}$, then we can find a diagonal partition $\mathbb{S}^x_y(\alpha, \beta)$ which induces $\bar{a}$ on both partition elements. If this true, it is equivalent from sender's point of view to substitute refinement of $[\alpha, \beta] \times \{0, 1\}$ with $\mathbb{S}^x_y(\alpha, \beta)$ into the equilibrium partition. By such substitutions one by one on each coarsening element, we will end up with a polar equilibrium. It remains to show that when it comes to coarsening element G, from the sender's point of view it is enough to consider only diagonal partitions. The remainder of the proof assumes that $\bar{a}$ is an aggregate action on every partition element of $[\alpha, \beta] \times \{0, 1\}$ and then shows that there exists a diagonal partition of $[\alpha, \beta] \times \{0, 1\}$ which induces $\bar{a}$ on both partition elements.

The proof relies on the next two lemmas.

Lemma 2. *If each element m_i of partition of $[\alpha, \beta] \times \{0, 1\} = \bigcup_{i=1}^n m_i$ induces aggregate action a, then there exists partition $[\alpha, \beta] \times \{0, 1\} = m \cup m'$ inducing the same action a on both m and m'.*

Lemma 3. *If $q_1 + q_2 \leq 1$ $\big(q_1 + q_2 \geq 1\big)$, for every $x \in [\alpha, \beta]$ $\big(y \in [\alpha, \beta]\big)$, then there exist $y(x) \in [\alpha, \beta]$ $\big(x(y) \in [\alpha, \beta]\big)$ such that diagonal partition $\mathbb{S}^x_{y(x)}(\alpha, \beta)$ $\big(\mathbb{S}^{x(y)}_y(\alpha, \beta)\big)$ induces the same action on both elements of partition. Moreover,*

$$\mathbb{A} = \{a \big| \exists x, y \in [\alpha, \beta],\ s.t. \forall m \in \mathbb{S}^x_y(\alpha, \beta) \quad a \text{ is aggregate action on } m\} = [min(\mathbb{A}), max(\mathbb{A})]$$

Now return to the proof sketch. The main idea of the rest of the proof is to show that, the highest and lowest achievable aggregate actions on $[\alpha, \beta]$ can be achieved by diagonal partitions and everything in between will then be achievable by Lemma 3. Lemma 2 shows that we can restrict our attention to partitions with only two elements. Namely we can characterize partition by sets $U, D \subseteq [0, 1]$, where one element of the partition $m = U \times \{1\} \cup D \times \{0\}$ and another is $m' = \Theta \setminus m$.

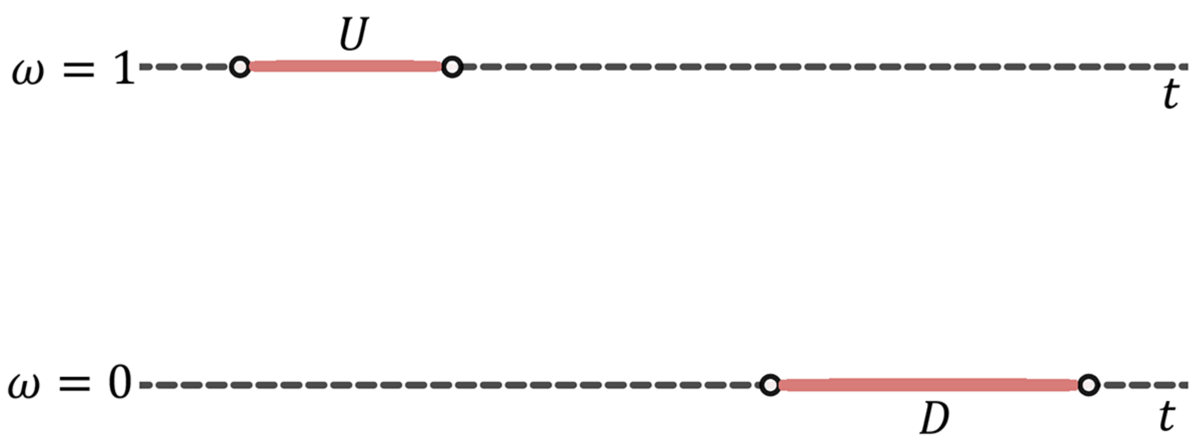

Figure 7: Partition with two elements, Union of red line segments is one partition elements, the rest is another partition element. What matters is measure of U ad D sets and average value of t on this set based on F distribution.

Furthermore, this characterization boils down to size u, d of sets U and D and μ and ν averages on U and D.

$$u = \int_U dF, \qquad d = \int_D dF \qquad \mu = \int_U \frac{t}{u} dF \qquad \nu = \int_D \frac{t}{d} dF$$

Note that in equilibrium aggregate actions must be the same:

$$\frac{q_1 u\mu + (1 - q_1)d\nu}{q_1 u + (1 - q_1)d} + \frac{q_2 u\mu + (1 - q_2)d\nu}{q_2 u + (1 - q_2)d} = \frac{\tau - (q_1 u\mu + (1 - q_1)d\nu)}{1 - (q_1 u + (1 - q_1)d)} + \frac{\tau - (q_2 u\mu + (1 - q_2)d\nu)}{1 - (q_2 u + (1 - q_2)d)}$$

We are looking for values of u, d, μ, ν, such that the above equality holds and the left-hand side (equivalently right-hand side) is maximal.

Now, assume that the highest aggregate action (same as a left-hand side) is achieved when U and D sets have given sizes u and d. In this case, finding the partition with the highest (lowest) achievable aggregate action boils down to choosing μ and ν. The Maximization (minimization) problem w.r.t μ and ν is linear with linear constraint; consequently, there is a corner solution. So, for any fixed u and d, we can WLOG assume either μ or ν is at a corner. The last step of the proof is that, if for a given u and d, either μ or ν is interior, then this u and d does not give maximum achievable action. If μ is highest, then U is the rightmost set with size u, and ν is lowest, means that D is the leftmost set with size d, meaning that the partition we consider is diagonal. This finishes the sketch of the proof of proposition 1.

Proposition 1 states that, if we consider the equivalence relation: *aggregate action equivalent* and partition the set of all equilibria into equivalence classes, in each equivalence class, we will have at least one polar equilibrium. Though there is no guarantee that the equilibrium of our game will be polar, from the sender's point of view, nothing is lost if we ignore all other equilibria.

Corollary 2. *If sender's bias isn't 0 and she is granted with ability to choose equilibrium, she will choose the one equivalent to a strict polar equilibrium.*

Proof WLOG assume $B > 0$, and consider a standard equilibrium with best ex-ante utility. Let it be a defined by a partition $0 = x_0 < x_1 <, ..., < x_n = 1$ and with action $\bar{a}_i = E(t|x_{i_1} < t < x_i)$ then $\exists \varepsilon,\ 0 < \varepsilon < B$ s.t. using diagonal partitions of $[x_i, x_{i+1}] \times \{0, 1\}$ the sender can achieve $\bar{b}_i = \bar{a}_i + \varepsilon$ on . The same $0 = x_0 < x_1 <, ..., < x_n = 1$ will define one-dimensional coarsening of equilibrium, because distance from $x_i + B$ is still middle between $\bar{b}_i$ and $\bar{b}_{i+1}$. Before state realizes the best action for sender in each partition element $[x_{i-1}, x_i)$ is $\bar{a}_i + B$, in standard equilibrium action on $[x_{i-1}, x_i)$ is $\bar{a}_i$ and in polar equilibrium we constructed, action is $\bar{b}_i = \bar{a}_i + \varepsilon$ which is by ε closer to her ideal action, thus her overall ex-ante utility will be higher in polar equilibrium than in the best standard equilibrium and she will chose the strict polar one.

Q.E.D

In the next section, we'll talk about the ordering of posterior beliefs.

6 Polarization

In Section 3 we saw that the equilibrium strategy we highlighted induced posterior beliefs on the payoff relevant variable which were ordered by First Order Stochastic Dominance. Moreover, the common prior was "between" the posteriors. We will show in this section that this result carries over to all two-dimensional polar partitions.

Say that *partition element m is (strictly polarizing* if one of the receiver's posterior on the payoff relevant variable induced by m (strictly) first order dominates the posterior of the other receiver induced by m i.e.

$$\textit{either} \quad F(t|q_1, m)(>) \geq F(t|q_2, m) \quad \textit{or} \quad F(t|q_2, m)(>) \geq F(t|q_1, m)^9$$

We'll say that *equilibrium is strictly polarizing* if every element m of equilibrium partition M is polarizing and at least one element of M is strictly polarizing, and the equilibrium is *strongly polarizing* if every element of the partition is strictly polarizing.

Every equilibrium we highlighted in Section was strongly polarizing. Recall that one dimensional partition elements induce the same posteriors, consequently if every element of the equilibrium partition is one-dimensional then there is no polarization. That is, equilibria in the standard cheap talk model are not strictly polarizing. However, this is not the case for the two-dimensional equilibria introduced here.

Proposition 2. *Every equilibrium is either equilibrium of standard model or is aggregate action equivalent to a strictly polarizing equilibrium.*

Proof The result follows from Proposition 1 and the following lemma shows that posterior beliefs induced by a diagonal partition element of any $[\alpha, \beta]$ interval are ordered by first order stochastic dominance.[10]

Lemma 4. *Let $\mathbb{S}^x_y(\alpha, \beta)$ be a diagonal partition of $[\alpha, \beta] \times \{0, 1\}$ s.t. $\{x, y\} \nsubseteq \{\alpha, \beta\}$,*

$$F(t|q_1, m) > F(t|q_2, m) \wedge F(t|q_1, m') < F(t|q_2, m')$$

Where $F(t|q_i, X) = F(t|q_i, (t, \omega) \in X)$ is receiver i-s posterior distribution after receiving signal X.

[9] $>$ here means that strict inequality holds for at least one value of t

[10] This lemma does not require any distributional assumptions or functional form restrictions for payoff.

Q.E.D.

The final result of this section provides additional restrictions on the equilibrium relationship between priors and posteriors.

Proposition 3. *Let G be a one-dimensional coarsening element of polar equilibrium and m, m' be a two-dimensional refinement elements of this polar equilibrium s.t. $m \cup m' = G \times \{0, 1\}$ then either*

$$\Big(F(t|q_1, m) \succ_{SOSD} F(t|G) \succ_{SOSD} F(t|q_2, m)\Big) \wedge \Big(F(t|q_1, m') \prec_{SOSD} F(t|G) \prec_{SOSD} F(t|q_2, m')\Big)$$

or

$$\Big(F(t|q_1, m) \prec_{SOSD} F(t|G) \prec_{SOSD} F(t|q_2, m)\Big) \wedge \Big(F(t|q_1, m') \succ_{SOSD} F(t|G) \succ_{SOSD} F(t|q_2, m')\Big)$$

7 Increasing Disagreement

In this section we'll explore how the set of equilibrium aggregate actions and sender payoffs change as the prior disagreement on the payoff irrelevant variable changes. In particular, we will see that increasing disagreement expands the set of equilibrium actions and sender payoffs.

For any payoff relevant interval $G = [\alpha, \beta] \subseteq [0, 1]$ denote by $\overline{M_G}(q_1, q_2)$ *the maximum achievable aggregate action on* $G \times \{0, 1\}$, similarly denote by $\underline{M_G}(q_1, q_2)$ *minimum achievable aggregate action on* G.

Lemma 5. *If $q_1 < q_2$, $\overline{M_G}$ is decreasing in q_1 and increasing in q_2 and $\underline{M_G}(q_1, q_2)$ is increasing in q_1 and decreasing in q_2*

Proof Sketch: For simplicity drop index G. Consider two senders. Sender 1 faces receivers with beliefs q_1, q_2 and sender 2 faces receivers with beliefs q_1', q_2, where $q_1' = q_1 - \varepsilon < q_1 < q_2$. Assume $a = \overline{M}(q_1, q_2)$ and $a' = \overline{M}(q_1', q_2)$ are the maximum aggregate actions achievable in some equilibrium. We want to show that $a' > a$. To show that M is increasing in q_2, recall that $\overline{M}(q_1, q_2) = \overline{M}(q_2, q_1) = \overline{M}(1 - q_2, 1 - q_1) = \overline{M}(1 - q_1, 1 - q_2)$, and we can repeat the proof of $a' > a$ for $\overline{M}(1 - q_2, 1 - q_1)$.
The idea of the proof is based on the construction of a partition which makes receivers of the second sender choose exactly the same action as the receivers of sender 1 choose in the equilibrium with highest aggregate action for sender 1.

The final results of this section describe how the set of relevant outcomes for the sender change with q_1 or q_2. First define the set of possible equilibrium aggregate actions. For any q_1, q_2 and B, the *set of possible aggregate actions* $\mathbb{EQ}(q_1, q_2, B)$ is a collection of all sets of aggregate actions, which can be induced in some equilibrium. More formally:

$$\mathbb{EQ}(q_1, q_2, B) = \Big\{ A \subset [0,1] |\ A = \{\bar{a}(m_1), ..., \bar{a}(m_k)\}\ where\ \{m_1, ...m_k\}$$
$$\text{is equilibrium partition and } \bar{a}(m) \text{ is aggregate action induced on m} \Big\}$$

Set of all possible equilibrium actions is set of actions which can be induced in some equilibrium with positive probability. More formally

$$\mathbb{EQ}_a(q_1, q_2, B) = \bigcup_{A \in \mathbb{EQ}(q_1, q_2, B)} A$$

Proposition 4. *If $q_1 < q_2$ then $\mathbb{EQ}(q_1, q_2, B)$ is decreasing in q_1 and increasing in q_2*

$$q_1' \leq q_1 < q_2 \leq q_2' \Rightarrow \Big(\mathbb{EQ}(q_1, q_2, B) \subseteq \mathbb{EQ}(q_1', q_2, B)\Big) \wedge \Big(\mathbb{EQ}(q_1, q_2, B) \subseteq \mathbb{EQ}(q_1, q_2', B)\Big)$$

Proof: If $A \in \mathbb{EQ}(q_1, q_2, B)$ then by Corollary 1, A generates the same one-dimensional coarsening for all values of payoff irrelevant beliefs. Now, take an arbitrary $a \in A$ then a is the action on some coarsening element G by Lemma 5 the same a could be achieved on G when believes are q_1', q_2 or q_1, q_2', where $q_1' \leq q_1 < q_2 \leq q_2'$. Since $a \in A$ was arbitrary we can repeat the same for all $a \in A$ to get that $A \in \mathbb{EQ}(q_1', q_2, B)$ and $A \in \mathbb{EQ}(q_1, q_2', B)$. This finishes the proof.

