MEMORY AND MARKETS*

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Abstract

In many environments, including credit and online markets, past records about participants are collected, published, and erased after some time. We study the effects of erasing past records on trade and welfare in a dynamic market where each seller's quality follows a Markov process and buyers leave feedback about sellers. When the average quality of sellers is low, unlimited records always lead to a market breakdown. Appropriately deleting past records, instead, fosters experimentation and can sustain trade in the long run. Positive and negative records play opposite roles with different intensity, and welfare is maximized for short positive records and long but bounded negative ones. Analogous results obtain within an information design approach, in which an information intermediary simply recommends to uninformed buyers whether to buy from each seller. Our findings have implications for the design of privacy regulations, credit bureaus, and feedback systems in online platforms.

JEL Codes: D82, D53, G20, G28, K35, L14, L15.

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1 Introduction

In many environments information intermediaries collect and make publicly available information on agents' history. Electronic platforms like eBay and Amazon, collect feedback about buyers and sellers, credit bureaus record credit history for borrowers, the legal system keeps track of criminal records, and Internet search engines effectively record information about an individual's past. Because of privacy laws, regulations, or intermediaries' own rules, after some time access to part or all of this information is denied. A number of natural questions arise: Can it be beneficial to limit the disclosure of past information to market participants? If so, for how long should past information be available? And do the answers depend on the type of recorded information?

This paper studies how the timespan of past records provided by an information intermediary affects the amount of trade and information prevailing in a market in the long run. We develop a continuous-time model of a dynamic market populated by sellers (borrowers, employees) of unknown quality, and competitive buyers (lenders, employers). At any point in time the seller can be of "high" or "low" quality and his quality changes according to a Markov process. After a purchase, the buyer learns the seller's quality and can leave "positive" or "negative" feedback with an information intermediary that makes the record accessible to future cohorts of buyers. Throughout the paper we focus on markets where the average quality of sellers is low and information on their past performance is crucial, as without it there is no trade. We study the effects of different lengths of public memory for positive and negative records, respectively, on equilibrium and welfare in the long run. Distinguishing the memory of positive from that of negative records turns out to be critical to understanding the effects of limited records on market outcomes.[1]

First, we show that when market conditions are poor, unlimited memory of past records has dramatic negative consequences in the long run: it necessarily leads to a market breakdown (Theorem 1). This happens even if the market starts with full information about sellers. The reason is that as soon as a seller gets a negative record, his expected quality drops and stays below the low population average, so that the buyers' willingness to pay for his product is below the seller's reservation value/cost. Since sellers' types change over

[1]The distinction between positive and negative records is self-evident in financial markets, where credit records may include "black" (credit remarks, past arrears, defaults and bankruptcies) and "white" information (patterns of repayments, open and closed credit accounts, new loans, debt maturity, guarantees and assets); and in some electronic markets (e.g. eBay's positive and negative feedbacks). But as we explain later, this distinction is relevant for most economic environments.

time, in the long run each good seller becomes bad and gets a negative record. Because of unlimited memory, from that moment on such sellers are out of the market forever. As time goes to infinity, each seller is almost surely excluded from trading. Hence, in this environment, the unlimited memory of past feedback necessarily leads to (almost) no trade in the long run.

Our second main result is that, under the same poor market conditions, a stationary equilibrium with a constant, positive fraction of sellers trading at each moment in time can instead be sustained if past records are deleted in a specific way. With limited records, the market breakdown can be avoided by retaining past negative records for a sufficiently long time while deleting positive records quickly or not recording them at all (Theorem 2). This is because, a longer memory of negative records allows the bad sellers to be identified and kept out of the pool of sellers with no records (empty history) for a longer time, thereby improving the average quality of that pool. This encourages buyers to offer higher prices to sellers with no records and allows them to trade and get feedback. A longer memory of positive records, instead, allows more good sellers to separate from the pool with no records, thereby discouraging trade with non-rated sellers and the production of information on their quality. This makes stationary equilibria with trade harder to sustain. The proposed short limits for positive records and long limits for negative ones encourage buyers to "experiment" by buying from sellers with no records, producing new information and improving long-run outcomes.

We also find that positive and negative records differ in terms of the intensity of their effects: negative records simply prevent the seller from trading for a given amount of time, while positive records have a nonlinear, self-enforcing effect because they induce further trade and new records right away.

We then show that in such a stationary equilibrium, social welfare is maximized when negative feedback is also deleted from public records after a specific amount of time, though later than positive feedback. In other words, if it does not compromise equilibrium existence, then it is socially beneficial to also limit the memory for negative records (Theorem 3). Keeping negative records for a long time excludes bad sellers from the market and sustains trade in the long run. However, excluding sellers for too long involves a social loss: after a sufficient time the seller's type is likely to have changed and it is socially optimal to trade with the seller in order to learn his true type. The positive social value of learning the seller's type, associated with the information generated by buyers that purchase and leave feedback, is not internalized by buyers. As a result, buyers do not buy from sellers that have old negative feedback even if this is socially optimal. Deleting old negative feedback is therefore beneficial: once negative feedback is deleted the seller is pooled with good sellers and can sell in the market. In other words, due to the information externality, the market may not "forgive" sellers with negative feedback and a policy intervention requiring the market to "forget" negative feedback can be beneficial.[2]

Finally, we consider a more general information design problem in which the intermediary does not disclose past feedback but instead recommends a seller for trade or not, depending on the seller's posterior probability of being of high type. In the spirit of the Bayesian persuasion literature we allow the intermediary to commit to a threshold recommendation policy: only sellers with posteriors above a certain threshold are recommended for trade. We show that stationary threshold recommendation policies are equivalent to limited record policies, and that all our results about trade in the long run and welfare maximizing memory policies also hold in this setting.

We believe our results have important policy implications. Informational problems are among the main sources of market and government failures. Increasing the amount of information available is often regarded as a natural remedy. Privacy concerns, on the other hand, have been at the center of several recent debates on electronic markets and the Internet. To some, a privacy regulation that, for example, mandates the removal of data from the public domain is simply unjustified because it opens the door to more fraud.[3] Many disagree with this view, and the EU has recently introduced a highly debated, far reaching privacy regulation, the General Data Protection Regulation (GDPR), that among other things mandates the cancellation by any entity - wherever located - of any personal data on European citizens after a limited time span.[4] We believe our results bring novel, theoretically grounded arguments to this important debate.

Our results are relevant for many markets, as we discuss in detail in Section 8. In the case of credit markets, there are also several empirical studies showing that removals of bankruptcy flags from borrowers credit history and changes in credit record retention significantly affect

[2]Here we use the language of Elul and Gottardi (2015) which shows that forgetting past defaults can be beneficial in a credit market with moral hazard.

[3]From Posner's blog, 8th May 2005. http://www.becker-posner-blog.com/2005/05/index.html
See also Posner (1983) and Nock (1993) .

[4]The EU GDPR 2016/679 became effective on May 25, 2018 (e.g. New York Times, 15 and 24 May 2018), and also entitles European citizens to have their personal data erased if no longer relevant to the original purposes or if they withdraw their consent.

credit markets (Musto (2004), Bos and Nakamura (2014) and Liberman et al. (2017)), and can even have sizable spillover effects on labor markets (Bos et al. (2016)).

Related Literature

Our paper is related to several strands of literature. The focus on the limited provision of information about history relates our paper to the literature on reputation and credit bureau regulation discussed right below. Yet the economic mechanisms that we highlight are very different from the asymmetric information problems (moral hazard and adverse selection) that are at the core of reputation models, and relate our paper more closely to the literature on information design and social learning, which studies dynamic information disclosure. The closest papers to ours in terms of economic mechanisms are Che and Hörner (2017) and Kremer et al. (2014), that focus on how an informed planner can induce short-term experimentation by market participants by over-recommending the less attractive new alternative to early users. The focus of our paper is very different, as we study the effects of the retention policy about user generated feedback in a stationary equilibrium, rather than strategic recommendations to induce short-term experimentation with a new product. The question of optimal information retention policy does not arise in Che and Hörner (2017) and in Kremer et al. (2014), since in these papers once the planner receives the signal about the unknown product, the experimentation stops and it is never optimal to forget this information. In our paper, instead, the types of sellers are not fixed but are changing. As a result, it is optimal to forget any kind of information after an appropriate interval of time, and permanent experimentation is necessary to produce information and avoid market breakdown. Also differently from these papers, we explicitly distinguish between positive and negative feedback and show that it is crucial, as they have fundamentally different effects on market dynamics, so that retention policies for positive and negative feedback should optimally be different.[5]

Optimal partial disclosure by the information intermediary relates our paper to the literature on Bayesian persuasion, where a principal influences agent's behavior through strategic information disclosure. See Kamenica and Gentzkow (2011), Rayo and Segal (2010), and Ostrovsky and Schwarz (2010) for examples of static models, and Ely et al. (2015), Ely

[5]Our paper is also related to the literature on strategic experimentation in markets with market power, including Bergemann and Välimäki (1996, 2000), Bolton and Harris (1999), Keller and Rady (1999), and Keller et al. (2005), among others. These papers do not have an information intermediary and do not analyze the effects of information disclosure, that our paper focuses on.

(2017), Orlov et al. (2018), and Romanyuk and Smolin (2018) for examples of dynamic ones. Differently from this literature, in our paper the information is not exogenous, but is produced by market participants and is endogenous to the information disclosure policy of the intermediary. The information externality among buyers also makes our paper distinct from this literature.

As we mentioned, our paper shares the main research questions with the reputation literature that studies the effects of limiting access to past history on moral hazard problems with adverse selection in dynamic settings, often with reference to credit markets. Vercammen (1995) shows first that limiting the retention of past records may improve incentives in a reputational model of credit market by preventing borrowers with a long history of positive records from "sitting on their laurels". In the same spirit, Padilla and Pagano (2000) show in a two-stage model of a credit market that when a credit bureau publishes too much information it undermines borrowers reputational incentives leading to poor market outcomes. Ekmekci (2011) shows in a general reputation game that censoring past information prevents long-term learning, allowing reputation to become "permanent" where Cripps et al. (2004) showed it would be impermanent otherwise. Elul and Gottardi (2015) study a credit market with moral hazard and adverse selection and characterize situations where reputational effects make forgetting past defaults beneficial. Hörner and Lambert (2015) investigate the optimal rating system in a classic career concern setting, showing when and how past information should be discounted to provide incentives at the cost of worsening the information available to the market.[6] Differently from these papers, our key results are not driven by the information asymmetry, but by the uncertainty about each seller's type and the positive information externality generated by market transactions, which is not internalized by small competitive agents. Our analysis identifies novel mechanisms and leads to quite different policy insights.[7]

The literature on repeated games with restricted memory, where there is no uncertainty about agents types, is even further away from what we are doing here.[8] Finally, our work

[6]See also Padilla and Pagano (1997), Chatterjee et al. (2011), Bottero and Spagnolo (2012), and Liu and Skrzypacz (2014), among others.

[7]For instance, Elul and Gottardi (2015) argue in favor of forgetting defaults while retaining information about successful repayments forever. In our model such a policy leads to a market breakdown and the opposite policy is optimal, that is, short memory for repayments (positive records) and long memory for defaults (negative ones).

[8]Bhaskar and Thomas (2017), for example, shows that erasing defaults allows lenders to commit to an optimal punishment in a moral hazard model with no uncertainty about the borrower's type. Dellarocas (2006) offers an early analysis in this spirit. See also Barlo et al. (2009) and Doraszelski and Escobar (2012).

is also relevant to the economic literature on privacy, recently surveyed in Acquisti et al. (2015).

The structure of the paper is as follows. Section 2 presents our model. Section 3 studies the case of unlimited past records. Section 4 is devoted to limited records, while in Section 5 we analyze welfare. Section 6 obtains correspondent results within an information design framework. Section 7 discusses robustness and extensions, and Section 8 applications and policy recommendations. Section 9 briefly concludes.

2 Environment

Consider an economy populated by sellers, buyers and an information intermediary who interact in continuous time $t \in [0, \infty)$.

Sellers. There is a unit mass of infinitely lived sellers $i \in [0, 1]$. At each instant, a seller may be active on the market (i.e. may have a product to sell) or not. Seller i is active whenever there is a jump in a counting process $\{N_t^i\}_{t \geq 0}$ with Poisson intensity $m > 0$. Markets with many (few) transactions per unit of time can be described by a process with a high (low) intensity m. For instance, in the context of Ebay one can think that with a certain probability, a person may decide to sell an old gadget. Similarly, in the context of a credit market, with a certain probability a potential borrower (seller of debt) may need to borrow from (sell debt to) a bank (buyer of debt).

We normalize the value of the product to the seller to one (it can be the value the seller derives from alternative use, the cost of production or, in the case of the credit market, the amount of necessary investment). The product price $P_i(t)$ is determined by the buyers' willingness to pay, which in turn depends on their expectation of the seller's quality. The seller decides whether to sell the product ($s_i = 1$) or not ($s_i = 0$). His instantaneous payoff is $V_i(t, s_i) = s_i[P_i(t) - 1] + 1$. We assume that the seller is impatient (myopic), that is, the seller heavily discounts the future and cares only about his instantaneous payoff.