Q.E.D

One straightforward corollary of Proposition 4 is that If $q_1 < q_2$ then $\mathbb{EQ}_a(q_1, q_2, B)$ is decreasing in q_1 and increasing in q_2 too. Another important corollary is that, sender is not worse of if beliefs change from q_1, q_2 to q_1', q_2' where $q_1' \leq q_1 < q_2 \leq q_2'$.

Corollary 3. *If $q_1' \leq q_1 < q_2 \leq q_2'$, then for every equilibrium of game with beliefs q_1, q_2 there exist equilibrium of game with beliefs q_1', q_2' with the exact same interim and ex-ante utilities.*

Proof Since every $A \in \mathbb{EQ}(q_1, q_2, B)$ is repeatable in game with beliefs q_1', q_2' and

A pins down preparation and aggregate actions on this coarsening, interim and ex-ante utilities are also fully determined by A.

Q.E.D

To illustrate this result, consider an example with $q_1 = q, q_2 = 1 - q$, $F = U[0,1]$ and B high enough so that there are no equilibria with more than one one-dimensional coarsening element ($B > 0.375$ would guarantee this). When $q_1 + q_2 = 1$, then for every p partition $M = m_1 \cup m_2$ where

$$m_1 = \big([0,p) \times \{0\}\big) \cup \big([p,1] \times \{1\}\big) \quad \textit{and} \quad m_2 = \big([0,p) \times \{1\}\big) \cup \big([p,1] \times \{0\}\big)$$

Induces the same aggregate action on both m_1 and m_2, i.e. M is equilibrium partition, and these type of partitions generate all possible aggregate actions. When $q' > q > \frac{1}{2}$, $[1-q, q] \subset [1-q', q']$. On the Figure 8 is given aggregate value as a function of p for different of values of q. As we see from the graph

$$\mathbb{EQ}_a(0.7, 0.3, B) = [0.48, 0.52] \subset \mathbb{EQ}_a(0.9, 0.1, B) = [0.42, 0.58] \subset \mathbb{EQ}_a(1, 0, B) = [0.25, 0.75]$$

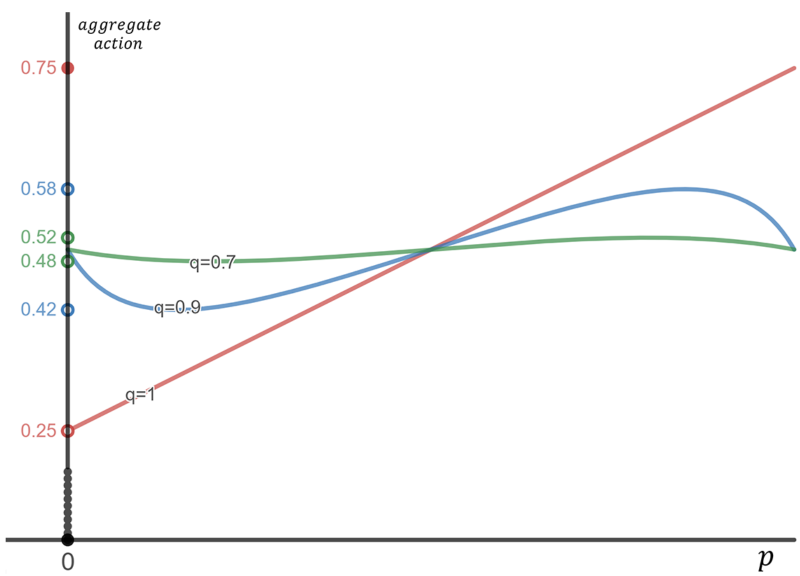

Figure 8: p is measured on horizontal axes. Further is q from 0.5, larger is the set of aggregate actions. Colored points on the vertical axes indicate highest and lowest values for each curve.

Before we conclude note that the level of bias above which informative equilibrium fail (equilibrium with more than one aggregate action) to exist, also increases with q_2 and decreases with q_1 when. Denote this threshold by $\overline{B}(q_1, q_2)$

Proposition 5. *If $B > 0$, $q'_1 \leq q_1 < q_2 \leq q'_2$ then $\overline{B}(q_1, q_2) \leq \overline{B}(q'_1, q'_2)$*

Proof Recall that border x between two coarsening elements or similarly border between two partition elements of standard cheap talk equilibrium is such that $x + B$ is middle point between actions on the left to x and on the right of x. When $B = \overline{B}(q_1, q_2)$, then $x = 0$, so B is $\frac{\bar{a}}{2}$ where $\bar{a}$ is aggregate action on $[0, 1]$. Meaning that $\overline{B}(q_1, q_2)$ is half of the highest achievable aggregate action which increases with q_2 and decreases with q_1 as shown in Lemma 5.

Q.E.D

Proposition 5 says that, when disagreement increases informative communication becomes possible for larger values of B. For example for uniform distribution the threshold value approaches $\frac{3}{8}$ when $q_1 \to 0$ and $q_2 \to 1$. in standard model it was equal to $\frac{1}{4}$.

If bias is $B = 0.25$ in standard model there is only one, babbling equilibrium. When $0.4 = q_1 = 1 - q_2$ we can construct an informative equilibrium (more than one possible aggregate action in equilibrium), however it would not be possible for $B = 0.26$. When disagreement increases it will become possible for $B = 0.26$ to have an informative equilibrium. For example belief values $0.2 = q_1 = 1 - q_2$ allows us construct such equilibrium.

8 Conclusion

We extended the standard cheap talk model in a minimal way so that it can capture the incentives of polarization. We added one receiver to be meaningful to talk about polarization, and then we added one dimension to the state space. The new dimension is binary and payoff irrelevant, and if it is ignored, the model becomes the standard cheap talk model. In contrast to payoff relevant variables, receivers have different beliefs about payoff irrelevant dimension.

This extension of the model increased the set of equilibria. We characterized new equilibria up to some equivalence relation. Namely, we proved that nothing is lost from the sender's point of view if we consider only polar equilibria. The strictly polar equilibria adds to the set of possible interim utilities the new elements. They can give higher ex-ante utility to sender. At the same time, it strictly polar equilibrium polarizes receivers with positive probability in the first-order stochastic dominance sense. Even slight disagreement is enough for influential equilibrium to exist, despite of level of Bias. Moreover, if receivers disagree more, there are more of the new equilibria with higher ex-ante utilities and the bias threshold above which there are no informative equilibria also increases.

In future work, it would be interesting to include receivers' preferences over equilibria in the analysis and think about ex-ante Pareto dominance relation on the set of equilibria. Our results depend on functional form assumptions on utility functions and aggregation rule. Quadratic loss utility allows us to work directly with expectations. What are the assumptions on utility functions and the distribution of t, which will guarantee the same results we got or at least some of them? The existence of new and influential equilibria can be guaranteed for most of the concave supermodular utility functions; however, whether the role of diagonal partitions will remain the same or not is questionable.

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Appendix

Proof of Proposition 0

Proposition 0 [Crawford and Sobel 1982] *Set of equilibrium partitions in which only one dimensional strategies are used is equivalent from all players point of view to the set of equilibrium partitions of standard cheap talk model*[11]*.*

Proof Our model is extension of standard[12] cheap talk model from Crawford and Sobel (1982), we add payoff irrelevant dimension and one receiver. As long as sender does not communicate privately, if sender will not use the payoff irrelevant dimension, there is nothing different form standard cheap talk model, since both players will process one dimensional signals in exactly the same way and posterior would be the same if they just ignore the second dimension of partition element $b_i \times s$ and update only using b_i. So their optimal response will be the same and equal to aggregate action and this means that we are back to standard cheap talk model and partition of $[0, 1]$ consisting of b_i as elements will be equilibrium partition of standard model.

Q.E.D.

Proof of Lemma 1

Lemma 1 *If M is equilibrium partition then M is a two stage communication protocol.*

Proof On the first step of the proof we will show that if different partition element induce posteriors which give positive probability to some subset of $[0, 1]$, then sender should be indifferent between sending these signals. After that we'll group all partition elements which induce same action in one group and finally we'll show that this group has form $[x_i, x_{i+1}] \times \{0, 1\}$.

Recall that two partitions which are measure 0 deviations from each other are equivalent and each of this partition is equivalent to the partition, elements of which are of type $m = \big(B_0 \times \{0\}\big) \cup \big(B_1 \times \{1\}\big)$, where B_0, B_1 are Borel sets. WLOG we will consider only partitions of this type.

One of the key step of the proof is to show that if m and m' payoff are state disjoint elements of the equilibrium partition meaning that $\mathbb{G}(m) \cap \mathbb{G}(m') \neq \emptyset$, then they induce same aggregate action.

Assume opposite. Take $t_0 \in int(\mathbb{G}(m) \cap \mathbb{G}(m'))$ (due to remark 1 in Model section, considered interior isn't empty) , and WLOG assume

$$U^s(\bar{a}(a_1^*(m), a_2^*(m)), t_0) > U^s(\bar{a}(a_1^*(m'), a_2^*(m'), t_0)$$

[11] More precisely cartesian product of equilibrium partition of standard cheap talk model and any subset of {0,1}

[12] The model they consider in section 4, where $F = U[0, 1]$ and utility function is quadratic loss functions

then if state $(t_0, \omega) \in m'$, sender has incentive to lie and signal m instead of m'. Generality is not lost, since if $\bar{a}$ and $\bar{a}'$ are different but t_0 were such that accidentally $U^s(\bar{a}, t_0) = U^s(\bar{a}', t_0)$ then we would consider $t_0 + \varepsilon \in int(\mathbb{G}(m) \cap \mathbb{G}(m'))$ instead of t_0.

Now if $A = \{\bar{a}(m) \big| m \in \{m_1, ..., m_k\}\} = \{\bar{a}_1, ..., \bar{a}_\ell\}$ is a set of actions in equilibrium we can rearrange our partition in the following way:

$$\Theta = \bigcup_{i \in \{1,...,k\}} m_i = \bigcup_{i=\{1,...,\ell\}} \Big(\bigcup_{j \in K_i} m_j \Big)$$

Where $\forall j \in K_i$, aggregate action induced on m_j is equal to $\bar{a}_i$.
In other words, we can group all partition elements on which the aggregate actions is the same.
Besides that, equilibrium actions also induce a partition of $[0, 1]$ in the following way:

$$[0, 1] = \bigcup_{j=\{1,...,\ell\}} \Big(\bigcup_{i \in \{K_j\}} \mathbb{G}(m_j^i) \Big) := \bigcup_{j=\{1,...,\ell\}} G_j$$

The fact that set of aggregated actions in equilibrium $A = \{\bar{a}_1, ..., \bar{a}_\ell\}$, is a finite set follows from **Lemma 1** in Crawford and Sobel (1982).

So an equilibrium of our game can be described as a partition of $[0, 1]$, into sets $\{G_1, ..., G_\ell\}$ and then for each $j \in \{1, ..., \ell\}$, partition of $G_j \times \{0, 1\}$ into $\{m_j^1, ..., m_j^{k_j}\}$

Since $U_{12}^S > 0$ we get that, if for states $t_1, t_2 \in G_i$, $\bar{a}_i$ is the optimal among $\{\bar{a}_1, ..., \bar{a}_\ell\}$, then it should be also optimal for $\forall t \in \big(min(t_1, t_2), max(t_1, t_2)\big)$, and consequently

$$\forall t \in (min(t_1, t_2), max(t_1, t_2)), \ t \in G_i$$

This means that $\exists z_1, z_2, \ s.t. \ G_i = [z_1, z_2] \subseteq [0, 1]$

Reinterpret once more an equilibrium, equilibrium strategy, first partition Θ into pairs of intervals,

$$\Theta = \bigcup_{j \in \{1,...,\ell\}} G_j \times \{0, 1\} = \bigcup_{i \in \{1,...,\ell\}} [z_{i-1}, z_i) \times \{0, 1\}$$

and then partitions each pair of interval into finer two-dimensional sets

$$G_j \times \{0, 1\} = [z_{j-1}, z_j) \times \{0, 1\} = \bigcup_{k \in \{1,...,k_j\}} m_j^k$$

where $\forall k \in \{1, ..., k_j\}$, message m_j^k induces same aggregated action $\bar{a}_j$
This finishes the proof proposition 1.

Q.E.D.

Proof of Lemma 2

Lemma 2 *If each element m_i of partition of $[\alpha, \beta] \times \{0, 1\} = \bigcup_{i=1}^{n} m_i$ induces aggregate action a, then there exists partition $[\alpha, \beta] \times \{0, 1\} = m \cup m'$ inducing same action a on both m and m'.*

proof: *Forthcoming ...*

Proof of Lemma 3

Lemma 2 *If $q_1 + q_2 \leq 1$ $\big(q_1 + q_2 \geq 1\big)$, for every $x \in [\alpha, \beta]$ $\big(y \in [\alpha, \beta]\big)$, there exist $y(x) \in [\alpha, \beta]$ $\big(x(y) \in [\alpha, \beta]\big)$ such that diagonal partition $\mathbb{S}^x_{y(x)}(\alpha, \beta)$ $\big(\mathbb{S}^{x(y)}_y(\alpha, \beta)\big)$ induces same action on both elements of partition. Moreover*

$$\mathbb{A} = \{a \big| \exists x, y \in [\alpha, \beta],\ s.t. \forall m \in \mathbb{S}^x_y(\alpha, \beta) \quad a \text{ is aggregate action on } m\} = [a_1, a_2]$$

Where $a_1 = min(\mathbb{A})$ and $a_2 = max(\mathbb{A})$

proof: WLOG assume that $q_1 + q_2 \leq 1$ and $q_1 < q_2$. Let us consider the following partition of $[0, 1] \times \{0, 1\}$:

$$m = [x, \beta] \times \{1\} \cup [\alpha, y) \times 0 \quad and \quad m' = [\alpha, x) \times \{1\} \cup [\beta, 1] \times 0$$

And prove that for $\forall x \in [\alpha, \beta],\ \exists y \in [\alpha, \beta]$ s.t.

$$\bar{a}(a^*_1(m(x, y)), a^*_2(m(x, y))) = \bar{a}(a^*_1(m'(x, y)), a^*_2(m'(x, y)))$$

Or we want to show that $\forall x \in [\alpha, \beta],\ \exists y \in [\alpha, \beta]\ \ s.t.\ \ g(x, y) = 0$ where:

$$g(x, y) = \bar{a}(a^*_1(m(x, y)), a^*_2(m(x, y))) - \bar{a}(a^*_1(m'(x, y)), a^*_2(m'(x, y)))$$

Note that, for quadratic loss function optimal action for a receiver is equal to expectation of the state according to posterior distribution. Also continuity of g is obvious.

Now note that if $\Pi_i = Pr((t, \omega) \in m | q_i, F)$ and $\tau = E(t|F)$, then from Bayes plausibility

$$\Pi_i a^*_i(m(x, y)) + (1 - \Pi_i) a^*_1(m'(x, y)) = \tau$$

In other words expectation of posterior mean should equal to prior mean.

If $F = \mathbb{U}[0, 1]$ is uniform on $[0, 1]$, then

$$a^*_i(m(x, y)) = \frac{q_i(1 - x) \cdot \frac{1+x}{2} + (1 - q_i) y \cdot \frac{y}{2}}{q_i(1 - x) + (1 - q_i) y}$$

$$a_i^*(m'(x,y)) = \frac{q_i x \cdot \frac{x}{2} + (1-q_i)(1-y)\cdot\frac{1+y}{2}}{q_i x + (1-q_i)(1-y)}$$

For uniform distribution above condition is quartic equation which is solvable in closed forms, for any q_1 and q_2. other distributions we should not expect it to be solvable in closed forms. We won't solve it but show that solution exists. For this let us draw level sets of g corresponding level 0.

$$g(x,y) = \frac{a_1^*(m) + a_2^*(m)}{2} - \frac{a_1^*(m') + a_2^*(m')}{2}$$

On Figure 9 we have graphs for fixed q_1, q_2 and for three different distributions, triangle distribution with density $f(t) = 2 - 2t$, uniform on $[0,1]$ and triangle distribution with density $f(t) = 2t$.

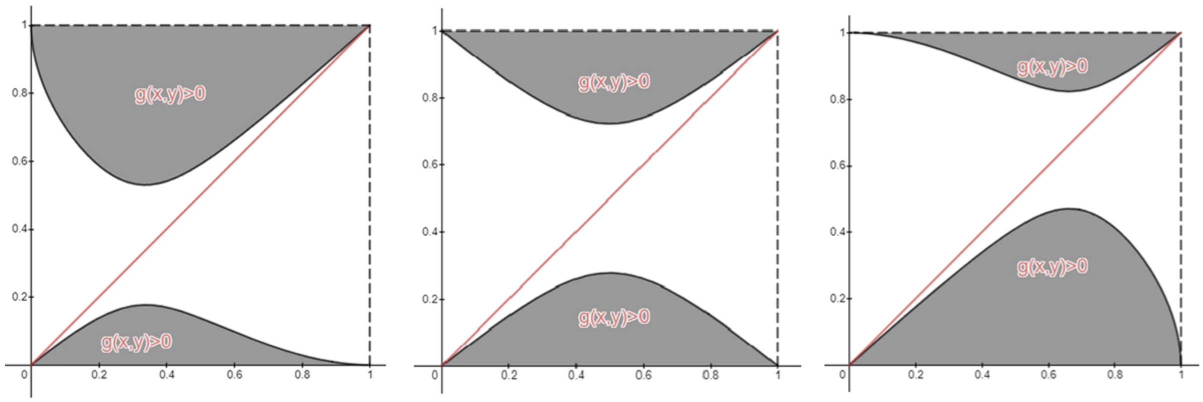

Figure 9: Grey areas are where $g > 0$, white areas where $g < 0$ and on the border we have equality. This demonstrate existence of equilibrium for these particular distributions.

As we see in all three cases for every x, there is y such that m and m' induce same aggregate action. Here $q_1 + q_2 < 1$. What we see on the graph besides existence of solution is that $y = x$ line never intersects the set $\{(x,y)|g(x,y) > 0\}$ and $g(x,0) > 0$ and $g(x,1) > 0$ ($(x,0)$, $x, 1$ are in grey area). The proof is based on this observation and not on actually solving equation derived from equilibrium condition.

For general F, it will be more convenient if we rewrite the problem in other variables, namely in u and d, which are the measures of $[x,\beta]$ and $[\alpha,y)$ sets, respectively.

$$u = \int_x^\beta dF \qquad d = \int_\alpha^y dF \Rightarrow x = Q(1-u) \quad and \quad y = Q(d)$$

Where $Q = F^{-1}$ is inverse of CDF F i.e. quantille function

And receivers actions are:

$$a_i^*(m(u,d)) = \frac{q_i u \cdot \int_{Q(1-u)}^{\beta} \frac{t}{u} dF + (1-q_i)d \cdot \int_{\alpha}^{Q(d)} \frac{t}{d} dF}{q_i u + (1-q_i)d}$$

$$a_i^*(m'(u,d)) = \frac{q_i(1-u) \cdot \int_0^{1-Q(1-u)} \frac{t}{1-u} dF + (1-q_i)(1-d) \cdot \int_{Q(d)}^{\beta} \frac{t}{1-d} dF}{q_i(1-u) + (1-q_i)(1-d)}$$

Let us rewrite it once more in terms of x and y

$$a_i^*(m(x,y)) = \frac{q_i u(x) \cdot \mu(x) + (1-q_i)d(y)\nu(y)}{q_i u(x) + (1-q_i)d(y)}$$

$$a_i^*(m'(x,y)) = \frac{q_i(1-u(x)) \cdot \mu'(x) + (1-q_i)(1-d(y)) \cdot \nu'(y)}{q_i(1-u(x)) + (1-q_i)(1-d(y))}$$

Where

$$\mu(x) = \int_x^{\beta} \frac{t}{1-F(x)} dF \qquad \nu(y) = \int_0^{y} \frac{t}{F(y)} dF,$$

$$\mu'(x) = \int_0^{1-x} \frac{t}{F(x)} dF, \quad \nu'(y) = \int_{1-y}^{1} \frac{t}{1-F(y)} dF$$

We'll omit argument in μ, ν, u, d whenever it's clear what is the argument.
Note that Bayesian plausibility[13] implies $u\mu + (1-u)\mu' = \tau$ and $d\nu + (1-d)\nu' = \tau$, Using this we can get rid of μ', ν' in $g(x,y)$

$$g(x,y) = \frac{q_1 u\mu + (1-q_1)d\nu}{q_1 u + (1-q_1)d} + \frac{q_2 u\mu + (1-q_2)d\nu}{q_2 u + (1-q_2)d} -$$

$$-\frac{\tau - (q_1 u\mu + (1-q_1)d\nu)}{1-(q_1 u + (1-q_1)d)} - \frac{\tau - (q_2 u\mu + (1-q_2)d\nu d)}{1-(q_2 u + (1-q_2)d)}$$

Now our goal would be to show that $\forall x \ \exists y_1, y_2$ s.t. $g(x,y_1) > 0 > g(x,y_2)$. More precisely we'll find two different y_1-s.
For arbitrary $x \in [\alpha, \beta]$ consider three different y-s, $y=0$, $y=1$ and $y=x$. m in these three cases looks as on the picture bellow.

What will be sign of $g(x,y)$ for different y-s above? .
Let's start from $g(x,0)$. Both receivers' action for partition element m is

$$a_1(m(x,0)) = a_2(m(x,0)) = \mu(x) = \int_x^{\beta} \frac{t}{1-F(x)} dF > \tau$$

From Bayesian plausibility we'll get that actions of both receivers in m' will be less than τ, hence $g(x,0) > 0$.

[13] see Kamenica and Gentzkow (2011)

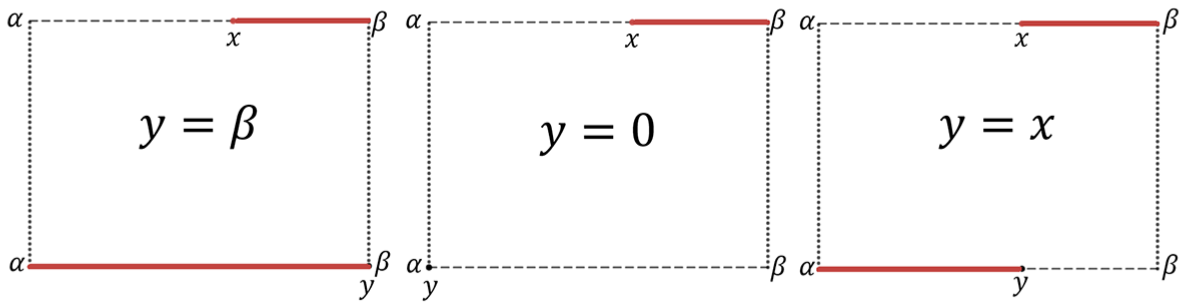

Figure 10: Three different y-s, $y = \beta, y = \alpha$ and $y = x$

Similarly for $g(x, 1)$, both receivers' action for partition element m' is

$$a_1(m'(x,1)) = a_2(m'(x,1)) = \mu'(x) = \int_0^{1-x} \frac{t}{F(x)} dF < \tau$$

From Bayesian plausibility we'll get that actions of both receivers in m will be more than τ, hence $g(x, 1) > 0$

And now the most difficult and main part, $y = x$, what will be the sign of $g(x, x)$? As we saw for uniform and triangle distributions $g(x, x) < 0$, does this hold for other distributions too?
We need to compare $a_1^*(m(x,x)) + a_2^*(m(x,x))$ to $a_1^*(m'(x,x)) + a_2^*(m'(x,x))$. Recall that:

$$2\bar{a}(m) = a_1^*(m(x,x)) + a_2^*(m(x,x)) = \frac{q_1 u\mu + (1-q_1)(1-u)\nu}{q_1 u + (1-q_1)(1-u)} + \frac{q_2 u\mu + (1-q_2)(1-u)\nu}{q_2 u + (1-q_2)(1-u)}$$

and

$$2\bar{a}(m') = a_1^*(m'(x,x)) + a_2^*(m'(x,x)) = \frac{q_1(1-u)\nu + (1-q_1)u\mu}{q_1(1-u) + (1-q_1)u} + \frac{q_2(1-u)\nu + (1-q_2)u\mu}{q_2(1-u) + (1-q_2)u}$$

Where

$$\mu = \int_x^\beta \frac{t}{1-F(x)} dF \qquad \nu = \int_0^x \frac{t}{F(x)} dF$$

Now note that both $\bar{a}(m)$ and $\bar{a}(m')$ are weighted sums of μ and ν where $\mu > \nu$, thus $\bar{a}(m)$ will be higher if it gives higher weight to μ and lower if it gives lower weight to μ. $g(x, x) \leq 0$ iff $\bar{a}(m) \leq \bar{a}(m')$. Weights of μ in $\bar{a}(m)$ and $\bar{a}(m')$ are as follows:

$$w = \frac{q_1 u}{q_1 u + (1-q_1)(1-u)} + \frac{q_2 u}{q_2 u + (1-q_2)(1-u)}$$

and

$$w' = \frac{(1-q_1)u}{q_1(1-u)+(1-q_1)u} + \frac{(1-q_2)u}{q_2(1-u)+(1-q_2)u}$$

And we want to show that $w \le w'$

$$w \le w'$$

$$\Updownarrow$$

$$\frac{q_1u}{q_1u+(1-q_1)(1-u)}+\frac{q_2u}{q_2u+(1-q_2)(1-u)} \le \frac{(1-q_1)u}{q_1(1-u)+(1-q_1)u}+\frac{(1-q_2)u}{q_2(1-u)+(1-q_2)u}$$

$$\Updownarrow \qquad (divide\ both\ sides\ on\ u,\ if \quad u=0, \quad then \quad w=w')$$

$$\frac{q_2}{q_2u+(1-q_2)(1-u)}-\frac{(1-q_2)}{q_2(1-u)+(1-q_2)u} \le \frac{(1-q_1)}{q_1(1-u)+(1-q_1)u}-\frac{q_1}{q_1u+(1-q_1)(1-u)}$$

$$\Updownarrow$$

$$\frac{(1-u)(q_2^2-(1-q_2)^2)}{\big(q_2u+(1-q_2)(1-u)\big)\big(q_2(1-u)+(1-q_2)u\big)} \le \frac{(1-u)((1-q_1)^2-q_1^2)}{\big(q_1(1-u)+(1-q_1)u\big)\big(q_1u+(1-q_1)(1-u)\big)}$$