Product quality. The buyers' valuation of seller i's product (product quality) $\theta_i(t)$ is stochastic: it can be high ($\theta^H > 1$) or low ($\theta^L = 0$). We refer to θ_i as seller i's type; a good seller's product has quality θ^H and a bad seller's product has quality θ^L. Note that we do not specify whether the seller knows his type or not, as our analysis holds in both cases. The quality of each seller may change over time. For instance, there can be innovations in products offered, changes in the seller's management or ownership, or an evolution of buyers' preferences. Seller i's product quality follows an exogenous time-homogeneous Markov process $\theta_i(t), t \in [0, \infty)$, with an initial probability distribution $\pi_i(0) = Pr(\theta_i(0) = \theta^H)$. For convenience, we introduce the following assumption:

Assumption 1. *At $t = 0$ there is a mass $\mu > 0$ of good sellers $\int_0^1 \pi_i(0)di = \mu$, and for $t \geq 0$ for any seller $i \in [0, 1]$*

$$\frac{d\pi_i(t)}{dt} = -\varphi(\pi_i(t) - \mu),$$

here $\varphi \in (0, \infty)$ parametrizes the intensity of type changes.

Assumption 1 ensures that the fraction of good sellers in the population is constant. Indeed, for $t \geq 0$ the probability of seller $i \in [0, 1]$ being of high type is:

$$\pi_i(t) = \pi_i(0)e^{-\varphi t} + \mu(1 - e^{-\varphi t}) \ , \tag{1}$$

which, together with $\int_0^1 \pi_i(0)di = \mu$, implies $\int_0^1 \pi_i(t)di = \mu$ for any $t \in [0, \infty)$.

Assumption 2. *The average quality of sellers in the population is low, $\mu\theta^H < 1$.*

Assumption 2 implies that without any information, the average seller will not trade. As mentioned earlier, this makes the information disclosed by the intermediary and its feedback retention policy crucial for the market.[9]

Buyers. At each moment $t \in [0, \infty)$, many competitive risk-neutral buyers are matched to active sellers. A buyer is never matched to the same seller twice. Alternatively, we could assume that buyers consume only once in their lifetime, or that they are short-lived, so in each instant buyers are different. We do not model competition between buyers explicitly, but follow Holmström (1999) and Mailath and Samuelson (2001) in simply assuming that buyers are ready to buy a product for a price equal to its expected quality. If the price is above the seller's valuation, the seller sells the product to one of the buyers that he chooses randomly. Before the buyer purchases the product from seller i, he does not know its quality and relies on past feedback h_i^t about seller i provided by the information intermediary (described below). The buyers have no other information about the seller except past feedback and the prior; they believe that the seller is high quality with probability $\mu(h_i^t, \pi_i(0))$, and these beliefs are updated using Bayes' rule. To shorten notation, we will often write $\mu(h_i^t)$ for the

[9]When the average quality of sellers is high trade takes whatever information is available. This case is still of interest because then limited records can generate endogenous, deterministic trade cycles, but characterizing these cycles requires a different, more tractable model; see e.g. Kovbasyuk et al. (2017).

beliefs omitting the prior $\pi_i(0)$. After the buyer purchases the product from seller i at time t, he learns the quality of the product $\theta_i(t)$ and derives utility $\theta_i(t)$.

Information intermediary. After a buyer has purchased the product from seller i and learned its quality, he can leave his feedback f_i^t on it with the information intermediary. The feedback can be positive $f_i^t = S$ (Satisfied) or negative $f_i^t = D$ (Dissatisfied), or there may be no feedback $f_i^t = N$ (No), with no loss of generality. If the seller does not trade there is no feedback $f_i^t = N$. At each $t \in [0, \infty)$ for each seller $i \in [0, 1]$, the information intermediary records feedback $f_i^t \in \{S, D, N\}$ and makes records of past feedback $h_i^t : [0, t) \to \{S, D, N\}$ available to the buyers.

Note that after the purchase the buyer has no incentive to leave feedback. Yet, in reality many buyers do leave feedback. We abstract from the buyer's motivation to leave honest feedback, and simply introduce the following:

Assumption 3. *After purchasing a high quality product the buyer leaves positive feedback S, and after purchasing a low quality product he leaves negative feedback D.*[10]

Market equilibrium. Informally, an equilibrium at each moment in time is characterized by the information the intermediary provides, the buyers' beliefs about sellers and the prices they offer to active sellers, active sellers' decisions to sell, and the feedback that buyers leave about sellers. Formally, for any prior information about sellers $\pi_i(0)$, $i \in [0, 1]$, an equilibrium for each $t \in [0, \infty)$ is characterized by public histories h_i^t of past feedback published by the information intermediary for all sellers $i \in [0, 1]$; buyers' beliefs about sellers' types $\mu(h_i^t, \pi_i(0))$; realizations of Poisson shocks dN_t^i that determine the sellers active at t; prices $P_j(t) \in R^+$ offered by buyers to each active seller j; the optimal selling decision $s_j^t \in \{0, 1\}$ for each active seller j; and feedback $f_i^t \in \{S, D, N\}$ recorded for each seller $i \in [0, 1]$ by the information intermediary. Buyers use Bayes' rule to update their beliefs about sellers.

Let us describe the equilibrium. If a Poisson shock hits seller i at time t, he becomes active and gets matched with many competitive buyers. Buyers use history of past feedback h_i^t to form their belief about the seller's quality $\mu(h_i^t)$, and offer a price equal to the expected value of the product $P_i(t) = \mu(h_i^t)\theta^H$.

Having observed the price, each active seller decides whether to sell the product $s_i = 1$ or not $s_i = 0$ in order to maximize his instantaneous payoff $V_i(t, s_i) = s_i[P_i(t) - 1] + 1$. It

[10] In Section 7.1 we discuss how our results would change if buyers were to leave feedback only with some probability $\lambda \in (0, 1]$.

immediately follows that an active seller sells his product $s_i = 1$ whenever $P_i(t) = \mu(h_i^t)\theta^H \geq 1$. Note that the seller's payoff does not depend on his type θ_i, that is, both types of sellers sell whenever they can get a price of at least one. If the seller decides to sell, the buyer perfectly learns the product's quality and leaves the corresponding feedback $f_i^t = S$ if $\theta_i^t = \theta^H$ and $f_i^t = D$ if $\theta_i^t = \theta^L$. If the seller is not active or if he does not sell, there is no feedback $f_i^t = N$.

As one can see, the equilibrium behavior of all agents can be easily described once one knows the buyers' beliefs $\mu(h_i^t)$ that in turn depend on the information policy of the intermediary. This is the key object of our analysis. We first consider the case of unlimited memory of past feedback and then turn to the case when the intermediary deletes past feedback.

3 Unlimited records

According to Assumption 2, the average quality of sellers is low and there can't be trade unless the buyers are able to tell apart at least some good sellers from the bad ones. In such a situation, one may expect that providing information to the buyers and retaining it indefinitely would be beneficial. For instance, one may think that the availability of perfect information about sellers at $t = 0$ and of full history of feedback at any $t > 0$ would facilitate trade. It turns out this is not the case in the long run.

Theorem 1. *If the intermediary provides full history of past feedback for any seller, then the fraction of sellers trading in equilibrium converges to zero with time.*

All omitted proofs can be found in the appendix. The finding that the full provision of past information is detrimental for trade in the long run, even if the market starts with perfect information, is striking; even more so when contrasted with the potential positive effects on trade in the long run of limited past information retention policies analyzed in the next section.

The logic behind Theorem 1 is very simple. With time, good sellers happen to become bad and get negative feedback. From that moment they are excluded from the market forever because buyers' posterior about their quality never exceeds the unconditional probability of a high-quality seller in population μ and, according to Assumption 2, the unconditional expected quality of a seller is low ($\mu\theta^H < 1$). As time goes to infinity, each seller is almost surely excluded from trading.

What would happen if market conditions are good and Assumption 2 is violated? Then past performance information is not crucial anymore for trade to be sustained, and unlimited memory cannot lead to a market breakdown. The reason is that buyers are willing to trade with sellers of average expected quality, and in the long run sellers with negative ratings today will converge to such average quality and start trading again. As mentioned, our interest here is on markets where information asymmetries are important and therefore past information crucial, hence in the reminder of the paper we will stick to the assumption that the average quality of sellers is low (Assumption 2 is satisfied).

Having illustrated the potential long-run drawbacks of full provision of past information, we can turn to questions: Can deleting past records after a certain period of time facilitate trade in the market? After what period of time should past records be optimally deleted? And should the time of deletion depend on the type of the record (positive versus negative)?

4 Limited records

In this section we first define a stationary equilibrium, and then derive conditions that guarantee the existence of a stationary equilibrium with trade. From now on, we assume that the information intermediary does not provide all past feedback on sellers to the buyers. Instead, the information intermediary deletes negative and positive feedback after timespans $T^- \leq \bar{T}$ and $T^+ \leq \bar{T}$ correspondingly. Clearly, the intermediary does not have to physically delete past information, it can simply choose not to disclose it to the public.

4.1 Relevant records and stationary equilibrium

First, we establish an intermediate result about relevant information that simplifies the analysis considerably. Note that the recorded history of a seller h_i^t can be a rather complex object, as it contains all positive feedback left in the last T^+ periods and all negative feedback left in the last T^- periods. However, the Markov nature of the seller's stochastic quality $\theta_i(t)$ guarantees that just the latest record available at t contains the sufficient information to determine the belief $\mu(h_i^t)$. For instance, if the latest record about seller i was left at $t-\tau$ and was positive $S(\tau)$, then the buyer knows that seller was of high quality at $t-\tau$ but understands that the seller's type might have changed since then. At the same time, any previous record left for the same seller before $t-\tau$ is irrelevant, as it was already obsolete at $t-\tau$ when the latest record was left.

Positive records are deleted T^+ periods after they are left, hence, without loss of generality we can say that the seller has a record $r_i^t = S(\tau)$, $\tau \in [0, T^+]$ at time t if his latest record was positive and was left at $t - \tau$. Denote by $S = \{S(\tau) : \tau \in [0, T^+]\}$ the set of all possible positive records. Analogously, for negative records we say that the seller has a record $r_i^t = D(\tau^-)$, $\tau^- \in [0, T^-]$ at time t if his latest record was negative and was left at $t - \tau^-$. Also let $D = \{D(\tau^-) : \tau^- \in [0, T^-]\}$ be the set of all possible negative records. Finally, the seller may not have any record, because his past records were deleted and he received no good feedback in the last T^+ periods and no bad feedback in the last T^- periods, in which case we say that the seller has a record $r_i^t = N$. Denote by $G = S \cup D \cup N$ the set of all possible records that a seller might have. The above arguments imply the following:

Lemma 1. *For any seller $i \in [0,1]$ at any time $t \in [0, \infty)$, the latest record $r_i^t \in G$ contains all type-relevant public information about the seller.*

In what follows we use records $r \in G$ instead of histories. This considerably simplifies the analysis: the joint distribution of sellers' types and records contains all relevant information about the economy at any moment of time t and, therefore, pins down the buyers' beliefs about sellers, the prices offered by the buyers and the sellers' selling decisions. This distribution is characterized by the following components: $\boldsymbol{\Delta}_t = \{\rho_t^N, \rho_t^S(.), \rho_t^D(.), \eta_t^N, \eta_t^S(.), \eta_t^D(.)\}$. Here, ρ_t^N and η_t^N are masses of good and bad sellers with an N record at time t, density functions $\rho_t^S(\tau)$ and $\eta_t^S(\tau)$, $\tau \in [0, T^+]$ determine densities of good and bad sellers with records $S_t(\tau)$ at time t, and density functions $\rho_t^D(\tau^-)$ and $\eta_t^D(\tau^-)$, $\tau^- \in [0, T^-]$ determine densities of good and bad sellers with records $D(\tau^-)$ at time t. Note that Assumption 1 guarantees that $\rho_t^N + \int_0^{T^+} \rho_t^S(\tau) d\tau + \int_0^{T^-} \rho_t^D(\tau^-) d\tau^- = \mu$ and $\eta_t^N + \int_0^{T^+} \eta_t^S(\tau) d\tau + \int_0^{T^-} \eta_t^D(\tau^-) d\tau^- = 1 - \mu$ for any $t \geq 0$. Let us now define the stationary equilibrium in our environment.

Stationary equilibrium. Stationary equilibrium is a market equilibrium in which the joint distribution of sellers' types and records is time invariant,[11] that is, $\boldsymbol{\Delta}_t = \boldsymbol{\Delta}$.

In the subsequent analysis we focus on stationary equilibria. We find conditions under which different stationary equilibria exist and study the properties of these equilibria.[12] First, note that a stationary equilibrium with no trade exists. This observation echoes the negative result of Theorem 1 as trade collapses in the long run, and is very intuitive. If no

[11] In principle, our model may have non-stationary equilibria that are outside the scope of this paper. Non-stationary equilibria are the focus of the companion paper Kovbasyuk et al. (2017).

[12] Note that in a stationary equilibrium the records of individual sellers constantly change, but this is not important from the aggregate point of view as far as the join distribution of sellers' types and records $\boldsymbol{\Delta}$ is constant.

seller trades, there is no feedback. When there is no feedback, in the long run all sellers look identical to the buyers and are believed to be of a high quality with the same probability μ. Assumption 2 implies that the buyers' willingness to pay is below the sellers reservation value $\mu\theta^H < 1$, and there is no trade in equilibrium.