Now note that $q_1+q_2<1$ and $q_1<q_2$, meaning that $q_1<\frac{1}{2}$, hence $q_1<1-q_1$. If $q_2\le\frac{1}{2}$, than left hand side of the last inequality is non-positive and right hand side is positive, so inequality holds. So consider only $q_2>\frac{1}{2}$. Also divide both sides of inequality on $1-u$. If $u=1$ (weak)inequality holds.

$$\Updownarrow$$

$$\begin{aligned} &\frac{2q_2-1}{u(1-u)(q_2^2+(1-q_2)^2)+(u^2+(1-u)^2)(q_2(1-q_2))} \le \\ &\le \frac{1-2q_1}{u(1-u)(q_1^2+(1-q_1)^2)+(u^2+(1-u)^2)(q_1(1-q_1))} \end{aligned} \tag{1}$$

To prove (1) first note that $2q_2-1$ cannot be more than $1-2q_1$ since

$$2q_2-1>1-2q_1 \Rightarrow 2q_2+2q_1>2\ contradicts\ q_1+q_2<1.$$

So, inequality (1) will be proved if we show that denominator of the right hand side is smaller than denominator of left hand side.

$$(1)$$

$$\Uparrow \quad (using\ 2q_1-1<1-2q_1)$$

$$u(1-u)(q_2^2+(1-q_2)^2)+(u^2+(1-u)^2)(q_2(1-q_2)) \ge u(1-u)(q_1^2+(1-q_1)^2)+(u^2+(1-u)^2)(q_1(1-q_1))$$

$$\Updownarrow$$

$$u(1-u)\Big(q_2^2-q_1^2+(1-q_2)^2-(1-q_1)^2\Big)+(u^2+(1-u)^2)\Big(q_2(1-q_2)-q_1(1-q_1)\ge 0$$

$$\Updownarrow$$

$$u(1-u)\Big((q_2-q_1)\big(2(q_1+q_2)-2\big)\Big)+(u^2+(1-u)^2)\Big((q_2-q_1)\big(1-(q_1-q_2)\big)\Big)\geq 0$$

$$\Updownarrow$$

$$-u(1-u)\Big((q_2-q_1)\big(2-2(q_1+q_2)\big)\Big)+\frac{u^2+(1-u)^2)}{2}\Big((q_2-q_1)\big(2-2(q_1-q_2)\big)\Big)\geq 0$$

$$\Updownarrow$$

$$\Big((q_2-q_1)\big(2-2(q_1+q_2)\big)\Big)\Big(\frac{u^2+(1-u)^2)}{2}-u(1-u)\Big)\geq 0$$

$$\Updownarrow \ (using\ q_1<q_2\ and\ q_1+q_2<1)$$

$$\frac{u^2+(1-u)^2)}{2}-u(1-u)\geq 0 \Leftrightarrow \frac{u^2-2u(1-u)+(1-u)^2)}{2}\geq 0$$

$$\Updownarrow$$

$$\Big(u-(1-u)\Big)^2\geq 0\ holds\ \forall u$$

This finishes the proof of inequality (1) which in turn means that $w \leq w'$, implying that $\bar{a}(m(x,x)) \leq \bar{a}(m'(x,x))$, hence $g(x,x) \leq 0$.
$\forall x,\ g(x,y)$ as function of y is continuous and we got that $g(x,0)>0$ and $g(x,1)>0$ and $g(x,x)\leq 0$ and using intermediate value theorem we can say that $\exists y_1, y_2$ s.t. $g(x,y_1)=g(x,y_2)=0$ and if $g(x,x)=0$ than $x=y_1=y_2$.
So we proved that $\forall x \in [\alpha,\beta]$ there is at least one y s.t. aggregate actions on m and m' are the same, where

$$m=[x,\beta]\times\{1\}\cup[\alpha,y)\times 0 \ \ and \ \ m'=[\alpha,x)\times\{1\}\cup[\beta,1]\times 0$$

if $q_1+q_2>1$ we need to just reverse roles of x and y and repeat the proof.

Q.E.D

Proof of Proposition 1

Proposition 1 *For every equilibrium there exists aggregate action equivalent polar equilibrium to it.*
Or in other words, polar equilibria are enough from sender's point of view.

Proof First we will show that:
Considering only one coarsening interval $[\alpha,\beta]$ is enough.
Note that it's enough to show the following. If $\bar{a}$ is aggregate action on arbitrary interval $[\alpha,\beta]$ for every element of two-dimensional refinement element of $[\alpha,\beta]\times\{0,1\}$, then we can find a diagonal partition $\mathbb{S}^x_y(\alpha,\beta)$ which induces $\bar{a}$ on both partition elements. If this true, it's equivalent from sender's point of view to substitute refinement

of $[\alpha, \beta] \times \{0,1\}$ with $\mathbb{S}_y^x(\alpha, \beta)$ in equilibrium partition. By such substitutions one by one on each one-dimensional coarsening element we will end up with polar equilibrium. So remains to show that when it comes to coarsening element G, from sender's point of view it's enough to consider only diagonal partitions. For remainder of the proof assume that $\bar{a}$ is a aggregate action on every partition element of $[\alpha, \beta] \times \{0,1\}$ and we want to show that there exists diagonal partition of $[\alpha, \beta] \times \{0,1\}$ which induces $\bar{a}$

It's enough to show that we can achieve highest and lowest possible aggregate action with just diagonal partitions

WLOG assume that $q_1 + q_2 \leq 1$ and note that from Lemma 2, we have that diagonal partitions can generate any aggregate in $[a_1, a_2]$, where a_1 is minimum achievable value of aggregate action induced by some diagonal partition a_2 is maximum achievable value of aggregate action induced by some diagonal partition. If we show that for any aggregate action $\bar{a}$ which can be induced by some partition, $a_1 \leq \bar{a} \leq a_2$, then $\bar{a}$ also can be achieved by diagonal partition.

Based on Lemma 1 we can claim that for any aggregate equilibrium action a there exist partition N, N'

$$[\alpha, \beta] \times \{0,1\} = N \cup N'$$

s.t. induced aggregate action on both N and N' is a.

Rewriting the problem in other variables

Assume that

$$N = U \times \{1\} \cup D \times \{0\}$$

Where U and D are measurable subsets of $[\alpha, \beta]$ with measures $u \in [\alpha, \beta]$ and $d \in [\alpha, \beta]$ respectively. Let us denote receiver i's perceived probability of state being in N by Π_i,

$$\Pi_i = q_i u + (1 - q_i) d$$

Now introduce the following notations:

$$\mu = E(t|t \in U) \ \ and \ \ \nu = E(t|t \in D)$$

μ is average on U and ν is average on D. Note that, μ and ν are the same for both receivers.

Now note that by Bayesian plausibility

$$\mu \times u + \mu' \times (1 - u) = \tau \ \ and \ \ \nu \times d + \nu' \times (1 - d) = \tau$$

where:

$$\mu' = E(t|t \notin U) \ \ and \ \ \nu' = E(t|t \notin D)$$

So

$$\mu' = \frac{\tau - \mu u}{1 - u} \ \ and \ \ \nu' = \frac{\tau - \nu d}{1 - d}$$

Receiver i's action a_i when state is in N is:

$$a_i(\mu,\nu,u,d)=\frac{q_i u\mu+(1-q_i)d\nu}{q_i u+(1-q_i)d}$$

And when state is in N' receiver i's action is:

$$a'_i(\mu,\nu,u,d)=\frac{q_i(1-u)\mu'+(1-q_i)(1-d)\nu'}{q_i(1-u)+(1-q_i)(1-d)}=\frac{q_i(1-u)\frac{\tau-\mu u}{1-u}+(1-q_i)(1-d)\frac{\tau-\nu d}{1-d}}{q_i(1-u)+(1-q_i)(1-d)}=$$

$$=\frac{\tau-(q_i u\mu+(1-q_i)d\nu)}{1-(q_i u+(1-q_i)d)}$$

If $\{N,N'\}$ is equilibrium partition then aggregate action for both N and N' must be the same:

$$\bar{a}(\mu,\nu,u,d)=\bar{a}'(\mu,\nu,u,d)\Rightarrow\frac{a_1(\mu,\nu,u,d)+a_2(\mu,\nu,u,d)}{2}=\frac{a'_1(\mu,\nu,u,d)+a'_2(\mu,\nu,u,d)}{2}\Rightarrow$$

$$\Rightarrow\frac{q_1u\mu+(1-q_1)d\nu}{q_1u+(1-q_1)d}+\frac{q_2u\mu+(1-q_2)d\nu}{q_2u+(1-q_2)d}=\frac{\tau-(q_1u\mu+(1-q_1)d\nu)}{1-(q_1u+(1-q_1)d)}+\frac{\tau-(q_2u\mu+(1-q_2)d\nu)}{1-(q_2u+(1-q_2)d)}$$

Note that, $\bar{a}$ depends on μ,ν,u,d, on the lengths of U and D sets and on averages on these sets not on set itself (Any two set with identical size and average will give same aggregate action).
Let us fix u and d and consider $\bar{a}$ as function of μ and ν. What can we say about μ and ν if are looking for extreme values (highest and lowest possible) of $\bar{a}$?

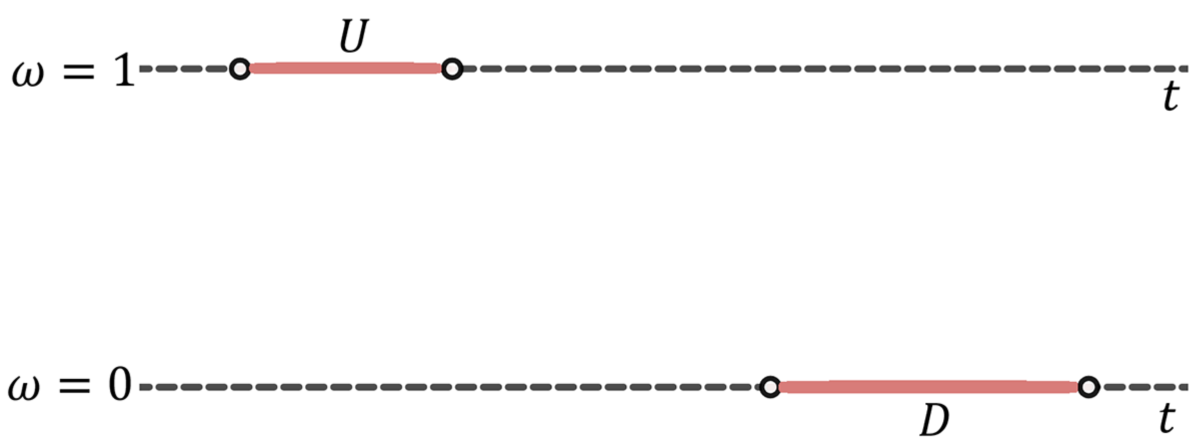

Figure 11: strategy with two partition elements, the one element of the partition is is union of U and D sets and another is the rest of the Θ.

Set N looks something like $U\cup D$ on picture above and U and D does not have to be intervals, but WLOG we can assume that they are intervals.

So, the questions is for given sizes of sets U and D (u and d respectively), what are the values of μ and ν which maximizes or minimizes $\bar{a}$.
One obvious constraint on μ and ν are imposed by distribution and sizes of sets, namely:

$$L(u) \leq \mu \leq R(u)$$

$$L(d) \leq \nu \leq R(d)$$

Where

$$L(u) = \int_{\alpha}^{Q(u)} \frac{t}{u} dF \quad L(d) = \int_{\alpha}^{Q(d)} \frac{t}{d} dF \quad R(u) = \int_{Q(1-u)}^{\beta} \frac{t}{u} dF \quad R(d) = \int_{Q(1-d)}^{\beta} \frac{t}{d} dF$$

Where Q is quantile function. For uniform distribution constraint looks as follows:

$$\frac{u}{2} \leq \mu \leq 1 - \frac{u}{2}$$
$$\frac{d}{2} \leq \nu \leq 1 - \frac{d}{2}$$

Another constraint on μ and ν is equilibrium condition ($\bar{a} = \bar{a}'$) and finally we get following optimization problem:

$$\begin{aligned} \max_{\mu,\nu} \ \bar{a} \ \ or \ \ \min_{\mu,\nu} \ \bar{a} \\ subject \ \ to: \\ \bar{a_1}(\mu,\nu) + \bar{a_2}(\mu,\nu) = \bar{a_1}'(\mu,\nu) + \bar{a_2}'(\mu,\nu) \\ L(u) \leq \mu \leq R(u) \\ L(d) \leq \nu \leq R(d) \end{aligned} \tag{2}$$

For fixed u and d optimization problem is linear
Note that a_1, a_2, a_1', and a_2' are linear in μ and ν. So the objective function $\bar{a}$ is linear in μ and ν and constraint $\bar{a_1}(\mu,\nu) + \bar{a_2}(\mu,\nu) = \bar{a_1}'(\mu,\nu) + \bar{a_2}'(\mu,\nu)$ is also linear in μ and ν. So we can rewrite our optimization problem in the following way:

$$\begin{aligned} \max_{\mu,\nu} \ \ or \ \ \min_{\mu,\nu} \ \ A_1\mu + A_2\nu \\ subject \ \ to: \\ B_1\mu + B_2\nu = B \\ L(u) \leq \mu \leq R(u) \\ L(d) \leq \nu \leq R(d) \end{aligned} \tag{3}$$

Where A_1, A_2, B_1, B_2 and B depend on u, d and q_1, q_2, but are constants w.r.t μ, ν. Also note that $A_1 + A_2 = 2$ and $(\nu, \mu) = (\tau, \tau)$ always satisfy $B_1\mu + B_2\nu = B$, meaning that $B = \tau(B_1 + B_2)$. Our optimization problem rewrites as follows:

$$\begin{aligned} \max_{\mu} \quad or \quad \min_{\mu} \quad A_1\mu + A_2 \times \frac{B - B_1\mu}{B_2} = \mu(A_1 - \frac{A_2 B_1}{B_2}) + \frac{A_2 B}{B_2} \\ subject \;\; to: \\ L(u) \leq \mu \leq R(u) \\ L(d) \leq \frac{B - B_1\mu}{B_2} \leq R(d) \end{aligned} \tag{4}$$

Depending on the sign of $A_1 - \frac{A_2 B_1}{B_2}$, for maximum μ should be largest possible or smallest possible and reversed for minimum.
If for extreme value of μ ($L(u)$ or $R(u)$), $L(d) \leq \frac{B-B_1\mu}{B_2} \leq R(d)$ holds, then up to difference of measure 0, $U = [0, y]$ or $U = [x, 1]$ where $y = Q(u), x = Q(1 - u)$.

Now, if for extreme values of μ ($L(u)$, or $R(u)$), $L(d) \leq \frac{B-B_1\mu}{B_2} \leq R(d)$ does not hold then μ should be at highest or lowest possible (depending on sign of $A_1 - \frac{A_2 B_1}{B_2}$ and whether we maximize or minimize)for which $L(d) \leq \frac{B-B_1\mu}{B_2} \leq R(d)$ condition holds with equality on one side and this means that ν will be at extreme, meaning that up to difference of measure 0, $D = [0, y]$ or $D = [x, 1]$ where $y = Q(d), x = Q(1 - d)$

All in all conclusion here is that for fixed values of measure of U and D, to reach highest and lowest values of $\bar{a}$, at least one of U and D should be extreme right interval $[x, 1]$, or extreme left interval $[0, y]$.

Possible scenarios of solution of above linear programming problem
Figure 12 summarizes 4 possible cases when only U or D is extreme right or left interval, i.e. when only one of μ, ν reaches maximum possible and is not constrained with equilibrium condition. For convenience graphs are made for uniform distribution (what changes with distribution are highest and lowest possible values of μ and ν and average τ),

Negatively sloped red line is constraint corresponding to the equilibrium condition (aggregate action on both N and N' should be the same). Negatively sloped green lines are indifference curves of sender's utility function (as a function of μ and ν for given u and d), and depending on relation of slopes of green line and red line point of minimum and point of maximum is determined. On 1 and 2, at optimal points μ reaches its highest and lowest possible points ($\mu = R(u)$ or $\mu = L(u)$ meaning that U is extreme right or left interval) but ν does not and it's reversed on 3 and 4 (meaning that D is extreme right or left interval)
Note that these lines are drawn for fixed u and d, choosing μ and ν is equivalent to choosing location of sets of sizes u and d. Slopes of objective function $(A_1\mu + A_2\nu)$ and constraints (both equilibrium constraint and distributional limitation of μ and ν) changes with u and d.

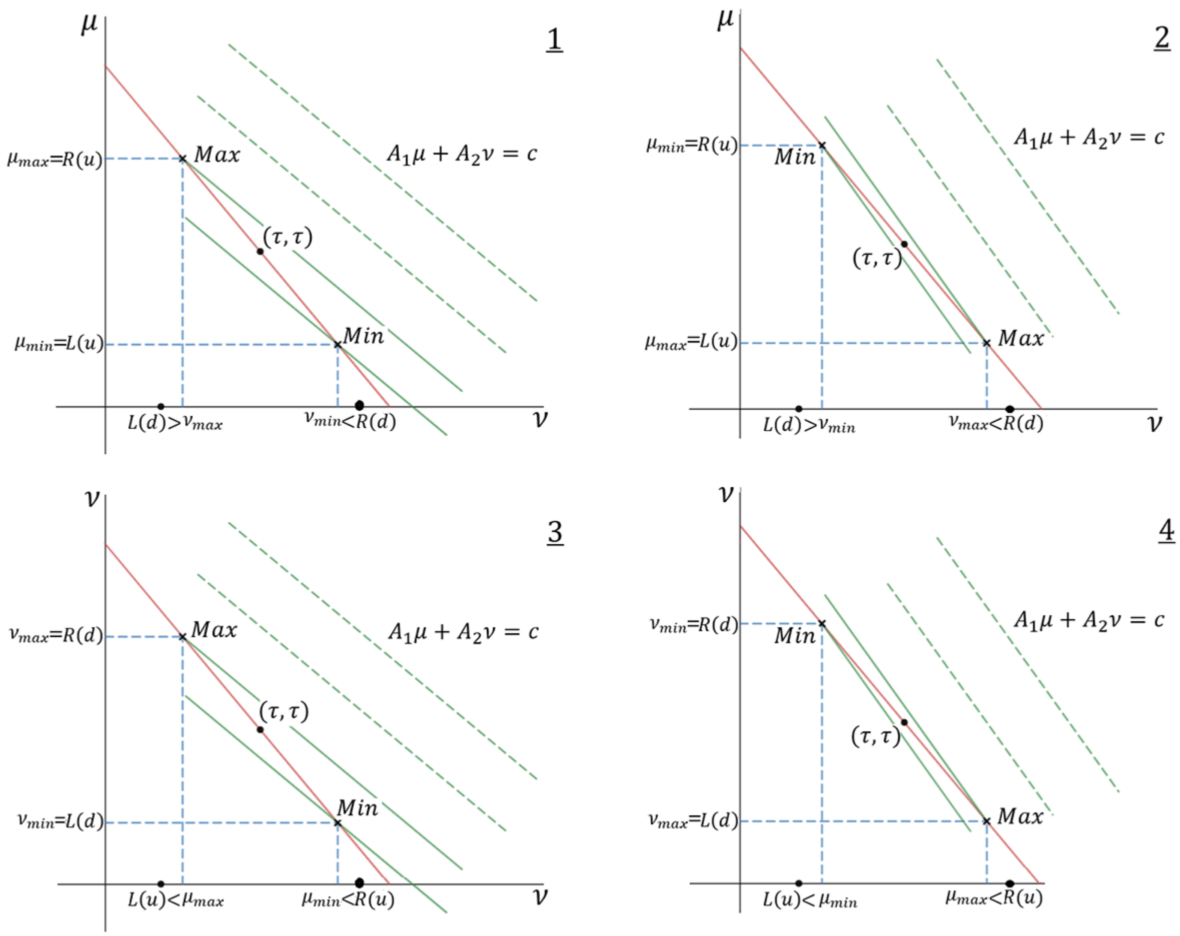

Figure 12: For fixed lengths u and d of U and D sets, optimization problem becomes linear program w.r.t μ and ν

Idea of the proof is to show that for none of the cases (1,2,3,4) on the picture, can describe scenario for optimal u and d. Instead it must look like 5 or 6 on the Figure 13 bellow.

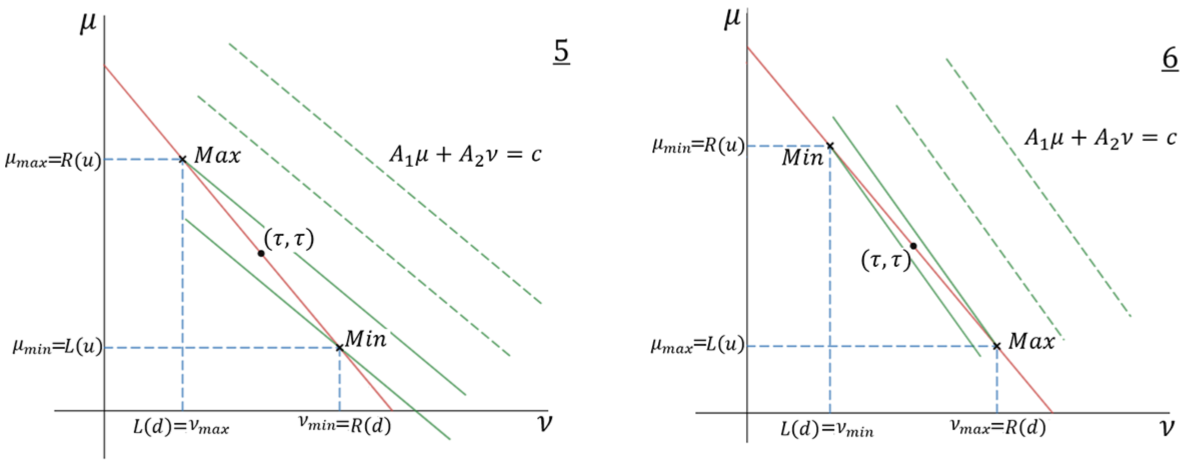

Figure 13: Both, μ and ν bind

On (5) and (6) both distributional constraints bind. One up and another down.

$$L(u) < \mu = R(u) \quad and \quad L(d) = \nu < R(d)$$

or

$$L(u) = \mu < R(u) \quad and \quad L(d) < \nu = R(d)$$

WLOG we can ignore cases 3 and 4, since it will differ from 1 and 2 just by labeling. We will show that for optimal u and d graph 1 on the picture above is impossible (case for graph 2 is analogous and we'll omit it) and instead we have (5).

Construction profitable perturbation when either μ or ν isn't binding
For this assume that case is 1 and look for perturbation of u and d s.t. sender can achieve higher aggregate action.
For clarity let us draw case 1 again.

This perturbation is simply marginal increase of d and keeping u the same. With increase of d constraint becomes steeper. We will show this little bit later, for the moment assume so. What happens' with A_1, A_2?.Recall the expressions for A_1, A_2

$$A_1 = \frac{q_1 u}{q_1 u + (1-q_1)d} + \frac{q_2 u}{q_2 u + (1-q_2)d)} = \frac{q_1 u}{\Pi_1} + \frac{q_2 u}{\Pi_2}$$

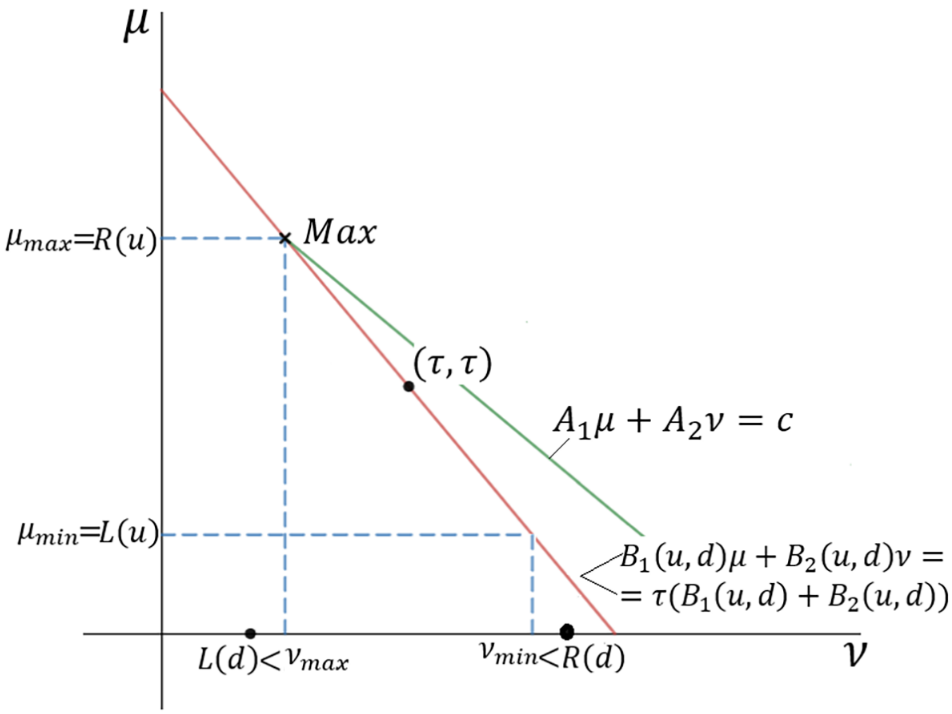

Figure 14: Leftmost U gives highest aggregate action

$$A_2 = \frac{(1-q_1)d}{q_1u+(1-q_1)d} + \frac{(1-q_2)d}{q_2u+(1-q_2)d} = \frac{(1-q_1)d}{\Pi_1} + \frac{(1-q_2)d}{\Pi_2}$$

Note A_1 is decreasing in d and A_2 is increasing in d. So objective function will also become little more steep. With the increase of d our picture would look like one on Figure 15

$$A_2 = \frac{(1-q_1)d}{q_1u+(1-q_1)d} + \frac{(1-q_2)d}{q_2u+(1-q_2)d} = \frac{(1-q_1)}{q_1\frac{u}{d}+(1-q_1)} + \frac{(1-q_2)}{q_2\frac{u}{d}+(1-q_2)} \uparrow^d$$

Also note here that $A_1 + A_2 = 2$

μ will remain the same, ν will increase to ν'. Since $\nu > L(d)$ was not binding we can make small enough change in d so that $\nu' > L(d')$. A_1 decreased to A_1' but A_2 increased to A_2'. What would happen to the value of objective function? If it increases then our perturbation is profitable.

$$A_1'\mu + A_2'\nu' - \Big(A_1\mu + A_2\nu\Big) = A_1'\mu + A_2'\big(\tau - \frac{B_1'}{B_2'}(\mu-\tau)\big) - \Big(A_1\mu + A_2\big(\tau - \frac{B_1'}{B_2'}(\mu-\tau)\big)\Big) =$$

$$= (A_1'-A_1)\mu + (A_2'-A_2)\tau + \big(\frac{A_2B_1}{B_2} - \frac{A_2'B_1'}{B_2'}\big)(\mu-\tau) =$$