4.2 Stationary equilibrium with trade

In this section we consider limited records, and derive condition that support a stationary equilibrium with trade. We start by describing some general features of the stationary equilibrium with trade.

Active sellers with negative records do not trade. Naturally, sellers with negative records do not sell because the buyers believe their products to be of low quality (lower than the population mean μ) and offer prices below the reservation value of the sellers.[13]

Active sellers with an N record sell. If active sellers with an N record were not selling in equilibrium then with time all sellers would get an N record and would stop selling, which contradicts the definition of a stationary equilibrium with trade.

Active sellers with positive records sell. Intuitively, a seller with a positive record $S(\tau)$, $\tau \in [0, T^+]$ has a higher expected quality and gets higher price offers than a seller with an N record and must be selling in equilibrium.

Let us now state an important intermediate result.

Lemma 2. *For a stationary equilibrium with trade to exist, positive records must be deleted sooner than negative ones: $T^+ < T^-$.*

The intuition is the following. In a stationary equilibrium with trade there must be trade with sellers with an N record. Since the average quality of sellers is low, $\mu\theta^H < 1$, the memory policy must ensure that the average quality of sellers with an N record is higher, $\mu(N)\theta^H \geq 1$. To do that, the information intermediary must exclude bad sellers from the pool with an N record for longer than the good ones, that is $T^- > T^+$.

Now we characterize all possible limited records that support a stationary equilibrium with trade.

Theorem 2. *1. A stationary equilibrium with trade exists if and only if: i) positive feedback is erased after a sufficiently short interval of time, $T^+ < \bar{T}^+$, ii) negative feedback is*

[13]Formally, a seller i who got a negative feedback at $t - \tau^-$ and no feedback since then, is of high type with probability $\pi_i(t) = \mu(1 - e^{-\varphi\tau^-})$. Since $\mu\theta^H < 1$, his expected quality is below his reservation value and he does not sell.

erased after a sufficiently long interval of time, $T^- \geq \underline{T}^-(T^+)$, and iii) the seller's type is sufficiently persistent:

$$\frac{\varphi}{m} < \frac{\mu(\theta^H - 1)}{1 - \mu\theta^H} \ . \tag{2}$$

$\bar{T}^+$ is determined by

$$\frac{1-\mu}{\mu}\frac{\varphi}{\varphi+m}[1-\mu\theta^H] = e^{-m\bar{T}^+}[\mu\theta^H + (1-\mu)\theta^H \frac{m}{m+\varphi}e^{-\varphi\bar{T}^+} - 1] \ , \tag{3}$$

and $\underline{T}^-(T^+)$ solves

$$\frac{1-\mu}{\mu}\frac{\varphi}{\varphi + m(1-e^{-\varphi\underline{T}^-})} = \theta^H[\mu e^{-mT^+} + (1-\mu)\frac{\varphi + me^{-(m+\varphi)T^+}}{m+\varphi}] - e^{-mT^+} \ . \tag{4}$$

2. There can be at most one stationary equilibrium with trade.

The proof is in the Appendix. Conditions (4) and $T^- \geq \underline{T}^-(T^+)$ guarantee that a seller with an N record trades, whereas conditions $T^+ < \bar{T}^+$, (2) and (3) make sure that one can find $T^+ \geq 0$ and $T^- \geq 0$ that satisfy (4) and $T^- \geq \underline{T}^-(T^+)$.

The intuition behind the result is as follows. First, condition (2) is very natural because for records of past feedback to have any value, the sellers' types must be persistent enough. Indeed, if the sellers' types change very quickly, past feedback becomes obsolete very quickly and it does not matter how long the information intermediary keeps it for. Second, a stationary equilibrium with trade exists if and only if the average quality of the non-rated (unknown) pool of sellers with N records is high. Otherwise, the pool of sellers with an N record can become a "black hole" for the market: if these sellers do not sell, then there may be no trade in the long run, as all sellers almost surely enter this pool with time and can't exit it. The conditions provided in Theorem 2 guarantee that the expected quality of sellers with an N record is high enough to sustain trade. Essentially, long memory for negative feedback T^- keeps low quality sellers outside of the non-rated pool of sellers with an N record, thereby increasing the average quality of this pool. Short memory for positive feedback T^+, brings high quality sellers to the non-rated pool quickly, and also increases the average quality of this pool.

At first, the fact that limited memory can lead to better long-run outcomes (stationary equilibrium with trade) than full memory (collapse of trade) is surprising. Yet, this result has a clear and robust rationale behind it. When trading, buyers produce a positive informational externality - the feedback on the seller. With perfect information about past feedback, buyers do not experiment enough - they only buy from sufficiently good sellers. Because of low experimentation and the Markov nature of sellers' types, past feedback becomes obsolete with time, and in the long run the market ends up, essentially, with no information but the unconditional fraction of good sellers in the market μ. Limited memory creates a pool of sellers with unknown history, and if the quality of this pool is sufficiently high, buyers will be ready to experiment by buying from the sellers in this pool and producing information. Essentially, by forcing good records to be forgotten, one can induce the market to produce enough new information to countervail the natural loss of information due the Markov nature of sellers' types, so that sufficient information is available to the market in the long run to sustain trade.

Example 1 (only negative records). In order to illustrate Theorem 2, consider a simple example. Suppose that (2) holds and positive feedback is not recorded $T^+ = 0$ (no positive memory). It is easy to check that in this case $T^+ < \bar{T}^+$, and condition $T^- \geq \underline{T}^-$ is necessary and sufficient for an equilibrium to exist. Using (4) condition $T^- \geq \underline{T}^-$ can be rewritten as:

$$\mu\theta^H \geq \frac{1-\mu}{1+\frac{m}{\varphi}(1-e^{-\varphi T^-})} + \mu \ .$$

When negative memory is short $T^- = 0$ the above condition it is not satisfied, because $\mu\theta^H < 1$. However, for long negative memory $T^- \to \infty$ it becomes $\mu\theta^H \geq \frac{\varphi}{m+\varphi}(1-\mu) + \mu$ and is equivalent to (2), which is satisfied if $\frac{\varphi}{m}$ is small. In other words, if the seller's type changes slowly relative to the intensity of trade in the market, then introducing a long memory for negative feedback can support trade in the long run and prevent the market breakdown.

It turns out the result illustrated in the above example is more general. From Theorem 2 it becomes clear that the existence condition is least stringent when negative memory is longest $T^- = \bar{T}$ and positive memory is shortest $T^+ = 0$. The intuition is as follows: long positive memory allows many good sellers to maintain positive feedback for a long time in a stationary equilibrium, which depletes the average quality of the pool of unknown sellers with an N record. Conversely, short memory for positive records improves the average quality of this pool and relaxes the existence condition for the stationary equilibrium equilibrium with trade. The long memory for negative feedback also helps to keep bad sellers out of the pool of sellers with an N record and to improve its average quality.[14]

Example 2 (fixed types). Hidden in Theorem 2 is another difference between positive and negative records, in addition to the main one that they influence equilibrium existence condition in opposite ways. To illustrate this second difference between positive and negative records, consider the special case where the sellers' types are almost permanent, that is, their types change with a very low intensity $\varphi \to 0$. Substituting for $\underline{T}^-$ from (4) into $T^- \geq \underline{T}^-$ and taking the limit when $\varphi \to 0$, we obtain the existence condition for a stationary equilibrium with trade:

$$(1 + mT^-)e^{-mT^+} \geq \frac{1-\mu}{\mu(\theta^H - 1)} .$$

Clearly, longer retention of negative feedback T^- relaxes the existence condition, while longer retention of positive feedback T^+ has the opposite effect. However, the strength of the two effects is also different. While the positive effect of T^- is linear, the negative effect of T^+ is exponential, that is, potentially much stronger. To see why this is the case, consider a good and a bad seller that have an N record and happen to trade at a given moment. The bad seller gets negative feedback after selling the product and leaves the pool of sellers with an N record. He is effectively excluded from trade for the time the negative feedback is retained T^-, after which the feedback is deleted and he enters back into the pool of sellers with an N record. Now consider the good seller. After selling the product he gets positive feedback and leaves the pool of sellers with an N record. Given that he has a good record, he can potentially trade again and get new positive feedback. The positive feedback is retained for T^+ periods and if the seller trades before the positive feedback is deleted, he gets new positive feedback which will be retained for another T^+ periods and he will be able to trade again. By repeating this argument, one can see that once a good seller leaves the pool of sellers with an N record, he can spend much longer than T^+ periods outside of this pool before entering again. This is why retaining positive records has a stronger negative effect on the average quality of the sellers with an N compared to the positive effect of retaining negative records. This is an additional reason why it is important to treat positive and negative records separately when analyzing models with feedback.

[14]Note, that for an equilibrium to exist, it must be technologically possible to keep past records for a sufficiently long time.

5 Welfare analysis

In equilibrium, two types of direct social losses may occur: a) when a bad product is purchased and the buyer suffers, and b) when a good product is not purchased and the potential buyer forgoes consumer surplus. Suppose a stationary equilibrium with trade exists. At each instance in such an equilibrium, only sellers with positive records $S(\tau)$, $\tau \in [0, T^+]$ and sellers with no records N trade. The value of the product to the seller is 1. A good product generates utility $\theta^H > 1$ to the buyer, the bad product generates no utility. The mass of good sellers trading during a short time interval dt is

$$m[\rho(N) + \int_0^{T^+} \rho(S(\tau))d\tau]dt,$$

the mass of bad sellers trading is

$$m[\eta(N) + \int_0^{T^+} \eta(S(\tau))d\tau]dt.$$

The surplus generated over a small interval of time dt can be expressed as follows:

$$Wdt = \left\{ [\rho(N) + \int_0^{T^+} \rho(S(\tau))d\tau](\theta^H - 1) - \eta(N) - \int_0^{T^+} \eta(S(\tau)d\tau \right\} mdt.$$

We take as a measure of welfare the flow of surplus per unit of time W. Note that in a stationary equilibrium welfare does not depend on t.

Lemma 3. *Welfare in a stationary equilibrium with trade is given by:*

$$W = m\frac{\mu\theta^H - 1 + m\frac{\mu}{\varphi}(1 - e^{-\varphi T^-})(\theta^H - 1)}{1 + m(1-\mu)T^- + m\frac{\mu}{\varphi}(1 - e^{-\varphi T^-})}. \tag{5}$$

The proof can be found in the Appendix. In the proof we characterize the joint stationary distribution of sellers' types and records, and then use this distribution to calculate masses of good and bad sellers trading in equilibrium, and to compute welfare.

Theorem 3. *1. If condition (2) does not hold, then only the no-trade stationary equilibrium exists and welfare is zero.*
2. If (2) holds, then the highest welfare is achieved in a stationary equilibrium with trade in which negative feedback is deleted after the timespan T^-_W, such that:

$$1 - \mu\theta^H - (\theta^H - 1)\frac{m\mu}{\varphi} + \mu[\theta^H + m(\theta^H - 1)(T_W^- + \frac{1}{\varphi})]e^{-\varphi T_W^-} = 0, \tag{6}$$

and positive feedback is deleted after a shorter timespan $T^+ \in [0, T_W^+]$. *Here* $T_W^+ < T_W^-$ *solves* $\underline{T}^-(T_W^+) = T_W^-$ *given by (4).*

The result is intuitive. If there is no trade, then welfare is zero. According to Theorem 2, condition (2) must hold for a stationary equilibrium with trade to exist. In a stationary equilibrium with trade it is beneficial to exclude sellers with negative records from trade. Indeed, a seller who got negative feedback τ^- periods ago is high quality with probability:

$$\pi_i(D(\tau^-)) = \pi_i(-\tau^-)e^{-\varphi\tau^-} + \mu(1 - e^{-\varphi\tau^-}) = \mu(1 - e^{-\varphi\tau^-}) . \tag{7}$$

Trade with such a seller is associated with a direct social welfare loss if $\mu(1-e^{-\varphi\tau^-})\theta^H < 1$. Assumption 2 implies that for any $\tau^- < T^- \leq \bar{T}$ this inequality holds, therefore trading with sellers with negative feedback involves a direct social cost. However, there is also an indirect benefit of trading with these sellers which comes from learning their actual types. This information has a positive social value as it enables future buyers to distinguish the sellers and trade only with those that have sufficiently high expected quality. In essence, there is a positive option value of a trade with a seller with negative feedback. For a seller who got negative feedback time T_W^- ago, the option value of learning his type compensates for the expected direct loss generated by the trade. That is why it is not socially optimal to exclude sellers with negative records from trade forever. When negative records are erased after time T_W^-, sellers with negative records older than T_W^- are able to trade and the social welfare is highest.

The memory for positive feedback does not affect welfare as long as it is not too long $T^+ \in [0, T_W^+]$, otherwise the stationary equilibrium with the highest welfare is not feasible because existence condition (4) is violated. If positive memory is sufficiently short, it's exact length does not matter for welfare. Intuitively, a seller with an N record can trade as well as a seller with a positive record $S(\tau)$, $\tau \in [0, T^+]$. Therefore, it does not matter when the positive record is erased as the seller can trade anyway.