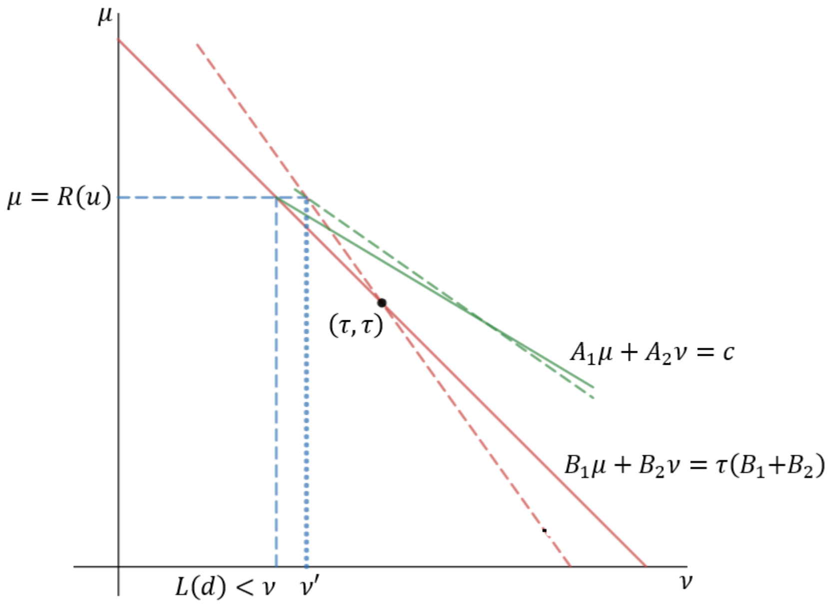

Figure 15: Solid lines are before increase of d, dashed lines are after increase of d

$$= (2 - A_2' - (2 - A_2))\mu + (A_2' - A_2)\tau + \left(\frac{A_2 B_1}{B_2} - \frac{A_2' B_1'}{B_2'}\right)(\mu - \tau) =$$

$$= (A_2 - A_2')(\mu - \tau) + \left(\frac{A_2 B_1}{B_2} - \frac{A_2' B_1'}{B_2'}\right)(\mu - \tau) =$$

$$= \Big(\mu - \tau\Big) \times \Big(A_2\big(\frac{B_1}{B_2} + 1\big) - A_2'\big(\frac{B_1'}{B_2'} + 1\big)\Big)$$

Since $\mu > \tau$, the last expression will be positive if $A_2\big(\frac{B_1}{B_2} + 1\big)$ is decreasing in d. Below we'll show this. Recall that we haven't proved that constraint becomes steeper.

Since constraint always passes through (τ, τ) steeper is equivalent that intercept on ν axes decreases. Intercept is

$$\frac{\tau(B_1(u,d) + B_2(u,d))}{B_2(u,d)} \downarrow_d \Leftrightarrow \frac{B_1(u,d)}{B_2(u,d)} \downarrow_d$$

But we won't prove that $\frac{B_1(u,d)}{B_2(u,d)}$ is decreasing, instead we'll show that

$$A_2(u,d)\big(\frac{B_1(u,d)}{B_2(u,d)} + 1\big) \downarrow_d$$

Since A_2 is increasing $\frac{B_1(u,d)}{B_2(u,d)}$ must be decreasing in this case.

Recall expressions for B_1, B_2.

$$B_1 = \frac{q_1u}{q_1u+(1-q_1)d}+\frac{q_2u}{q_2u+(1-q_2)d}+\frac{q_1u}{(1-(q_1u+(1-q_1)d))}+\frac{q_2u}{1-(q_2u+(1-q_2)d)} =$$

$$\frac{q_1u}{\Pi_1}+\frac{q_2u}{\Pi_2}+\frac{q_1u}{1-\Pi_1}+\frac{q_2u}{1-\Pi_2} = A_1+\frac{q_1u}{1-\Pi_1}+\frac{q_2u}{1-\Pi_2}$$

$$B_2 = \frac{(1-q_1)d}{q_1u+(1-q_1)d}+\frac{(1-q_2)d}{q_2u+(1-q_2)d}+\frac{(1-q_1)d}{(1-(q_1u+(1-q_1)d))}+\frac{(1-q_2)d}{1-(q_2u+(1-q_2)d)} =$$

$$= \frac{(1-q_1)d}{\Pi_1}+\frac{(1-q_2)d}{\Pi_2}+\frac{(1-q_1)d}{1-\Pi_1}+\frac{(1-q_2)d}{1-\Pi_2} = A_2+\frac{(1-q_1)d}{1-\Pi_1}+\frac{(1-q_2)d}{1-\Pi_2}$$

Prove that $A_2(u,d)\Big(\dfrac{B_1(u,d)}{B_2(u,d)}+1\Big)$ **is decreasing in** d

$$A_2(u,d)\Big(\frac{B_1(u,d)}{B_2(u,d)}+1\Big) = \frac{A_2}{B_2}\Big(B_1+B_2\Big) = \frac{A_2}{B_2}\Big(2+\frac{\Pi_1}{1-\Pi_1}+\frac{\Pi_2}{1-\Pi_2}\Big) = \frac{A_2}{B_2}\Big(\frac{1}{1-\Pi_1}+\frac{1}{1-\Pi_2}\Big) =$$

$$\frac{1}{\frac{B_2}{A_2}}\Big(\frac{1}{1-\Pi_1}+\frac{1}{1-\Pi_2}\Big) = \frac{1}{1+\frac{d(\frac{1-q_1}{1-\Pi_1}+\frac{1-q_2}{1-\Pi_2})}{d(\frac{1-q_1}{\Pi_1}+\frac{1-q_2}{\Pi_2})}}\Big(\frac{1}{1-\Pi_1}+\frac{1}{1-\Pi_2}\Big) =$$

$$\frac{\frac{1-q_1}{\Pi_1}+\frac{1-q_2}{\Pi_2}}{\frac{1-q_1}{\Pi_1(1-\Pi_1)}+\frac{1-q_2}{\Pi_2(1-\Pi_2)}} \times \frac{1-\Pi_1+1-\Pi_2}{(1-\Pi_1)(1-\Pi_2)} =$$

$$\frac{(1-q_1)\Pi_2+(1-q_2)\Pi_1}{(1-q_1)\Pi_2(1-\Pi_2)+(1-q_2)\Pi_1(1-\Pi_1)} \times \Big(1-\Pi_1+1-\Pi_2\Big) =$$

$$1+\frac{(1-q_1)\Pi_2(1-\Pi_1)+(1-q_2)\Pi_1(1-\Pi_2)}{(1-q_1)\Pi_2(1-\Pi_2)+(1-q_2)\Pi_1(1-\Pi_1)}$$

So $A_2(u,d)\Big(\dfrac{B_1(u,d)}{B_2(u,d)}+1\Big)$ is decreasing in d is equivalent to that the following expression is decreasing in d

$$\frac{(1-q_1)\Pi_2(1-\Pi_1)+(1-q_2)\Pi_1(1-\Pi_2)}{(1-q_1)\Pi_2(1-\Pi_2)+(1-q_2)\Pi_1(1-\Pi_1)} \downarrow_d \text{(denote by } \frac{Z(d)}{Q(d)}\text{)}$$

Let's find first Z' and Q'

$$Z'(d) = (1-q_1)(1-q_2)(1-\Pi_1)-(1-q_1)^2\Pi_2+(1-q_1)(1-q_2)(1-\Pi_2)-(1-q_2)^2\Pi_1 =$$

$$(1-q_1)(1-q_2)(1-\Pi_1+1-\Pi_2)-(1-q_1)^2\Pi_2-(1-q_2)^2\Pi_1 \equiv Y-(1-q_1)^2\Pi_2-(1-q_2)^2\Pi_1$$

$$Q'(d) = (1-q_1)(1-q_2)(1-\Pi_2)-(1-q_1)(1-q_2)\Pi_2+(1-q_1)(1-q_2)(1-\Pi_1)-(1-q_1)(1-q_2)\Pi_2 \equiv$$

$$\equiv Y-(1-q_1)(1-q_2)\Pi_2-(1-q_2)(1-q_1)\Pi_1$$

$$\frac{Z(d)}{Q(d)}\downarrow_d \Leftrightarrow Z'(d)Q(d)-Z(d)Q'(d)<0$$

$$\Updownarrow$$

$$Y(1-q_1)\Pi_2(1-\Pi_2)+Y(1-q_2)\Pi_1(1-\Pi_1)-Y(1-q_1)\Pi_2(1-\Pi_1)-Y(1-q_2)\Pi_1(1-\Pi_2)-$$
$$-(1-q_1)^3\Pi_2^2(1-\Pi_2)+(1-q_1)^2(1-q_2)\Pi_2^2(1-\Pi_1)-$$
$$\cancel{-(1-q_1)^2(1-q_2)\Pi_1\Pi_2(1-\Pi_1)}+\cancel{(1-q_1)^2(1-q_2)\Pi_1\Pi_2(1-\Pi_1)}-$$
$$\cancel{-(1-q_1)(1-q_2)^2\Pi_1\Pi_2(1-\Pi_2)}+\cancel{(1-q_1)(1-q_2)^2\Pi_1\Pi_2(1-\Pi_2)}-$$
$$-(1-q_2)^3\Pi_1^2(1-\Pi_1)+(1-q_2)^2(1-q_1)\Pi_1^2(1-\Pi_2)<0$$

$$\Updownarrow$$

$$Y\Big(\Pi_1-\Pi_2\Big)\Big((1-q_1)\Pi_2-(1-q_2)\Pi_1\Big)+$$
$$+\Big(\big((1-q_1)\Pi_2\big)^2-\big((1-q_2)\Pi_1\big)^2\Big)\Big((1-q_2)(1-\Pi_1)-(1-q_1)(1-\Pi_2)\Big)<0$$

$$\Updownarrow$$

$$\Big((1-q_1)\Pi_2-(1-q_2)\Pi_1\Big)\times$$
$$\times\Big(Y\big(\Pi_1-\Pi_2\big)+\Big((1-q_1)\Pi_2+(1-q_2)\Pi_1\Big)\Big((1-q_2)(1-\Pi_1)-(1-q_1)(1-\Pi_2)\Big)\Big)<0$$

$$\Updownarrow \quad \text{substitute } Y=(1-q_1)(1-q_2)(1-\Pi_1+1-\Pi_2)$$

$$u(q_2-q_1)\times$$
$$\times\Bigg(\big((1-q_1)(1-q_2)(1-\Pi_1+1-\Pi_2)\big)\big(\Pi_1-\Pi_2\big)+$$
$$+\Big((1-q_1)\Pi_2+(1-q_2)\Pi_1\Big)\Big((1-q_2)(1-\Pi_1)-(1-q_1)(1-\Pi_2)\Big)\Bigg)<0$$

$$\Updownarrow \quad \text{Since } q_2>q_1$$

$$\Pi_1(1-\Pi_1)\Big((1-q_2)^2+(1-q_1)(1-q_2)\Big)-\Pi_2(1-\Pi_2)\Big((1-q_1)^2+(1-q_1)(1-q_2)\Big)<0$$

$$\Updownarrow$$

$$\Big(1-q_1+1-q_2\Big)\times\Bigg((1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)\Bigg)<0$$

$$\Updownarrow \quad \text{since } q_1<q_2\leq 1,\text{ if } q_2=0 \text{ inequality hodls, so assume } q_2<1$$

$$(1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)<0$$

So we need to show that $(1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)<0$. As a function of d, this is a quadratic function with coefficient in front of d^2 equal to:

$$-(1-q_2)(1-q_1)^2+(1-q_1)(1-q_2)^2<0 \quad \textit{since } q_1<q_2$$

So as a function of d, $(1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)<0$ reaches it's highest value at the vertex of the parabola. Let us find the vertex.

$$\frac{\partial((1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2))}{\partial d}=0 \Leftrightarrow \Pi_1=\Pi_2$$

and when $\Pi_1=\Pi_2=\Pi$

$$(1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)=\Pi(1-\Pi)\big((1-q_2)-(1-q_1)\big)<0$$

So the even at the maximum $(1-q_2)\Pi_1(1-\Pi_1)-(1-q_1)\Pi_2(1-\Pi_2)$ is negative which ends the prove of $A_2(u,d)\Big(\dfrac{B_1(u,d)}{B_2(u,d)}+1\Big)$ is decreasing in d and since $A_2(u,d)$ is increasing in d, $\dfrac{B_1(u,d)}{B_2(u,d)}$ is also decreasing in d. So slight increase of d and adjusting ν so that equilibrium condition holds, increases aggregate value. Note that even though we used $q_1<q_2$, the proof is independent from this assumption. If we assume reversed than parabola we considered will have a minimum which is positive.

Also note that, when ν is binding too, we cannot guarantee that when we perturb d in this way we can keep same μ, fact that ν was not binding allowed us to do so. This finishes the proof.

Q.E.D

Proof of Lemma 4

Lemma 4 *Let $\mathbb{S}^x_y(\alpha,\beta)$ be a diagonal partition of $[\alpha,\beta]\times\{0,1\}$ s.t. $\{x,y\}\nsubseteq\{\alpha,\beta\}$, then*

$$q_1<q_2 \ \Rightarrow$$

$$\Rightarrow F(t|q_1,(t,\omega)\in m)>F(t|q_2,(t,\omega)\in m)\bigwedge F(t|q_1,(t,\omega)\in m')<F(t|q_2,(t,\omega)\in m')$$

Where $F(t|q_i,(t,\omega)\in X)$ is receiver i-s posterior distribution after receiving signal X. .