6 General recommendation system

Instead of restricting the information intermediary to a particular memory policy, in this section we show that analogous results can be obtained adopting an information design approach and considering a general recommendation mechanism. Because actions of buyers are binary, they either purchase from a seller or not, it is without loss of generality to consider binary yes/no recommendations: The intermediary either recommends the seller ($r = 1$) or not ($r = 0$).

At each date t, the intermediary observes the full history of feedback for each seller h_i^t, $i \in [0, 1]$ and computes the posterior π_i, thus he knows posteriors $\pi_i \in [0, 1]$ for all sellers at t. The buyers, on the other hand, do not see these histories and rely on the intermediary's recommendations. Here we focus on threshold recommendation policies, in which the intermediary recommends sellers that have a posterior above a certain threshold $z \in (0, 1) : r(\pi) = 1$ for $\pi \geq z$ and $r(\pi) = 0$ for $\pi < z$.

Intuitively, if an intermediary that maximizes welfare recommends a seller with a posterior π, he will also recommend a seller with a posterior $\pi' > \pi$.

As is standard in the literature on Bayesian persuasion, we assume that the intermediary can commit to the recommendation policy, that is, he can commit to the threshold z at $t = 0$ and then implement it for $t \in [0, \infty)$. We are interested in stationary equilibria that characterize long-term market outcomes, and consider policies that are independent of t. Such policies are common in practice: privacy regulations, internet platforms, credit and criminal records rules typically require information to be deleted after a certain fixed time span, which does not depend on calendar time.

Recall that the average quality of seller's is low ($\mu\theta^H < 1$) and without an appropriate recommendation policy there would be no trade. Let us first establish an analog of Theorem 1 about market collapse in the long run.

Theorem 4. *A recommendation policy with a threshold $z \geq \mu$ leads to no trade in the long run.*

We omit the formal proof of this Theorem as it is similar to that of Theorem 1. The intuition is also similar. Sellers' types are Markov: Each seller will turn bad at some point, trade, get a negative feedback, and end up with a posterior $\pi = 0$. From that moment on the seller will never be able to trade, as the posterior about his type will converge to the mean (μ) from below, which is lower than z, i.e. the seller will never be recommended to trade again. With time the mass of sellers with a posterior above μ converges to zero, and the market collapses. Note also, that the posterior about the sellers that are not recommended stays strictly below μ, so that buyers follow the intermediary's recommendations not to buy.

By Theorem 4 considering policies $z < \mu$ is without loss of generality. It turns out that a policy with a threshold $z < \mu$ is equivalent to a policy from the class of limited record policies studied in Section 4. Consider a limited records policy with $T^+(z) = 0$ and $T^-(z)$, which solves

$$\mu(1 - e^{-\varphi T^-(z)}) = z. \tag{8}$$

Lemma 4. *A limited records policy $T^+(z)$ and $T^-(z)$ is equivalent to a threshold policy z.*

The proof is immediate. With record limits $T^+ = 0$ and $T^-(z)$, any seller who has received a bad feedback time span $\tau \in [0, T^-]$ ago, has a posterior $\pi = \mu(1 - e^{-\varphi\tau}) \leq z < \mu$ and is excluded from trade, which is equivalent to a no recommendation ($r = 0$) under the threshold recommendation policy. Analogously, a seller without a bad feedback under the limited records policy has an empty history (an N record) and can trade. Under the threshold recommendation policy such a seller would have a yes recommendation ($r = 1$) and would also trade. Therefore, the two policies are equivalent and induce the same market outcomes. QED.

This implies that Theorems 2 and 3 can be generalized and applied to threshold recommendation policies. We first state an analog of Theorems 2 about the existence of a stationary equilibrium with trade.

Theorem 5. *A stationary equilibrium with trade can be implemented with a threshold policy if*

$$\underline{z} = \frac{\varphi}{m} \frac{1 - \mu\theta^H}{\mu(\theta^H - 1)} < \mu,$$

and any threshold $z \in [\underline{z}, \mu)$ implements a stationary equilibrium with trade.

The proof is straightforward. By Lemma 4 a threshold policy z is equivalent to a limited records policy with limits $T^+ = 0$ and $T^-(z)$, which is given by (8). Condition $\underline{z} < \mu$ is equivalent to (2) in Theorem 2. Condition $T^+ < \overline{T}^+$ is satisfied with $T^+ = 0$. Also, condition $T^- \geq \underline{T}^-(0)$ is equivalent to $z \geq \underline{z}$, and $z < \mu$ is equivalent to $T^- < \infty$. Therefore any threshold policy $z \in [\underline{z}, \mu)$ implements a stationary equilibrium with trade. Conversely, if $z < \overline{z}$, or $z \geq \mu$ a stationary equilibrium with trade cannot be implemented by Theorem 2, or by Theorem 4 respectively. QED.

Analogously, one can find a threshold recommendation policy which maximizes the flow of welfare in a stationary equilibrium. If (2) holds Theorem 3 characterizes T_W^- which maximizes the flow of welfare. Let's define the equivalent threshold policy

$$z^* = \mu(1 - e^{-\varphi T_W^-}).$$

This allows us to establish the following result.

Theorem 6. *If $\underline{z} < \mu$, there exists a unique threshold policy with the threshold z^* which supports the stationary equilibrium with the highest welfare.*

The proof is straightforward and follows directly from Theorem 3 and Lemma 4.

This analysis illustrates that our results hold if one considers information design by the information intermediary, provided that the intermediary uses a threshold recommendation policy. As we argued before, the threshold policies are natural: the intermediary only recommends sellers with relatively high posteriors. Considering more general recommendation policies, i.e. non-monotone recommendation policies, is beyond the scope of this paper. Moreover, non-monotone policies do not seem realistic, and are unlikely to be optimal in a stationary environment.

7 Robustness and extensions

In this section we discuss the robustness of our findings by relaxing some of the many simplifying assumptions that make the model tractable, and develop some extensions that are of interest for particular applications.

7.1 Buyers do not always leave feedback

It is easy to verify that we can relax Assumption 3 that the buyer always leaves feedback after purchasing the product, and assume instead that a buyer leaves feedback only with a certain probability $\lambda \in (0, 1]$. In this case, the probability of trading and getting feedback is $m\lambda$ instead of m, so the results do not change qualitatively. For instance, the existence conditions of Theorem 2 when the feedback is left with probability λ can be obtained by replacing m with $m\lambda$. Other results can be also easily established for $\lambda \in (0, 1]$. This extension of our results is important for online market platforms such as Ebay and Amazon, where the probability of a buyer leaving feedback after a transaction can be as low as few percent.

7.2 Quality-insensitive buyers

In the analysis we assumed that no buyer values the low-quality product as much as the seller $\theta^L = 0 < 1$. In other words, the sale of a low-quality product is inefficient. As a result, in our equilibrium sellers with negative feedback are effectively excluded from the market. This is a stark prediction, as one may think that in reality there are always some buyers that will buy even from the low-quality sellers. This may be either because they value low-quality products more than other buyers and the seller, or because they make mistakes. In this section, we analyze the possibility that some buyers are not quality sensitive, i.e., they value the low-quality product as much as the high-quality product and their valuation is equal to the seller's reservation value of one. This way we ensure that any seller may have a chance to trade no matter what his feedback is.

We assume that each active seller, when matched with several buyers, may have a quality-insensitive buyer among them with probability $\beta \in (0, 1]$. The quality-insensitive buyer, whenever matched to an active seller i, bids his valuation $b_i = 1$ independently of the seller's posterior probability $\mu(r_i^t)$ of being high type. Note that the buyer does not want to bid less than 1 because the seller's reservation value is also 1 and he only trades when the price is at least 1. It immediately follows that any active seller with expected quality $\mu(r_i^t)\theta^H < 1$ can only sell to a quality-insensitive buyer, which happens with probability β. In other words, sellers with low expected quality trade with a low intensity $\beta m \leq m$. Apart from not valuing quality, the quality-insensitive buyers are exactly like other buyers: they learn the quality of the product after the purchase and leave feedback with the informational intermediary.[15]

Clearly, in such an environment there will always be trade, as quality-insensitive buyers are ready to purchase from any seller. We are interested in a stationary equilibrium in which sellers with an N record are able to sell to all buyers, not only the information-insensitive ones. We call this equilibrium the "high-trade stationary equilibrium".

Theorem 7. *The high trade stationary equilibrium exists if and only if memory for positive*

[15] If one is not comfortable with quality-insensitive buyers actually learning the true quality of the product, one can instead assume that ordinary buyers sometimes make mistakes: with probability β one of them offers a price of one for a product of expected quality below one.

feedback is short enough, memory for negative feedback is long enough, and the sellers' types are persistent enough: $T^+ < T^*$, $T^- \geq \hat{T}$, *and*

$$\frac{\varphi}{m} \leq \frac{\theta^H - 1}{1 - \mu\theta^h} \; . \tag{9}$$

Here T^* *solves* $A(T^*) = \theta^H(\mu + (1-\mu)\frac{m}{m+\varphi}e^{-\varphi T^*}) = 1$ *and* $\hat{T}$ *solves* $C(T^*, \hat{T}) = 0$,

$$\begin{aligned} C(T^+, T^-) = & \frac{Xe^{-mT^+}}{m} + \frac{e^{-m\beta T^-}}{m} - \\ & \theta^H(X(\frac{\mu e^{-mT^+}}{m} + \frac{(1-\mu)e^{-(m+\varphi)T^+}}{m+\varphi}) + \frac{\mu e^{-m\beta T^-}}{m} - \frac{\mu e^{-(m\beta+\varphi)T^-}}{m+\varphi}) \; , \end{aligned} \tag{10}$$

$$X = \frac{\mu}{1-\mu}(1 + \frac{m(1-\beta)}{m\beta+\varphi}(1 - e^{-(m\beta+\varphi)T^-})) \; . \tag{11}$$

This result is similar to Theorem 2: the existence condition for the high trade stationary equilibrium is less stringent when memory for positive feedback shortens (T^+ goes down), and when memory for negative feedback lengthens (T^- goes up).

8 Applications and policy

Until recently, little economic theory was available to guide policy on how long "public memory" should be in different environments and for different types of records. Regulation on data retention, on the other hand, has been in place for quite some time in many countries, so it is not surprising that it has remained rather generic. Even the mentioned new EU regulation, the GDPR, remains quite imprecise on the exact retention limits.[16]

In this section we discuss the policy implications of our results. As always, the implications should be taken with a grain of salt, as to allow to uncover the main mechanisms discussed earlier, our model abstracts from many issues that can be relevant in some markets, like moral hazard and reputation. Depending on a particular application, the optimal policy is

[16]It states that "*The data should be: [...] e) kept in a form which permits identification of data subjects for no longer than is necessary for the purposes for which the personal data are processed; personal data may be stored for longer periods insofar as the personal data will be processed solely for archiving purposes in the public interest, scientific or historical research purposes or statistical purposes in accordance with Article 89(1) subject to implementation of the appropriate technical and organisational measures required by this Regulation in order to safeguard the rights and freedoms of the data subject ('storage limitation');*" (Article 5(1)(e)), and that "*The personal data should be adequate, relevant and limited to what is necessary for the purposes for which they are processed. This requires, in particular, ensuring that the period for which the personal data are stored is limited to a strict minimum. Personal data should be processed only if the purpose of the processing could not reasonably be fulfilled by other means. In order to ensure that the personal data are not kept longer than necessary, time limits should be established by the controller for erasure or for a periodic review.*" (Recital 39).

likely to mix and balance recommendations coming from different models.

8.1 Credit and online markets, and criminal records

Credit markets exemplify the wild variation in adopted retention policies. Figure 1 plots the number of years after which positive and negative information about borrowers must be erased by credit bureaus for a handful of countries. Positive information generally contains the pattern of repayments, open and closed credit accounts and new loans, while negative information is about defaults, bankruptcies, delinquencies, arrears. As one can see from Figure 1, retention limits differ substantially even among similar countries.

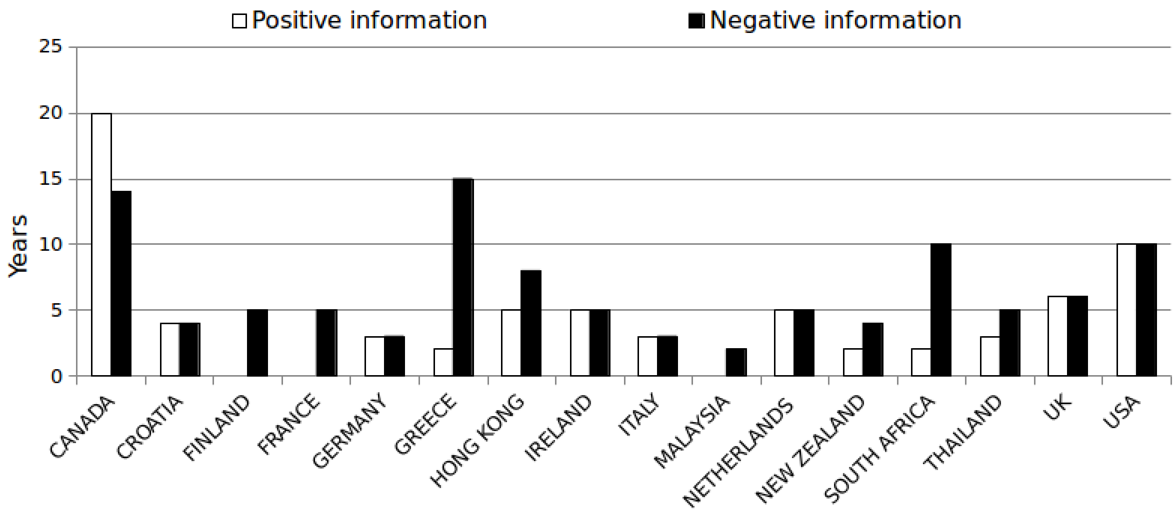

Figure 1: Retention periods for credit records

Our results suggest that for countries or credit market segments where lending is risky because the average quality of borrowers or the success chances of their project is poor, limiting credit bureau's ability to collect and distribute positive information may be a good policy option for reducing the risk of market breakdowns.

Internet platforms collecting feedback on participants also have to decide for how long to leave past records accessible, and how much memory to assign to summary indicators. For example, in 2008 eBay changed its reputational indicator and made it a function of feedback left in the last year only, instead of all past records, while still leaving all past feedback accessible but at the cost of some search effort. The effective absence of legal enforcement for most online transactions allows wide scope for fraud, and for this reason platforms consider their feedback system crucial to their survival. Our results then apply, and suggest that eBay should at least have considered the possibility of assigning a different memory to positive and negative feedback in its new reputation indicator (and possibly limiting the overall history of the feedbacks it makes accessible to buyers).[17]

Criminal records typically do not contain "positive" past records. Police and court databases tend to automatically store only negative past information, and in many countries there are (different) rules that limit for how long these can be accessed. These policies appear closer to what our model suggests could be optimal.[18]

8.2 The "right to be forgotten" on the Internet

For other environments, it may be more difficult to distinguish, ex ante, positive from negative information. This, however, does not mean that our results do not apply. For instance, our results are relevant for the debate on "the right to be forgotten" on the Internet, even though information is not already classified as "good" and "bad".

On May 13, 2014, the European Court of Justice ruled against Google in favor of Costeja (Judgement in case C-131/12), a case brought by a Spanish man, Mario Costeja Gonzalez, who requested the removal of a link to a digitized 1998 article in the La Vanguardia newspaper about an auction for his foreclosed home due to debt that he had subsequently repaid. The court ruled in favor of Costeja that search engines are responsible for the content they point to and that individuals have the right to ask search engines to remove links with personal information about them. This applies where the information is considered "excessive in relation to the purposes for which they were processed and in the light of the time that has elapsed." This ruling was highly controversial and sparked a lively transatlantic debate.[19] Still, its fundamental principle has been fully incorporated in Article 17 of the EU GDPR.

Suppose that it is in principle possible to erase past records from search results after some time. In particular, suppose that in our environment a regulation is introduced stating that, if a seller's record is older than T^f, then the seller can ask for that record to be erased from

[17]Moreover, on eBay the feedback is not verifiable, so the possibility and incentives to provide false feedback, may make a long memory of feedback not advisable.

[18]Still, there is a wild variation in the memory of criminal records, even within the US. For example, the minimum time after which misdemeanors can be expunged ranges from the one year of Hawaii to the five years of California, Kentucky and Mississipi, the ten years of Florida and the never of several other states (see http://ccresourcecenter.org/state-restoration-profiles/50-state-comparisonjudicial-expungement-sealing-and-set-aside/).

[19]For example, the New York Times published two articles in 2015 on that topic titled "Right to Be Forgotten Online Could Spread" and "Europes Expanding Right to Be Forgotten"'. See also Keller (2017) for further information.

the public information. How will the regulation affect our equilibrium?

We showed that in any stationary equilibrium with trade, the sellers without records must be expected to be of sufficiently high quality to induce buyers to trade, while no trade occurs with sellers with negative feedback. Consider a seller's incentive to require his most recent record to be erased when it becomes T^f periods old (records other than the most recent are payoff-irrelevant or are cancelled according to the same logic stated here).

If that record is negative, the seller will indeed request that it is erased, as getting an N record allows him to start trading again. If the most recent record is positive, the seller has no reason to request its cancellation, as it would not increase the chance of trading. The same reasoning applies to the records older than the most recent one: the seller will ask for negative records that are more than T^f periods old to be removed, but of course not for the positive ones.

Remark 1. A rule allowing sellers to have their records "forgotten" when they become older than some T^f is either irrelevant (when $T^f \geq T^-$) or will lead to a shorter memory for negative records without affecting the memory for positive records.

The right to be forgotten is unlikely to affect the length of positive records, as agents are unlikely to ask to delete positive information about them. As for negative records, if an equilibrium with trade is sustainable, that is if $T^f \geq T^-$, then the policy goes in the right direction if T^f is close to the optimal T^-_W, as defined in Theorem 3. If on the other hand, deletion of negative records makes the equilibrium with trade not sustainable, that is $T^f < T^-$, then the right to be forgotten can cause a market breakdown in the long run.

All in all, if people are given the right to control past information about them, they would know which of their past records are good or bad given their private situation and future plans, and are likely to ask for the removal of negative past information only. The Court's decision represents a step in the right direction by avoiding an excessively long memory for negative records, but it effectively puts no restriction on the memory for positive records, which may not always be desirable according to our analysis.

8.3 Need for regulation

Even though excessively long memory of records may have the negative aggregate/social effects we described, in many environments these records are still valuable to individual buyers, and private incentives exist for intermediaries to collect and distribute them. Therefore, our model suggests that privacy regulation limiting data retention of negative and positive records separately may be desirable in markets plagued by informational problems.

Our analysis also suggests that the current regulation, where present, may not be optimal, at least in markets where average quality is poor. For example, the current trend of credit bureaus increasingly collecting positive records (in addition to the usual negative ones) may end up having harmful long-term consequences in high risk credit market segments. These positive past records may make it very difficult for borrowers without any record to enter (or re-enter) the market and obtain credit in the first place.[20] Similarly, as discussed above, if regulators let individuals choose which information on Internet is deleted, then negative but not positive information will be deleted, with potentially harmful consequences for labor and other markets. In both cases, mandatory deletion of positive information may be beneficial.

8.4 Policy evaluation

Our analysis focuses on the long-run effects of limited memory of an information intermediary on information production and trade. However, it is possible that the short-term effects are very different from the long-run ones. For instance, an increase in memory for positive records can improve information and can be beneficial in the short term. However, as we have shown, such an increase can be detrimental for future information production and trade. This suggests that evaluating the effects of changes in data retention can be more challenging than assumed in recent empirical papers, as some of the effects may only come into play with time.

9 Conclusion

We studied a dynamic market where each seller's quality is uncertain and stochastically changes with time. In the model, as in many real markets, an information intermediary collects past feedback on sellers and publicly reports it in order to improve the information available to buyers. The model shows that even in the absence of moral hazard and adverse selection, when sellers' quality is unknown one has to think carefully about the length of "public memory" of past feedback.

[20]A recent New York Times article by Patricia Cohen (Oct. 10, 2014) documents how difficult it is for people with no credit history to obtain a first loan in the highly informed US credit market, leading them to form neighborhood-based rotating saving and credit associations to obtain their first records. This suggests that the long-term effects we highlight may actually manifest themselves rather quickly in certain markets.

A straightforward transparency argument - "disclosing more information is always better for market efficiency" - turns out to be flawed when the average quality of sellers is low: unlimited memory of sellers' past feedback always leads then to a market breakdown, even if the market starts with perfect information about sellers' types.

We then show that in the same market with a low average quality of sellers, limiting the memory of the information intermediary can have beneficial effects: it improves information production by the market so that stationary equilibria with trade become sustainable.

We also find that it is crucial to distinguish between positive and negative past records, as they affect the market equilibrium in opposite ways, and with different intensity. A stationary equilibrium with trade is easier to sustain if the memory of positive records is short and the memory of negative ones is long. A long memory for negative records keeps bad sellers out of the pool of sellers with no records for a long time, improving the average quality of that pool and thereby encouraging experimentation (i.e. trading with non-rated sellers). The opposite holds for positive records: the longer they are visible, the longer good sellers are separated from the pool of sellers with no records, the lower the average quality in that pool, and the less buyers are inclined to trade with non-rated sellers. The intensity of these effects is also different because negative records simply prevent sellers from trading for a period of time, while positive records have a nonlinear, self-enforcing effect by inducing further trade and thereby new records right afterwards.

Optimal regulation of the information intermediary when market conditions are poor may require limiting both positive and negative records, although to a different extent. As argued above, long positive records may prevent buyers from experimenting with non-rated sellers and lead to a market collapse. The memory of negative records affects welfare in a more subtle way, and there is an optimal length for it. A long memory of negative records helps to sustain trade in the long run, but if it is too long, the level of trade and learning becomes suboptimal.

Some of our recommendations for public memory may not arise spontaneously in markets. Indeed, all kinds of records, including positive ones, are valuable for buyers, and therefore private parties will have incentives to collect and sell this information. Limiting the length of public memory for such records may therefore require regulation.

We believe these results provide a novel perspective, relevant to a number of important contexts and useful to understand the possible aggregate effects of information retention rules and rating systems. They could be useful, for example, to evaluate the potential aggregate consequences of the current trend toward the accumulation of extensive (positive and negative) information by electronic platforms and private and public agencies (credit bureaus and registers, criminal and health records databases, etc.), and of privacy rules mandating limits to the retention or disclosure of such information.

These issues are at the core of a number of current policy debates, from the one on "the right to be forgotten" by Internet search engines and the new EU General Data Protection Regulation, to those on the optimal memory of credit bureaus and on of feedback systems of electronic trading platforms, to that on how long past (e.g. juvenile) offenses should be accessible to potential employers. Future work could extend our analysis building richer models better tailored to specific situations.

Appendix

Proof of Theorem 1. First, at any t competitive buyers offer a price $P_i = \mu(h_i^t)\theta^H$ to each active seller i, the seller i maximizes his payoff $V_i(t, s_i)$ and sells whenever $P_i \geq 1$. Under Assumption 1, for any $t > 0$ without any feedback, the probability of seller $i \in [0, 1]$ being of high type is given by

$$\pi_i(t) = \pi_i(0)e^{-\varphi t} + \mu(1 - e^{-\varphi t}).$$

Clearly, at $t = 0$ sellers with $\pi_i(0) < \frac{1}{\theta^H}$ do not sell because $P_i = \pi_i(0)\theta^H < 1$. These sellers will never trade and will never get any feedback because

$$\mu(h_i^t) = \pi_i(t) = \pi_i(0)e^{-\varphi t} + \mu(1 - e^{-\varphi t})$$

and Assumption (2) guarantees $\pi_i(t)\theta^H < 1$. If at $t = 0$ buyers' prior about all sellers is low $\pi_i(0) < \frac{1}{\theta^H}$ for all $i \in [0, 1]$, then clearly there is no trade ever. Suppose that some sellers at $t = 0$ have high prior probability of being high quality $\pi_i(0) \geq \frac{1}{\theta^H}$. Denote the total mass of these sellers by $x < 1$. Note that as soon as a seller sells a low-quality product, he gets negative feedback and is revealed to be of low type. From that moment the seller is excluded from trading forever, just like those sellers who have low prior probability of being high type at $t = 0$. Each high quality seller from $t = 0$ onward may randomly become low quality, and then sell and get negative feedback. This happens with intensity $(1 - \mu)\varphi m$. At $t = 0$ the total mass of sellers that could trade is $x < 1$, therefore for any $t \geq 0$, the mass of sellers that can trade does not exceed $\overline{\mu}(t) = xe^{-(1-\mu)\varphi mt}$. As $t \to \infty$, the mass $\overline{\mu}(t) \to 0$, therefore the fraction of sellers trading in equilibrium converges to zero with time. QED.

Proof of Lemma 2. Consider $T^+ \geq T^-$ and suppose that the market is in a stationary equilibrium with trade. Now let's show that this leads to a contradiction.

Denote by X any informative record other than N, so that all kinds of positive or negative record are pooled in X. Let $\rho(X)$ and $\eta(X)$ be masses of good and bad sellers with an X record. Analogously, define $\rho(N)$ and $\eta(N)$ for an N record. Denote by λ_ρ the intensity with which an X record of a good seller is deleted and replaced by an record N in a stationary equilibrium. Analogously, denote λ_η for the bad seller. Since in a stationary equilibrium a seller with an N record becomes active and trades with intensity m, for the masses of sellers with different records to stay constant the following equalities must hold:

$$m\rho(N) = \lambda_\rho \rho(X),\ m\eta(N) = \lambda_\eta \eta(X).$$

Since $T^+ \geq T^-$, positive records are deleted slower than the negative ones. A good seller with an X record is strictly more likely to have a positive record than a bad seller with an X record, therefore $\lambda_\rho < \lambda_\eta$. In other words, a good seller is less likely to loose an X record than a bad seller. The total masses of good and bad sellers are μ and $1-\mu$ correspondingly, hence one can express:

$$\lambda_\rho = \frac{m\rho(N)}{\mu - \rho(N)},\ \lambda_\eta = \frac{m\eta(N)}{1-\mu-\eta(N)}.$$

Then $\lambda_\rho < \lambda_\eta$ implies

$$\frac{\rho(N)}{\eta(N)} < \frac{\mu}{1-\mu}.$$

This is equivalent to $\mu(N) < \mu$, that is the fraction of good sellers among sellers with an N record is lower than the population mean. Since $\mu\theta^H < 1$ this means sellers with an N record are not selling in equilibrium, a contradiction. Hence, when $T^+ \geq T^-$ no stationary equilibrium with trade is possible, and $T^+ < T^-$ is a necessary condition for a stationary equilibrium with trade to exist. QED.