Proof Assume $q_1<q_2$. For simplicity, without loss of generality we'll assume $[\alpha,\beta]=[0,1]$. When message is $m=[0,y)\times\{0\}\cup[x,1]\times\{1\}$, then if $x<y$ posteriors are:

$$F_i(t|m) = \begin{cases} \dfrac{(1-q_i)F(t)}{\Pi_i} & 0 \le t \le x \\ \dfrac{F(t)-q_iF(x)}{\Pi_i} & x \le t \le y \\ \dfrac{q_iF(t)-q_iF(x)+(1-q_i)F(y)}{\Pi_i} & y \le t \le 1 \end{cases}$$

Where $\Pi_i = (1-q_i)y + q_i(1-x))$

$$F_1(t|m) > F_2(t|m)$$
$$\Updownarrow$$
$$\Big(\frac{(1-q_1)F(t)}{\Pi_1} > \frac{(1-q_2)F(t)}{\Pi_2} \quad when 0 \le t \le x\Big) \wedge$$
$$\wedge\Big(\frac{F(t)-q_1F(x)}{\Pi_1} > \frac{F(t)-q_2F(x)}{\Pi_2} \quad when\ x \le t \le y\Big) \wedge$$
$$\wedge\Big(1-F_2(t|m) > 1-F_1(t|m) \quad when\ y \le t \le 1\Big)$$

For the first part of the conjunction we have:

$$\frac{(1-q_1)F(t)}{\Pi_1} > \frac{(1-q_2)F(t)}{\Pi_2} \Leftrightarrow \Pi_2(1-q_1) > \Pi_1(1-q_2)$$
$$\Updownarrow$$
$$\big((1-q_2)y + q_2(1-x)\big)(1-q_1) > \big((1-q_1)y + q_1(1-x)\big)(1-q_2)$$
$$\Updownarrow$$
$$q_2(1-x) > q_1(1-x) \quad holds\ because\ q_1 < q_2\ and\ 1-x > 0$$

Now let us do the third part of the conjunction. Note that $1 - F_i(T|m) = Pr(T > t|q_i, m) = \frac{q_i(1-F(T))}{\Pi}$.

$$1-F_2(t|m) > 1-F_1(t|m) \Leftrightarrow \frac{q_2(1-F(T))}{\Pi_2} > \frac{q_1(1-F(T))}{\Pi_1}$$
$$\Updownarrow$$
$$q_2\Pi_1 > q\Pi_2 \Leftrightarrow q_2y > q_1y \quad holds\ since\ q_2 > q_1\ and\ y > 0$$

For the second part of the conjunction we have:

$$\frac{F(t)-q_1F(x)}{\Pi_1} > \frac{F(t)-q_2F(x)}{\Pi_2} \Leftrightarrow F(t)(\Pi_2 - \Pi_1) > F(x)(q_1\Pi_2 - q_2\Pi_1)$$
$$\Updownarrow$$

$$F(t)(q_2 - q_1)(1 - x - y) > -F(x)(q_2 - q_1)y$$

$$\Updownarrow \quad if\ 1 - x - y \geq 0\ inequality\ holds\ consider\ 1 - x - y < 0$$

$$F(t)(1 - x) > (F(t) - F(x))y$$

$$\Updownarrow \quad Note\ F(t) > F(x)\ if\ (F(t) = F(x)\ inequality\ holds$$

$$\frac{1 - x}{y} > 1 - \frac{F(x)}{F(t)}$$

Note that by the third part of the conjunction $F_1(y) > F_2(y)$ and since all this transitions from one inequality to another was equivalencies we have that

$$F_1(y) > F_2(y) \Rightarrow \frac{1 - x}{y} > 1 - \frac{F(x)}{F(y)}$$

But, since $F(t)$ is CDF it's increasing and since $t < y$, $F(t) < F(y)$

$$\frac{1 - x}{y} > 1 - \frac{F(x)}{F(t)} < \frac{1 - x}{y} > 1 - \frac{F(x)}{F(t)} < \frac{1 - x}{y}$$

This finishes the proof of case when $x < y$. Let us consider now the case when $x \geq y$. In this case posteriors are:

$$F_i(t|m) = \begin{cases} \dfrac{(1 - q_i)F(t)}{\Pi_i} & 0 \leq t \leq y \\ \dfrac{(1 - q_i)F(y)}{\Pi_i} & y \leq t \leq x \\ 1 - \dfrac{q_i(1 - F(t))}{\Pi_i} & x \leq t \leq 1 \end{cases}$$

Now if $t \in [0, y]$ and $t \in [x, 1]$ are the same as first and third parts of conjunction of $x < y$ case. As for $t \in [y, x]$ it is same as $t \in [0, y]$, $F(t)$ is just replaced with $F(y)$ which cancels anyway.
the message is m', we just replace q_i with $(1 - q_i)$ and this time $1 - q_1 > 1 - q_2$ and hence $F_2 > F_1$. This finishes the proof of Lemma 4.

Q.E.D

Proof of Proposition 3

Proposition 3 Let G be a one-dimensional coarsening element of polar equilibrium and m, m' be a two-dimensional refinement elements of this polar equilibrium s.t. $m \cup m' = G \times \{0, 1\}$ then either

$$\Big(F(t|q_1, m) \succ_{SOSD} F(t|G) \succ_{SOSD} F(t|q_2, m)\Big) \wedge \Big(F(t|q_1, m') \prec_{SOSD} F(t|G) \prec_{SOSD} F(t|q_2, m')\Big)$$

or

$$\Big(F(t|q_1,m) \prec_{SOSD} F(t|G) \prec_{SOSD} F(t|q_2,m)\Big) \wedge \Big(F(t|q_1,m') \succ_{SOSD} F(t|G) \succ_{SOSD} F(t|q_2,m')\Big)$$

Proof Without loss of generality we'll assume $G = [0,1]$, so $F(t|G) = F(t)$ When message is $m = [0,y) \times \{0\} \cup [x,1] \times \{1\}$.

We'll use the theorem, which state that if CDF -s H_1 and H_2 of two distribution of positive random variable cross just once, and $H_1(x) \leq H_2(x)$ before intersection, then $E(x|H_1) \geq E(x|H_2)$ implies that $H_1 \succ_{SOSD} H_2$. We'll show that our posteriors are either above prior or intersect at most once, and equilibrium condition will make sure that expectations also satisfy condition which allows us to apply theorem.

First consider the case when $x < y$. In this case Posterior for message m for receiver i is:

$$F_i(t|m) = \begin{cases} \dfrac{(1-q_i)F(t)}{\Pi_i} & 0 \leq t \leq x \\ \dfrac{F(t) - q_iF(x)}{\Pi_i} & x \leq t \leq y \\ \dfrac{q_iF(t) - q_iF(x) + (1-q_i)F(y)}{\Pi_i} & y \leq t \leq 1 \end{cases}$$

Where $\Pi_i = (1-q_i)y + q_i(1-x))$

If $\frac{1-q_i}{\Pi_i} > 1$ then for $t \in [0,x]$, $F_i(t|m) > F(t)$ and since $F' > 0$, $F_i' > F'$ too. This means that at first discontinuity point of F_i, $t = x$, $F_i(x) > F(x)$. In the range of $t \in [x,y]$, $F_i' > F'$, so F_i will still higher than F on $[x,y]$. Now on interval $[y,1]$, $F_i(y) > F(y)$ and $F_i(1) = F(1) = 1$, so $F_i - F > 0$ at y and 0 at $t = 1$. If for some point between y and 1 $F_i - F < 0$. Since both F_i and F are differentiable on $[y,1]$ there should be a local minimum point where $F'(t) - F_i'(t) = 0 \Rightarrow \frac{q_i}{\Pi} = 1$, but this means that $F'(t) - F_i'(t) = 0$ for all $t \in [y,1]$ and in this case $F_i(y) > F(y)$ and change in same rate so they cannot intersect, event at 1, meaning that F_i remains above F on $[y,1]$. All in all, if $\frac{1-q_i}{\Pi_1} > 1$, then $F_i(t|m) > F$ meaning that posterior of receiver i dominates F. Now we cannot have that for both $i = 1,2$, $\frac{1-q_i}{\Pi_1} > 1$ since in this case both a_1 and a_2 (actions of receivers when message is m) will be less than τ, and this cannot happen in equilibrium.

Now consider $F_i(t|m')$ when $x < y$

$$F_i(t|m') = \begin{cases} \dfrac{q_iF(t)}{1-\Pi_i} & 0 \leq t \leq x \\ \dfrac{q_iF(x)}{1-\Pi_i} & x \leq t \leq y \\ 1 - \dfrac{(1-q_i)(1-F(t))}{1-\Pi_i} & y \leq t \leq 1 \end{cases}$$

Now if both $\frac{q_i}{1-\Pi_i} < 1$, then F_i is below F up to point y, and then similarly to

$F(t|m)$ it is obvious that F_i stays below F, meaning that F_i dominates F with first order stochastic dominance. Now we cannot have that for both $i = 1, 2$, $\frac{q_i}{1-\Pi_1} < 1$ since in this case both a'_1 and a'_2 (actions of receivers when message is m') will be more than τ, and this cannot happen in equilibrium.
$\frac{q_i}{1-\Pi_i} < 1 \Leftrightarrow \frac{1-q_i}{\Pi)i} < 1$. So we cannot have at the same time $\frac{1-q_i}{\Pi_i} < 1$ for both receivers. Meaning that we have have either:

$$\frac{1-q_1}{\Pi_1} < 1 < \frac{1-q_2}{\Pi_1} \quad and \quad \frac{q_1}{1-\Pi_1} < 1 < \frac{q_2}{1-\Pi_2} < 1$$

or

$$\frac{1-q_1}{\Pi_1} > 1 > \frac{1-q_2}{\Pi_1} \quad and \quad \frac{q_1}{1-\Pi_1} > 1 > \frac{q_2}{1-\Pi_2} < 1$$

or $q_i = 1 - \Pi_i$. WlOG assume that it's the second case.

Now consider again $F_i(t|m)$ and case $\frac{1-q_i}{\Pi_i} < 1$. In this case when $t \in [0, x]$, F_i is bellow F. Now note that $F(1) - F_i(1) = 1$ and if F and F_i intersect twice between x and 1, there should be at least 2 local extreme points for function $F - F_i$ on $[x, 1]$, if we exclude kink of F_i which is at $t = y$, at least for one point $F'(t) - F'_i(t) = 0$. This cannot happen on $[x, y]$ interval, since $F'_i(t) = \dfrac{F'(t)}{\Pi} > F'(t)$ and if $F'(t) - F'_i(t) = 0$ for some $t \in [y, 1]$, then $q_i = P_i$ and $F(t) = F_i(t)$ for all $t \in [y, 1]$. So we have that F_i and F cross each other at most once.

So we have that when $x < y$, $\frac{q_1}{1-\Pi_1} > 1$ and $F_1(t|m)$ is above F and $\frac{q_2}{1-\Pi_2} < 1$ and $F_2(t|m)$ intersect with F at most once. $F_1(t|m) > F(t) \Rightarrow a_1 < \tau \Rightarrow a_2 > tau$. Then using theorem about single crossing CDF-s we can say that $F_2(t|m) \succ_{SOSD} F(t)$ and

$$F_1(t|m) > F(t) \Rightarrow F_1(t|m) \prec_{FOSD} F(t) \Rightarrow F_1(t|m) \prec_{SOSD} F(t)$$

What about $F(t|m')$? We have $\dfrac{q_1}{1-\Pi_1} > 1 > fracq_2 1 - \Pi_2 < 1$. In exactly similar way as for $F_i(t|m')$ we'll get that $F_1(t|m')$ is initially higher than $F(t)$ and intersects at most once. Also from lemma 3 we get that $F_1(t|m') > F_2(t|m')$ and respectively $a'_1 < a'_2 \Rightarrow a'_1 < \tau < a'_2$. Again using theorem about single crossing CDF-s we can say that $F_1(t|m') \prec_{SOSD} F(t)$.

$F_2(t|m')$ is initially lower than $F(t)$ and remains so up until 1 So we'll get that

$$F_2(t|m') < F(t) \Rightarrow F_2(t|m') \succ_{FOSD} F(t) \Rightarrow F_2(t|m') \succ_{SOSD} F(t)$$

As for the case when $x \geq y$, Since we have not used any assumptions on q_1, q_2 we can repeat the same proof $1 - q_1, 1 - q_2$ which just relabels the problem. This finishes the proof of proposition 4.