Proof of Theorem 2. For brevity, in what follows we call a *stationary equilibrium with trade* simply an *equilibrium*. In equilibrium active sellers with an N record must be selling, that is, their average quality must be high enough $\mu(N)\theta^H \geq 1$. The rest of the proof finds necessary and sufficient conditions for sellers with an N record to trade.

As argued in the text, in equilibrium the average quality of sellers with an N record is lower than the quality of sellers with an $S(T^+)$ record, that is, $\mu(N) < \pi_i(S(T^+))$. Here $\pi_i(S(T^+)) = \mu + (1-\mu)e^{-\varphi T^+}$ is the posterior probability of a seller with an $S(T^+)$ record being of high type. In order to have $\mu(N)\theta^H \geq 1$, one must have $\pi_i(S(T^+))\theta^H > 1$. Given that $\pi_i(S(T^+))$ decreases with T^+, the memory for positive records must be short enough for an equilibrium to exist:

$$\mu\theta^H + (1-\mu)\theta^H e^{-\varphi T^+} > 1. \tag{12}$$

Due to Lemma 2 we consider $T^+ \leq T^-$ without loss of generality. In a stationary equilibrium the distribution $\mathbf{\Delta}$ of sellers' types to records is constant. Recall that we denote masses of high quality sellers with records N, $S(\tau)$ and $D(\tau^-)$ in a stationary equilibrium by

$\rho(N)$, $\rho(S(\tau))$ and $\rho(D(\tau^-))$ correspondingly. Analogously, for low-quality sellers we denote masses $\eta(N)$, $\eta(S(\tau))$ and $\eta(D(\tau^-))$. From Assumption 1 it follows that the total mass of high-quality sellers in the population is constant and equal to μ, hence in a stationary equilibrium we have:

$$\rho(N) + \int_0^{T^+} \rho(S(\tau))d\tau + \int_0^{T^-} \rho(D(\tau^-))d\tau^- = \mu, \tag{13}$$

$$\eta(N) + \int_0^{T^+} \eta(S(\tau))d\tau + \int_0^{T^-} \eta(D(\tau^-))d\tau^- = 1 - \mu. \tag{14}$$

In a stationary equilibrium, at any time t a seller i with $D(\tau^-)$ does not sell. A seller with a $D(\tau^-)$ record was of low quality τ^- time ago, hence (1) allows us to express the posterior probability of this seller being of high type $\mu(1-e^{-\varphi\tau^-})$, which coincides with the fraction of high-quality sellers among those with a $D(\tau^-)$ record according to the law of large numbers. Denoting by $\eta^D = \eta(D(0))$ the total mass of sellers with a $D(0)$ record, for any $\tau^- \in [0, T^-]$ we get:

$$\begin{aligned} \rho(D(\tau^-)) &= \eta^D \mu(1 - e^{-\varphi\tau^-}), \\ \eta(D(\tau^-)) &= \eta^D (1 - \mu + \mu e^{-\varphi\tau^-}). \end{aligned} \tag{15}$$

Consider a seller j with an $S(\tau)$ record $\tau \in [0, T^+]$. He got positive feedback τ periods ago, that is, he was of a high type then $\pi_j(-\tau) = 1$. He hasn't traded since then, hence, using (1), the posterior probability of this seller being of high type is

$$\pi_j(S(\tau)) = \pi_j(-\tau)e^{-\varphi\tau} + \mu(1 - e^{-\varphi\tau}) = \mu + (1 - \mu)e^{-\varphi\tau}.$$

By the law of large numbers, this posterior probability also determines the fraction of high-quality sellers among those with an $S(\tau)$ record. Since $\mu + (1 - \mu)e^{-\varphi\tau} \geq \mu$ and $\mu\theta^H \geq 1$, sellers with an $S(\tau)$ record $\tau \in [0, T^+]$ can trade. Denote by $\rho^S = \rho(S(0))$ the total mass of sellers with an$S(0)$ record. Both types of sellers with records $S(\tau)$, $\tau \in [0, T^+]$ trade with the same intensity m and leave the state $S(\tau)$. When a high-quality seller trades he gets positive feedback and his record is updated to $S(\tau = 0)$. When a low-quality seller trades he gets negative feedback and his record is updated to $D(\tau^- = 0)$. Therefore, the total mass of sellers with an $S(\tau)$ record exponentially decays with τ, that is,

$$\rho(S(\tau)) + \eta(S(\tau)) = \rho^S e^{-m\tau}.$$

Using this fact for any $\tau \in [0, T^+]$ we get:

$$\begin{aligned} \rho(S(\tau)) &= \rho^S e^{-m\tau}(\mu + (1-\mu)e^{-\varphi\tau}), \\ \eta(S(\tau)) &= \rho^S e^{-m\tau}(1-\mu)(1-e^{-\varphi\tau}). \end{aligned} \tag{16}$$

Sellers with N and $S(\tau)$, $\tau \in [0, T^+]$ records trade with intensity m. Upon trading their type is revealed: high-quality sellers get an $S(0)$ record and low-quality sellers get a $D(0)$ record. At the same time each instance, all sellers in states $S(0)$ or $D(0)$ exit these states and in a stationary equilibrium we must have:

$$\begin{aligned} \rho^S &= \rho(S(0)) = m\rho(N) + m\int_0^{T^+} \rho(S(\tau))d\tau, \\ \eta^D &= \eta(D(0)) = m\eta(N) + m\int_0^{T^+} \eta(S(\tau))d\tau. \end{aligned} \tag{17}$$

Finally, in the stationary equilibrium masses of high- and low-quality sellers with N records must stay constant. Each instance, $S(T^+)$, $D(T^-)$ records are deleted and a mass $\rho(S(T^+)) + \rho(D(T^-))$ of high-quality sellers enters state N, at the same time high-quality sellers with an N record trade and leave this state with intensity m. Analogous reasoning is true for low-quality sellers. Finally, high-quality sellers with intensity $\varphi(1-\mu)$ change their type and become low-quality sellers, while low-quality sellers become high-quality sellers with intensity $\varphi\mu$, and we obtain:

$$\begin{aligned} \dot{\rho}(N) &= \rho(S(T^+)) + \rho(D(T^-)) - m\rho(N) - \varphi(1-\mu)\rho(N) + \varphi\mu\eta(N) = 0, \\ \dot{\eta}(N) &= \eta(S(T^+)) + \eta(D(T^-)) - m\eta(N) + \varphi(1-\mu)\rho(N) - \varphi\mu\eta(N) = 0. \end{aligned} \tag{18}$$

Summing the above two equations we get:

$$m(\rho(N) + \eta(N)) = \rho(S(T^+)) + \rho(D(T^-)) + \eta(S(T^+)) + \eta(D(T^-)). \tag{19}$$

Substitute for $\rho(S(T^+))$, $\rho(D(T^-))$, $\eta(S(T^+))$, $\eta(D(T^-))$ in (19) and we obtain:

$$m(\rho(N)+\eta(N))=\rho^S e^{-mT^+}+\eta^D=Z. \tag{20}$$

Since $\eta(N)=Z/m-\rho(N)$, we get from the first equation of (18) that:

$$(m+\varphi)\rho(N)=\rho^S(\mu e^{-mT^+}+(1-\mu)e^{-(m+\varphi)T^+})+\eta^D\mu(1-e^{-\varphi T^-})+\varphi\mu Z/m. \tag{21}$$

A seller with an N record can trade only if $\frac{1}{\mu(N)}=\frac{\rho(N)+\eta(N)}{\rho(N)}\leq\theta^H$, from (21) we get:

$$\frac{(m+\varphi)[\rho^S e^{-mT^+}+\eta^D]}{m\rho^S[\mu e^{-mT^+}+(1-\mu)e^{-(m+\varphi)T^+}]+m\eta^D\mu[1-e^{-\varphi T^-}]+\mu\varphi[\rho^S e^{-mT^+}+\eta^D]}\leq\theta^H. \tag{22}$$

Compute:

$$F(T^+)=\frac{1}{\rho^S}\int_0^{T^+}\rho(S(\tau))d\tau=\frac{\mu}{m}(1-e^{-mT^+})+\frac{1-\mu}{m+\varphi}(1-e^{-(m+\varphi)T^+}), \tag{23}$$

and use it in the first equation of (17) in order to obtain:

$$\rho^S=m\rho(N)+m\rho^S F(T^+). \tag{24}$$

Compute:

$$F^-(T^-)=\frac{1}{\eta^D}\int_0^{T^-}\rho(D(\tau^-))d\tau^-=\mu T^- -\frac{\mu}{\varphi}(1-e^{-\varphi T^-}). \tag{25}$$

Substitute in (13) to obtain:

$$\rho(N)+\rho^S F(T^+)+\eta^D F^-(T^-)=\rho^S/m+\eta^D F^-(T^-)=\mu. \tag{26}$$

Compute:

$$G(T^+)=\frac{1}{\rho^S}\int_0^{T^+}\eta(S(\tau))d\tau=\frac{1-\mu}{m}(1-e^{-mT^+})-\frac{1-\mu}{m+\varphi}(1-e^{-(m+\varphi)T^+}), \tag{27}$$

and use it in the second equation of (17) in order to obtain:

$$\eta^D=m\eta(N)+m\rho^S G(T^+). \tag{28}$$

Compute:

$$G^-(T^-) = \frac{1}{\eta^D} \int_0^{T^-} \eta(D(\tau^-)) d\tau^- = (1-\mu)T^- + \frac{\mu}{\varphi}(1 - e^{-\varphi T^-}). \tag{29}$$

Substitute into (14) and, using (28), obtain:

$$\eta(N) + \rho^S G(T^+) + \eta^D G^-(T^-) = \eta^D/m + \eta^D G^-(T^-) = 1 - \mu. \tag{30}$$

which delivers:

$$\eta^D = m \frac{1-\mu}{1 + mG^-(T^-)}. \tag{31}$$

Using (26) we get:

$$\rho^S = m \frac{\mu + m\mu G^-(T^-) - m(1-\mu)F^-(T^-)}{1 + mG^-(T^-)}. \tag{32}$$

Combining the above equations we get:

$$\frac{\eta^D}{\rho^S} = \frac{1-\mu}{\mu + \mu m T^- - mF^-(T^-)}. \tag{33}$$

Using (24), (28) and (33) we get:

$$\frac{\eta(N)}{\rho(N)} = \frac{\eta^D - m\rho^S G(T^+)}{\rho^S(1 - mF(T^+))} = \frac{\eta^D/\rho^S - mG(T^+)}{1 - mF(T^+)} \tag{34}$$

$$\frac{\rho(N) + \eta(N)}{\rho(N)} = 1 + \frac{\eta(N)}{\rho(N)} = 1 + \frac{\frac{1-\mu}{\mu + \mu m T^- - mF^-(T^-)} - mG(T^+)}{1 - mF(T^+)} \leq \theta^H. \tag{35}$$

Finally, substituting for $F(T^+)$, $F^-(T^-)$, $G(T^+)$ from (23), (25) and (27) we get a necessary and sufficient condition for a stationary equilibrium with trade to exist:

$$1 + \frac{\eta(N)}{\rho(N)} = \frac{\frac{1-\mu}{\mu} \frac{1}{1 + \frac{m}{\varphi}(1 - e^{-\varphi T^-})} + e^{-mT^+}}{\mu e^{-mT^+} + (1-\mu)\frac{\varphi + me^{-(m+\varphi)T^+}}{m+\varphi}} \leq \theta^H,$$

which can be rewritten as

$$\frac{1-\mu}{\mu} \frac{\varphi}{\varphi + m(1 - e^{-\varphi T^-})} \leq \theta^H \left[\mu e^{-mT^+} + (1-\mu)\frac{\varphi + me^{-(m+\varphi)T^+}}{m+\varphi} \right] - e^{-mT^+}. \tag{36}$$

In order to prove that this condition is sufficient, we need to show that it implies the necessary condition (12). Note that the left-hand side of (36) is decreasing in T^-. In order for (36) to hold for some finite T^-, we must have that in the limit when $T^- \to \infty$ the condition holds as a strict inequality, that is:

$$\frac{1-\mu}{\mu}\frac{\varphi}{\varphi+m}[1-\mu\theta^H] < e^{-mT^+}\left[\mu\theta^H + (1-\mu)\theta^H \frac{m}{m+\varphi}e^{-\varphi T^+} - 1\right]. \qquad (37)$$

Note that $1-\mu\theta^H > 0$ by Assumption 2, hence the condition implicitly requires $\mu\theta^H + (1-\mu)\theta^H \frac{m}{m+\varphi}e^{-\varphi T^+} - 1 > 0$, which in turn implies (12).