Q.E.D

Proof of Lemma 5

lemma 5 *If $q_1 < q_2$, $\overline{M_G}$ is decreasing in q_1 and increasing in q_2 and $\underline{M_G}(q_1, q_2)$ is increasing in q_1 and decreasing in q_2*

Proof For simplicity we'll drop index G. Consider two senders. The first on (call her sender 1) faces pair of receivers with beliefs q_1, q_2 and the second one (call her sender 2) faces pair of receivers with beliefs q_1', q_2, where $q_1' = q_1 - \varepsilon < q_1 < q_2$. Assume $a = M(q_1, q_2)$ and $a' = M(q_1', q_2)$ are maximum aggregate actions achievable in equilibrium. We want to show that $a' > a$. To show that M is increasing in q_2, recall that $M(q_1, q_2) = M(q_2, q_1) = M(1 - q_2, 1 - q_1) = M(1 - q_1, 1 - q_2)$, and we can repeat the proof of $a' > a$ for $M(1 - q_2, 1 - q_1)$.
For sake of simplicity, when we speak about partition which induces highest achievable action, we'll say "*sender is choosing the strategy or partition which induces highest aggregate action as if it's her goal*"

Stating problem in terms of u, d, μ and ν and then translating it in terms of Π_1, Π_2, H_1, H_2
Now recall that, equilibrium strategy (partition N_0, N_1) can be characterized by sets of $U \subset [\alpha, \beta] \times \{1\}$, $D \subset [\alpha, \beta] \times \{1\}$ or equivalently by lengths of U and D sets, u and d and averages on this sets $\mu_0 = E(t|t \in U)$ and $\nu_0 = E(t|t \in D)$. Let $\mu_1 = E(t|t \notin U)$ and $\nu_1 = E(t|t \notin D)$ and $\tau = E(t)$, then $u, d, \mu_0, \nu_0, \mu_1, \nu_1$ describe equilibrium strategy if following condition holds:

$$\frac{q_1 u\mu_0 + (1-q_1)d\nu_0}{q_1 u + (1-q_1)d} + \frac{q_2 u\mu_0 + (1-q_2)d\nu_0}{q_2 u + (1-q_2)d} =$$

$$= \frac{q_1(1-u)\mu_1 + (1-q_1)(1-d)\nu_1}{q_1(1-u) + (1-q_1)(1-d)} + \frac{q_2(1-u)\mu_1 + (1-q_2)(1-d)\nu_1}{q_2(1-u) + (1-q_2)(1-d)}$$

$$\Updownarrow \qquad using \quad \mu_0 u + \mu_1(1-u) = \tau \quad and \quad \nu_0 d + n_1(1-d) = \tau$$

$$\frac{q_1 u\mu_0 + (1-q_1)d\nu_0}{q_1 u + (1-q_1)d} + \frac{q_2 u\mu_0 + (1-q_2)d\nu_0}{q_2 u + (1-q_2)d} = \frac{\tau - (q_1 u\mu_0 + (1-q_1)d\nu_0)}{1 - (q_1 u + (1-q_1)d)} + \frac{\tau - (q_2 u\mu_0 + (1-q_2)d\nu_0)}{1 - (q_2 u + (1-q_2)d)}$$

Now note that it's equivalent to consider Π_1 and Π_2 instead of u and d, where Π_i is receiver i's perceived probability of state being in N_0 and $1 - \Pi_i$ of state being in N_1. The reason is that for given q_1, q_2, Π_1 and Π_2 uniquely determine u and d from the following system of equation:

$$\Pi_1 = q_1 u + (1-q_1)d \;\; and \;\; \Pi_2 = q_2 u + (1-q_2)d$$

$$\Downarrow$$

$$u = \frac{\Pi_2(1-q_1) - \Pi_1(1-q_2)}{q_2 - q_1} \;\; and \;\; d = \frac{\Pi_1 q_2 - \Pi_2 q_1}{q_2 - q_1}$$

Now, note that we are allowed to have only $u \geq 0$ and $d \geq 0$, these translates to Π_1

and Π_2 in the following way:

$$\frac{q_1}{q_2} \leq \frac{\Pi_1}{\Pi_2} \leq \frac{1-q_1}{1-q_2} \quad and \quad \frac{q_1}{q_2} \leq \frac{1-\Pi_1}{1-\Pi_2} \leq \frac{1-q_1}{1-q_2}$$

So sender 1 is free to choose any Π_1, Π_2 (and then matching μ_0, ν_0) which satisfies above conditions and sender 2 can choose any Π_1, Π_2 for which,

Sender 2 is less restricted in her choice of Π_i, let her choose same Π_i as sender 1 and prove that associated u', d', μ, ν are feasible.

$$\frac{q_1'}{q_2} \leq \frac{\Pi_1}{\Pi_2} \leq \frac{1-q_1'}{1-q_2} \quad and \quad \frac{q_1'}{q_2} \leq \frac{1-\Pi_1}{1-\Pi_2} \leq \frac{1-q_1'}{1-q_2}$$

Since, $q_1' < q_1$ sender 2's condition on Π_1, Π_2, is less restrictive. Consequently if sender 1 can choose Π_1, Π_2, sender 2 can choose those Π_1, Π_2 too.
Now assume that for sender 1, maximum achievable aggregate action is achieved by some partition N_0, N_1. Assume actions when state is in N_0 is described by μ_0, ν_0, u, d in the following way:

$$a_i^0(\mu_0, \nu_0, u, d) = \frac{q_i u \mu_0 + (1-q_i) d \nu_0}{q_i u + (1-q_i) d} = \frac{H_i}{\Pi_i}$$

For state in N_1 action as a function of μ_0, ν_0, u, d is given in following way:

$$a_i^1(\mu_0, \nu_0, u, d) = \frac{\tau - (q_i u \mu_0 + (1-q_i) d \nu_0)}{1 - (q_i u + (1-q_i) d)} = \frac{\tau - H_i}{1 - \Pi_i}$$

Note that because of **Proposition 2**, without loss of generality we can assume that $\mu_0 = R(u)$ and $\nu_0 = L(d)$ and $d > u$. Where

$$R(x) = \int_{Q(1-x)}^{\beta} \frac{t}{x} dF(t) \qquad \textit{average on the right tail of size } x$$

$$L(x) = \int_{\alpha}^{Q(x)} \frac{t}{x} dF(t) \qquad \textit{average on the left tail of size } x$$

and Q is quantile function of distribution F. Can we construct an equilibrium strategy for sender 2, with the same actions as a_i^0 and a_i^1?

Finding sender 2's u, d, μ, ν

Similarly to sender 1, sender 2 also can use Π_1, Π_2, then in this case lengths of U and D sets for sender 2 will be:

$$u' = \frac{\Pi_2(1-q_1') - \Pi_1(1-q_2)}{q_2 - q_1'} \quad and \quad d' = \frac{\Pi_1 q_2 - \Pi_2 q_1'}{q_2 - q_1'}$$

Before we continue note that if A, B, A_1, B_1, x are positive then:

$$\frac{A + A_1 x}{B + B_1 x} \geq \frac{A}{B} \quad \Leftrightarrow \quad BA_1 x \geq AB_2 x \quad \Leftrightarrow \quad \frac{A_1}{B_1} \geq \frac{A}{B}$$

$$u' = \frac{\Pi_2(1 - q_1') - \Pi_1(1 - q_2)}{q_2 - q_1'} = \frac{\Pi_2(1 - q_1) - \Pi_1(1 - q_2) + \Pi_2\varepsilon}{q_2 - q_1 + \varepsilon}$$
$$and \quad u = \frac{\Pi_2(1 - q_1) - \Pi_1(1 - q_2)}{q_2 - q_1}$$
$$\Downarrow$$
$$u' \geq u \Leftrightarrow u \leq \Pi_2 = q_2 u + (1 - q_2) d$$

And since $u < d$, $u < \Pi_2$, hence $u' > u$. Similarly for d', we have:

$$d' = \frac{\Pi_1 q_2 - \Pi_2 q_1'}{q_2 - q_1'} = \frac{\Pi_1 q_2 - \Pi_2 q_1' + \Pi_2\varepsilon}{q_2 - q_1' + \varepsilon}$$
$$and \quad d = \frac{\Pi_1 q_2 - \Pi_2 q_1}{q_2 - q_1}$$
$$\Downarrow$$
$$d' \geq d \Leftrightarrow d \leq \Pi_2 = q_2 u + (1 - q_2) d$$

And since $u < d$, $d > \Pi_2$ and consequently $d' < d$.

Given, u' and d', sender 2 can choose μ_0' from $[L(u'), R(u')]$ and ν_0' from $[L(d'), R(d')]$. Note here that

Now if sender 2 choose Π_1, Π_2, can she choose μ_0', ν_0' so that following holds?

$$q_1' u' \mu_0' + (1 - q_1') d' \nu_0' = H_1 \quad and \quad q_2 u' \mu_0' + (1 - q_2) d' \nu_0' = H_2$$

$$\Downarrow$$

$$\nu_0' = \frac{H_1 q_2 - H_2 q_1'}{d'(q_2 - q_1')} = \frac{H_1 q_2 - H_2 q_1'}{\Pi_1 q_2 - \Pi_2 q_1'}$$

and

$$\mu_0' = \frac{H_2(1 - q_1') - H_1(1 - q_2)}{u'(q_2 - q_1')} = \frac{H_2(1 - q_1') - H_1(1 - q_2)}{\Pi_2(1 - q_1') - \Pi_1(1 - q_2)}$$

Here note that for sender 1, or before q_1 changed to q_1'

$$\nu_0 = \frac{H_1 q_2 - H_2 q_1}{\Pi_1 q_2 - \Pi_2 q_1} \qquad and \qquad \mu_0 = \frac{H_2(1 - q_1) - H_1(1 - q_2)}{\Pi_2(1 - q_1) - \Pi_1(1 - q_2)}$$

Feasibility of new μ, ν

If we show that $\mu_0' \in [L(u'), R(u')]$ and $\nu_0' \in [L(d'), R(d')]$. then effectively we constructed strategy for sender 2 which induces same actions for her as N_0, N_1 were inducing for sender 1 (or when receivers beliefs were q_1, q_2), i.e a_i^0 and a_i^1.

Recall that, μ_0, ν_0 were optimal for sender 1 and because of Proposition 1, $\mu_0 = R(u) > R(u')$ ($R(u)$ is decreasing and we showed $u' > u$) and $\nu_0 = L(d) > L(d')$. ($L(d)$ is increasing and we showed that $d' < d$).
If we show that $\nu_0' > \nu_0$, since $\nu_0 = L(d) > L(d')$, we'll get $\nu_0' > L(d')$. To show $\nu_0' > \nu_0$, note that:

$$\nu_0 = \frac{H_1 q_2 - H_2 q_1}{\Pi_1 q_2 - \Pi_2 q_1}$$

and

$$\nu_0' = \frac{H_1 q_2 - H_2 q_1'}{\Pi_1 q_2 - \Pi_2 q_1'} = \frac{H_1 q_2 - H_2 q_1 + H_2 \varepsilon}{\Pi_1 q_2 - \Pi_2 q_1 + \Pi_2 \varepsilon}$$

$$\Downarrow$$

$$\nu_0' \geq \nu_0 \Leftrightarrow \nu_0 \leq \frac{H_2}{\Pi_2} = \frac{(1-q_2)d}{q_2 u + (1-q_2)d} \times \nu_0 + \frac{q_2 u}{q_2 u + (1-q_2)d} \times \mu_0$$

Since μ_0 is average on rightmost interval and ν_0 is average on leftmost interval $\mu_0 > \tau > \nu_0$ and consequently weighted sum of μ_0 and ν_0 ($a_2 = \frac{H_2}{\Pi_2}$ is weighted sum of μ_0 and ν_0) is more than ν_0, so $\nu_0' > \nu_0$.
For $\nu_0' < R(d')$ note that ν_0, d, μ_0, u are continuous functions of q_1, q_2 (they depend and are continuous in H_i, Π_i too, but we keep them fixed) and $\nu_0 = L(d) < R(d)$ we can always take ε small enough so that $\nu_0' < R(d) < R(d')$.

For $\mu_0' < R(u')$ we cannot repeat the same what we did for $\nu_0' > L(d')$ since both μ_0' and $R(u')$ are more than $\mu_0 = R(u)$, but we can repeat the same for μ_1' and $L(1-u')$.

Similarly to

$$\nu_0 = \frac{H_1 q_2 - H_2 q_1}{\Pi_1 q_2 - \Pi_2 q_1} \qquad \textit{and} \qquad \mu_0 = \frac{H_2(1-q_1) - H_1(1-q_2)}{\Pi_2(1-q_1) - \Pi_1(1-q_2)}$$

We have:

$$\nu_1 = \frac{(\tau - H_1)q_2 - (\tau - H_2)q_1}{(1-\Pi_1)q_2 - (1-\Pi_2)q_1} \qquad \textit{and} \qquad \mu_1 = \frac{(\tau - H_2)(1-q_1) - (\tau - H_1)(1-q_2)}{(1-\Pi_2)(1-q_1) - (1-\Pi_1)(1-q_2)}$$

Hence,

$$\mu_1' = \frac{(\tau - H_2)(1-q_1) - (\tau - H_1)(1-q_2) + (\tau - H_2)\varepsilon}{(1-\Pi_2)(1-q_1) - (1-\Pi_1)(1-q_2) + (1-\Pi_2)\varepsilon}$$

Now note that

$$a_2^1 = \frac{\tau - H_2}{1-\Pi_2} = \frac{(1-q_2)(1-d)}{q_2(1-u) + (1-q_2)(1-d)} \times \nu_1 + \frac{q_2(1-u)}{q_2(1-u) + (1-q_2)(1-d)} \times \mu_1$$

Since μ_1 is average on leftmost interval and ν_1 is average on rightmost interval $\mu_1 < \tau < \nu_1$ and consequently weighted sum of μ_1 and ν_1 ($a_2^1 = \frac{\tau - H_2}{1 - \Pi_2}$ is weighted sum of μ_1 and ν_1) is more than ν_1, so $\mu_1' > \mu_1$.
We also got that $u' > u$, hence

$$L(1-u') < L(1-u) = \mu_1 < \mu_1'$$

So remains to show that if $(1-u)\mu_1' + u\mu_0' = \tau$ and $L(1-u') < \mu_1'$ then $\mu_0' < R(u')$.
For this note that:

$$(1-u')\mu_1' + u'\mu_0' = \tau \quad and \quad (1-u)L(1-u') + u'R(u') = \tau$$

$$\Downarrow$$

$$(1-u')\big(\mu_1' - L(1-u')\big) = u'\big(R(u') - \mu_0'\big)$$

Left hand side of last equality is positive and consequently we must have $R(u') > \mu_0'$.
As for $\mu_0' > L(u')$ we can use similar argument what we have used for $\nu_0' < R(d')$, by choosing small enough ε $\mu_0' > L(u')$ can be guaranteed.

To summarize the proof, we took arbitrarily small deviation from q_1, $q_1' = q_1 - \varepsilon$ and proved that for q_1', q_2 there is equilibrium strategy in which receivers take exactly same actions as they took when beliefs were q_1, q_2 and aggregate actions were highest. Moreover based on lemma 2, since $\nu_0 > L(d')$ we know that equilibrium partition generated by (μ_0, ν_0, d', u') does not give maximum achievable aggregate action for beliefs q_1', q_2. Thus, maximum achievable aggregate action for beliefs q_1', q_2 is higher than same for q_1, q_2. This finishes the proof of lemma 4.

Q.E.D