Now we need to show that the necessary and sufficient existence condition (36) is equivalent to the requirements of Theorem 2.

Given that the left-hand side is decreasing in T^- we can rewrite (36) as $T^- \geq \underline{T}^-$ provided that there exists $\underline{T}^- < \infty$ which solves (36) as equality. This is possible if and only if for $T^- \to \infty$ (36) holds as a strict inequality, that is, (37) holds. In other words, (37) is a necessary condition, and together with $T^- \geq \overline{T}^-$ these two conditions are sufficient.

Denote by

$$\zeta(T^+) = e^{-mT^+}\left[\mu\theta^H + (1-\mu)\theta^H \frac{m}{m+\varphi}e^{-\varphi T^+} - 1\right]$$

the term on the right-hand side of (36) and of (37), differentiating one obtains

$$\frac{\partial\zeta(T^+)}{\partial T^+} = -me^{-mT^+}[\mu\theta^H - 1 + (1-\mu)\theta^H e^{-\varphi T^+}] < 0$$

because (37) requires

$$\mu\theta^H + (1-\mu)\theta^H \frac{m}{m+\varphi}e^{-\varphi T^+} - 1 > 0.$$

Given that the right-hand side of (37) is decreasing in T^+, we can rewrite (37) as $T^+ < \overline{T}^+$, provided that there exists $\overline{T}^+ > 0$ such that (37) holds as equality. Such $\overline{T}^+ \geq 0$ exists if and only if (37) holds for $T^+ = 0$, that is, if and only if

$$\frac{\varphi}{m} < \frac{\mu(\theta^H - 1)}{1-\mu\theta^H},$$

which is equivalent to (2).

To sum up we have established that if (2) holds, then there exists $\overline{T}^+ > 0$ that solves:

$$\frac{1-\mu}{\mu}\frac{\varphi}{\varphi+m}[1-\mu\theta^H] = e^{-m\overline{T}^+}\left[\mu\theta^H + (1-\mu)\theta^H\frac{m}{m+\varphi}e^{-\varphi\overline{T}^+} - 1\right]. \tag{38}$$

If also $T^+ < \overline{T}^+$, then there exists $\underline{T}^- < \infty$ that solves:

$$\frac{1-\mu}{\mu}\frac{\varphi}{\varphi+m(1-e^{-\varphi T^-})} = \theta^H\left[\mu e^{-mT^+} + (1-\mu)\frac{\varphi+me^{-(m+\varphi)T^+}}{m+\varphi}\right] - e^{-mT^+}. \tag{39}$$

If also $T^- \geq \underline{T}^-$, then a stationary equilibrium with trade exists because (36) is satisfied. In other words, conditions (2), $T^+ < \overline{T}^+$ and $T^- \geq \underline{T}^-$ are sufficient. They are also necessary, because if one of them is violated then (36) is not satisfied.

Finally, in a stationary equilibrium with trade the distribution $\mathbf{\Delta}$ is uniquely defined, hence at most one stationary equilibrium with trade exists. QED.

Proof of Lemma 3. From (23) and from (27) we get

$$\int_0^{T^+} \rho(S(\tau))d\tau = \rho^S F(T^+),$$

$$\int_0^{T^+} \eta(S(\tau))d\tau = \rho^S G(T^+)$$

correspondingly. From (24) we express

$$\rho(N) = \rho^S(\frac{1}{m} - F(T^+)).$$

Using (28) and (33) we express

$$\eta(N) = \rho^S(\frac{1}{m}\frac{1-\mu}{\mu+m\mu T^- - mF^-(T^-)} - G(T^+)).$$

This allows us to express welfare as:

$$W = \left(\theta_H - 1 - \frac{1-\mu}{\mu+m\mu T^- - mF^-(T^-)}\right)\rho^S. \tag{40}$$

Finally, substituting for ρ^S, $F^-(T^-)$ and $G^-(T^-)$ from (32), (25) and (29) one obtains (5). QED.

Proof of Theorem 3. If there is no trade, the welfare is zero. Condition (2) is necessary for a stationary equilibrium with trade to exist, assume it holds and the market is in a stationary equilibrium.

To see how welfare changes with T^-, denote

$$B(T^-) = 1 + m(1-\mu)T^- + m\frac{\mu}{\varphi}(1 - e^{-\varphi T^-})$$

and rewrite welfare as

$$W = m\left(\theta^H - 1 - \frac{(1-\mu)\theta^H + (\theta^H - 1)m(1-\mu)T^-}{B(T^-)}\right).$$

Compute the derivative:

$$\frac{\partial W}{\partial T^-} = -m\frac{(\theta^H - 1)m(1-\mu)}{B(T^-)} + m\frac{((1-\mu)\theta^H + (\theta^H - 1)m(1-\mu)T^-)(m(1-\mu) + m\mu e^{-\varphi T^-})}{B(T^-)^2}$$

After manipulations one obtains

$$\frac{\partial W}{\partial T^-} = \frac{m^2(1-\mu)}{B(T^-)^2}h(T^-),$$

here

$$h(T^-) = 1 - \mu\theta^H - (\theta^H - 1)\frac{m\mu}{\varphi} + \mu[\theta^H + m(\theta^H - 1)(T^- + \frac{1}{\varphi})]e^{-\varphi T^-}.$$

Note that $h(T^-)$ is decreasing because for $T^- > 0$ one has

$$\frac{\partial h(T^-)}{\partial T^-} = -\varphi\mu\left[\theta^H + m(\theta^H - 1)T^-\right]e^{-\varphi T^-} < 0.$$

Also note that for $T^- = 0$ one has $h(0) = 1 > 0$. On the other hand, for $T^- \to \infty$ we have

$$h(T^-) \to 1 - \mu\theta^H - (\theta^H - 1)\frac{m\mu}{\varphi} < 0,$$

because (2) implies

$$\frac{1 - \mu\theta^H}{\mu(\theta^H - 1)} < \frac{m}{\varphi}.$$

It follows that there is a unique $T_W^- > 0$ such that $h(T_W^-) = 0$.

Given that the sign of $\frac{\partial W}{\partial T^-}$ is determined by the sign of $h(T^-)$, welfare $W(T^-)$ is maximized for $T^- = T_W^-$ that satisfies $h(T_W^-) = 0$, that is, it solves (6).

Now we need to verify that the stationary equilibrium with trade can be supported when $T^- = T_W^-$, that is conditions i) $T^+ < \bar{T}^+$ and ii) $T_W^- \geq \underline{T}(T^+)$ of Theorem 2 hold.

Since T_W^+ solves $T_W^- = \underline{T}(T_W^+)$ as equality, and the right hand side of (4) decreases with T^+, condition ii) is satisfied for any $T^+ \in [0, T_W^+]$ and violated for $T^+ > T_W^+$. Note that for any $T^+ \in [0, T_W^+]$ condition i) $T^+ < \bar{T}^+$ is also satisfied (for details see the proof of Theorem 2). As a result $T^+ \in [0, T_W^+]$ ensures that T_W^- can be sustained in equilibrium. Lemma 2 implies $T_W^+ < T_W^-$.

In a stationary equilibrium with trade $\frac{\partial W}{\partial T^+} = 0$, therefore the welfare is maximized for $T^- = T_W^-$ and any $T^+ \in [0, T_W^+]$. QED.

Appendix for Online Publication

Proof of Theorem 7. Each active seller is matched with many buyers, and with probability $\beta \leq 1$ there is a quality-insensitive buyer among them. Suppose seller i is active and has a quality-insensitive buyer among other buyers he is matched with. The quality-insensitive buyer valuation of a product is equal to one independently of the product's quality, therefore he always offers a price of one to the seller. Other buyers offer the price $P_i^t = \mu(r_i^t)\theta^H$. It follows that, in equilibrium, active sellers with $\mu(r_i^t)\theta^H \geq 1$ trade at price $P_i^t = \mu(r_i^t)\theta^H$ with probability one, while active sellers with $\mu(r_i^t)\theta^H < 1$ trade at price $P_i^t = 1$ with probability β and do not trade with probability $1 - \beta$.

In a high-trade stationary equilibrium, active sellers with an N record must be able to sell to all buyers: $\mu(N)\theta^H \geq 1$. In this equilibrium, sellers with $S(\tau)$, $\tau \in [0, T^+]$ records must be able to sell to all buyers, that is $\mu(S(\tau))\theta^H \geq 1$. Suppose otherwise $\mu(S(\tau))\theta^H < 1$ for some $\tau < T^+$, then $\mu(S(T^+))\theta^H < 1$ because

$$\mu(S(\tau)) = \mu + (1 - \mu)e^{-\varphi\tau}$$

decreases with τ. It follows that sellers with $S(T^+)$ can't sell to all buyers. Moreover,

$$\mu(D(T^-)) = \mu(1 - e^{-\varphi T^-}) < \mu(S(T^+)) = \mu + (1 - \mu)e^{-\varphi T^+}$$

implies that sellers with $D(T^-)$ record also can't sell to all buyers. In a high-trade stationary equilibrium, sellers get an N record only if in the previous instance they had an $S(T^+)$ or $D(T^-)$ record that got deleted, which implies $\mu(N) \leq \mu(S(T^+))$. This means that sellers with an N record can't sell to all buyers: $\mu(N)\theta^H < 1$, i.e. a contradiction. It follows that in a high-trade stationary equilibrium $\mu(S(\tau))\theta^H \geq 1$ must hold for any $\tau \in [0, T^+]$. Given that $\mu(S(\tau))\theta^H$ decreases with τ, we get the first necessary condition for a high-trade equilibrium to exist:

$$\mu(S(\tau))\theta^H = \theta^H(\mu + (1 - \mu)e^{-\varphi T^+}) \geq 1. \tag{41}$$

As before, $\rho(r)$ and $\eta(r)$ denote masses of high- and low-quality sellers with record $r \in G$ correspondingly. In a stationary equilibrium these masses must not change with time and equations (13) and (14) hold.

A seller with record $S(\tau)$, $\tau \in [0, T^+]$ trades whenever he gets active. Hence, masses of high- and low-quality sellers with $S(\tau)$, $\tau \in [0, T^+]$ records are given by (16), where, $\rho^S = \rho(S(0))$.

In the previous analysis, sellers with $D(\tau^-)$, $\tau^- \in [0, T^-]$ were not able to sell, but now these sellers can sell to quality-insensitive buyers and only to these buyers. Indeed $\mu(D(\tau^-))\theta^H \leq 1$ for any $\tau^- \in [0, T^-]$, because

$$\mu(D(\tau^-)) = \mu(1 - e^{-\varphi T^-}) \leq \mu(D(T^-)) = \mu(1 - e^{-\varphi T-})$$

which, together with Assumption 2 and $T^- \leq \overline{T}$, implies $\mu(D(\tau^-))\theta^H < 1$ for any $\tau^- \in [0, T^-]$. Therefore, in a stationary equilibrium, at any time t a seller i with a $D(\tau^-)$ record $\tau^- \in [0, T^-]$ may become active with Poisson arrival rate m and try to sell. He can sell only if he meets a quality-insensitive seller, which happens with probability β. Essentially, these sellers trade with intensity $m\beta$ and get a new record: bad sellers get a $D(0)$ record and good sellers get an $S(0)$ record. Denote by $\eta^D = \eta(D(0))$ the mass of sellers with $D(0)$ at any moment in time, from the above argument it follows that the total mass of sellers with a $D(\tau^-)$ record is given by $\Delta^D(\tau^-) = \eta(D(\tau^-)) + \rho(D(\tau^-)) = \eta^D e^{-m\beta\tau^-}$. According to the law of large numbers and (7), the fraction of high-quality sellers among those with a $D(\tau^-)$ record is given by $\mu(1 - e^{-\varphi\tau^-})$. For any $\tau^- \in [0, T^-]$ we get:

$$\begin{aligned} \rho(D(\tau^-)) &= \eta^D e^{-m\beta\tau^-} \mu(1 - e^{-\varphi\tau^-}), \\ \eta(D(\tau^-)) &= \eta^D e^{-m\beta\tau^-} (1 - \mu + \mu e^{-\varphi\tau^-}). \end{aligned} \tag{42}$$

Sellers with N and $S(\tau)$, $\tau \in [0, T^+]$ records trade with intensity m, while sellers with $D(\tau^-)$, $\tau^- \in [0, T^-]$ records trade with intensity βm. Upon trade their type is revealed, high-quality sellers get an $S(0)$ record and low-quality sellers get a $D(0)$ record. At the same time each instance, all sellers in states $S(0)$ or $D(0)$ exit these states and in a stationary equilibrium we must have:

$$\begin{aligned} \rho^S = \rho(S(0)) &= m\rho(N) + m \int_0^{T^+} \rho(S(\tau))d\tau + m\beta \int_0^{T^-} \rho(D(\tau^-))d\tau^-, \\ \eta^D = \eta(D(0)) &= m\eta(N) + m \int_0^{T^+} \eta(S(\tau))d\tau + m\beta \int_0^{T^-} \eta(D(\tau^-))d\tau^-. \end{aligned} \tag{43}$$

In the stationary equilibrium, masses of high- and low-quality sellers with N records must stay constant. Each instance, $S(T^+)$, $D(T^-)$ records are deleted and a mass

$$Z = \Delta(S(T^+)) + \Delta(D(T^-)) = \rho^S e^{-mT^+} + \eta^D e^{-m\beta T^-}$$

of sellers gets an N record, at the same time sellers with an N record trade and leave this state with intensity m:

$$m(\rho(N) + \eta(N)) = Z = \rho^S e^{-mT^+} + \eta^D e^{-m\beta T^-}. \tag{44}$$

For convenience, we express

$$Z = a_Z \rho^S + b_Z \eta^D,$$

$$\eta(N) = Z/m - \rho(N),$$

here

$$a_Z = e^{-mT^+},\ b_Z = e^{-m\beta T^-}.$$

Note that (23) and (27) hold, hence we can write

$$\int_0^{T^+} \rho(S(\tau))d\tau = \rho^S F(T^+),$$

$$\int_0^{T^+} \eta(S(\tau))d\tau = \rho^S G(T^+).$$

Denote:

$$F'^-(T^-) = \frac{1}{\eta^D}\int_0^{T^-} \rho(D(\tau^-))d\tau^- = \frac{\mu}{m\beta}(1 - e^{-m\beta T^-}) - \frac{\mu}{m\beta + \varphi}(1 - e^{-(m\beta+\varphi)T^-}), \tag{45}$$

$$G'^-(T^-) = \frac{1}{\eta^D}\int_0^{T^-} \eta(D(\tau^-))d\tau^- = \frac{1-\mu}{m\beta}(1 - e^{-m\beta T^-}) + \frac{\mu}{m\beta + \varphi}(1 - e^{-(m\beta+\varphi)T^-}). \tag{46}$$

In a stationary equilibrium, the masses of high- and low-quality sellers with an N record must be constant and (18) holds. Using the first equation from (18) we can express:

$$m\rho(N) + \varphi(1-\mu)\rho(N) - \varphi\mu[Z/m - \rho(N)] = \rho(S(T^+)) + \eta(D(T^-)),$$

rearranging and substituting for $\rho(S(T^+))$, $\eta(D(T^-)$ from (16), (42) we obtain:

$$(m+\varphi)\rho(N) = \rho^S(\mu e^{-mT^+} + (1-\mu)e^{-(m+\varphi)T^+}) + \eta^D \mu e^{-m\beta T^-}(1 - e^{-\varphi T^-}) + \varphi\mu Z/m. \quad (47)$$

To shorten notations, we rewrite the above expression as

$$\rho(N) = \rho^S a_X + \eta^D b_X + \frac{\varphi\mu}{(\mu+\varphi)m}(a_Z\rho^S + b_Z\eta^D),$$

here

$$a_X = \frac{(\mu e^{-mT^+} + (1-\mu)e^{-(m+\varphi)T^+})}{m+\varphi},$$

$$b_X = \frac{\mu e^{-m\beta T^-}(1 - e^{-\varphi T^-})}{m+\varphi}.$$

Further,

$$\rho(N) = a_\rho \mu^S + b_\rho \eta^D,$$

where

$$a_\rho = a_X + \frac{\varphi\mu}{(m+\varphi)m}a_Z, \; b_\rho = b_X + \frac{\varphi\mu}{(m+\varphi)m}b_Z.$$

Given that

$$\eta(N) = Z/m - \rho(N)$$

we can express

$$\eta(N) = a_\eta \mu^S + b_\eta \eta^D,$$

where

$$a_\eta = -a_X + \frac{1}{m}(1 - \frac{\varphi\mu}{m+\varphi})a_Z,$$

$$b_\eta = -b_X + \frac{1}{m}(1 - \frac{\varphi\mu}{m+\varphi})b_Z.$$

After substitutions we get:

$$
\begin{aligned}
a_\rho &= \frac{\mu}{m} e^{-mT^+} + \frac{1-\mu}{m+\varphi} e^{-(m+\varphi)T^+}, \\
b_\rho &= \frac{\mu}{m} e^{-m\beta T^-} - \frac{\mu}{m+\varphi} e^{-(m\beta+\varphi)T^-}, \\
a_\eta &= \frac{1-\mu}{m} e^{-mT^+} - \frac{1-\mu}{m+\varphi} e^{-(m+\varphi)T^+}, \\
b_\eta &= \frac{1-\mu}{m} e^{-m\beta T^-} + \frac{\mu}{m+\varphi} e^{-(m\beta+\varphi)T^-}.
\end{aligned}
\tag{48}
$$

In a high-trade stationary equilibrium, a seller with an N record must be able to sell to all buyers, that is, we must have

$$
\mu(N) = \frac{\rho(N)}{\rho(N) + \eta(N)} \geq \frac{1}{\theta^H},
$$

or equivalently

$$
\frac{\eta(N)}{\rho(N)} \leq \theta^H - 1.
$$

Denoting $X = \frac{\rho^S}{\eta^D}$ we can express:

$$
\frac{\eta(N)}{\rho(N)} = \frac{a_\eta \rho^S + b_\eta \eta^D}{a_\rho \rho^S + b_\rho \eta^D} = \frac{a_\eta X + b_\eta}{a_\rho X + b_\rho}. \tag{49}
$$

The existence condition for a high-trade equilibrium $\frac{\eta(N)}{\rho(N)} \leq \theta^H - 1$ can be rewritten as:

$$
(a_\rho + a_\eta)X + b_\rho + b_\eta \leq \theta^H (a_\rho X + b_\rho). \tag{50}
$$

which, after substitutions, becomes:

$$
\begin{aligned}
&C(T^+, T^-) = \frac{Xe^{-mT^+}}{m} + \frac{e^{-m\beta T^-}}{m} - \\
&\theta^H \left(X\left(\frac{\mu e^{-mT^+}}{m} + \frac{(1-\mu)e^{-(m+\varphi)T^+}}{m+\varphi}\right) + \frac{\mu e^{-m\beta T^-}}{m} - \frac{\mu e^{-(m\beta+\varphi)T^-}}{m+\varphi} \right) \leq 0.
\end{aligned}
\tag{51}
$$

To complete the proof, an expression for X remains to be found. The second equation in (43) can be rewritten in the following way:

$$
\eta(N) + \rho^S G(T^+) + \eta^D G'^-(T^-) = \eta^D/m + (1-\beta)\eta^D G'^-(T^-).
$$

Using (14) we get:

$$\eta(N) + \rho^S G(T^+) + \eta^D G'^-(T^-) = \eta^D/m + (1-\beta)\eta^D G'^-(T^-) = 1-\mu, \tag{52}$$

which implies:

$$\eta^D = \frac{(1-\mu)m}{1+m(1-\beta)G'^-(T^-)}. \tag{53}$$

Analogously, the first equation in (43) can be rewritten in the following way:

$$\rho(N) + \rho^S F(T^+) + \eta^D F'^-(T^-) = \rho^S/m + (1-\beta)\eta^D F'^-(T^-).$$

Using (13) we get:

$$\rho(N) + \rho^S F(T^+) + \eta^D F'^-(T^-) = \rho^S/m + (1-\beta)\eta^D F'^-(T^-) = \mu, \tag{54}$$

which, together with (53), implies:

$$\rho^S = \mu m - \frac{m(1-\beta)m(1-\mu)F'^-(T^-)}{1+m(1-\beta)G'^-(T^-)}. \tag{55}$$

Using (53) and (55), we express

$$\frac{\rho^S}{\eta^D} = \frac{\mu}{1-\mu} + \frac{m(1-\beta)}{1-\mu}(\mu G'^-(T^-) - (1-\mu)F'^-(T^-)).$$

Substituting for $G'^-(T^-)$ and $F'^-(T^-)$ from (45) and (46) we get

$$\mu G'^-(T^-) - (1-\mu)F'^-(T^-) = \frac{\mu}{m\beta+\varphi}(1-e^{-(m\beta+\varphi)T^-}).$$

Therefore we obtain (11):

$$X = \frac{\rho^S}{\eta^D} = \frac{\mu}{1-\mu}\left(1 + \frac{m(1-\beta)}{m\beta+\varphi}(1-e^{-(m\beta+\varphi)T^-})\right).$$

Together, equations (41),(51) provide necessary and sufficient conditions for the existence of a high-trade stationary equilibrium. Let us consider the effect of T^- on these conditions. Condition (41) is not affected by T^-. Let us differentiate $C(T^+, T^-)$ given by (51) with respect to T^-, taking into account that X given by (11) also depends on T^-:

$$\frac{dC}{dT^-} = \frac{\partial C}{\partial T^-} + \frac{\partial C}{\partial X}\frac{\partial X}{\partial T^-}.$$

Using

$$\frac{\partial X}{\partial T^-} = \frac{\mu}{1-\mu} m(1-\beta)e^{-(m\beta+\varphi)T^-}$$

we obtain:

$$\frac{dC}{dT^-} = -\beta e^{-m\beta T^-} - \theta^H \left(-\beta\mu e^{-m\beta T^-} + \frac{\mu(m\beta+\varphi)}{m+\varphi} e^{-(m\beta+\varphi)T^-}\right) +$$
$$\left(\frac{e^{-mT^+}}{m} - \theta^H \left(\frac{\mu e^{-mT^+}}{m} + \frac{(1-\mu)e^{-(m+\varphi)T^+}}{m+\varphi}\right)\right) \frac{\mu}{1-\mu} m(1-\beta)e^{-(m\beta+\varphi)T^-} =$$
$$-\beta e^{-m\beta T^-}(1-\mu\theta^H)$$
$$\frac{\mu e^{-(m\beta+\varphi)T^-}}{1-\mu}\left((1-\beta)[(e^{-mT^+}(1-\mu\theta^H - \frac{(1-\mu)\theta^H m e^{-\varphi T^+}}{m+\varphi})] - \frac{(1-\mu)\theta^H(m\beta+\varphi)}{m+\varphi}\right) < 0.$$

Indeed, $\mu\theta^H < 1$ implies

$$-\beta e^{-m\beta T^-}(1-\mu\theta^H).$$

Moreover,

$$\begin{aligned}(1-\beta)[(e^{-mT^+}(1-\mu\theta^H - \frac{(1-\mu)\theta^H m e^{-\varphi T^+}}{m+\varphi})] - \frac{(1-\mu)\theta^H(m\beta+\varphi)}{m+\varphi} &= \\ (1-\beta)[e^{-mT^+}(1-\mu\theta^H - (1-\mu)\theta^H e^{-\varphi T^+}] &+ \\ (1-\beta)\varphi\frac{(1-\mu)\theta^H}{m+\varphi}e^{-(m+\varphi)T^+} - (m\beta+\varphi)\frac{(1-\mu)\theta^H}{m+\varphi} &< 0,\end{aligned} \tag{56}$$

Because

$$(1-\beta)\varphi e^{-(m+\varphi)T^+} < m\beta+\varphi,$$

condition (41) implies

$$1-\mu\theta^H - (1-\mu)\theta^H e^{-\varphi T^+} \leq 0.$$

We have shown that $C(T^+,T^-)$ decreases with T^-, that is, the existence condition for the high-trade stationary equilibrium is easier to satisfy when negative records are kept for a long time.

Consider now the effect of T^+. Note, that X depends on T^- but not on T^+. Let's differentiate $C(T^+,T^-)$ given by (51) with respect to T^+:

$$\frac{\partial C}{\partial T^+} = -Xe^{-mT^+}\left(1-\theta^H(\mu+(1-\mu)e^{-\varphi T^+})\right). \tag{57}$$

Condition (41) implies

$$1 \leq \theta^H(\mu + (1-\mu)e^{-\varphi T^+}),\ \frac{\partial C}{\partial T^+} > 0,$$

unless

$$1 = \theta^H(\mu + (1-\mu)e^{-\varphi T^+}).$$

Hence, as T^+ increases the existence conditions (41) and (51) are harder to satisfy.

Let us find the smallest T^+, denoted T^*, which rules out the existence of the high-trade equilibrium. From the previous analysis, we know that increasing T^- always relaxes the existence condition (51), therefore if the condition does not hold for $T^- \to \infty$ it will not hold for any finite T^-. As $T^- \to \infty$, $X \to \frac{\mu}{1-\mu}\frac{m+\varphi}{m\beta+\varphi}$ and (51) becomes:

$$C(T^+, \infty) = \frac{Xe^{-mT^+}}{m}\left(1 - \mu\theta^H \quad - \frac{(1-\mu)\theta^H me^{-\varphi T^+}}{m+\varphi})\right) \leq 0. \tag{58}$$

Take T^*, which solves

$$A(T^*) = \theta^H(\mu + (1-\mu)\frac{m}{m+\varphi}e^{-\varphi T^*}) = 1.$$

Clearly, for any $T^+ > T^*$, (58) is not satisfied. This in turn implies that for any finite T^- and $T^+ \geq T^*$, the existence condition (51) is violated, while for $T^+ < T^*$ one can find $T^- = \hat{T} < \infty$ such that (51) holds as equality $C(T^*, \hat{T}) = 0$:

$$\begin{aligned} \frac{Xe^{-mT^+} + e^{-m\beta\hat{T}}}{\mu\theta^H} &= X[e^{-mT^+} + \frac{(1-\mu)m}{\mu(m+\varphi)}e^{-(m+\varphi)T^+}] + e^{-m\beta\hat{T}} - \frac{m}{m+\varphi}e^{-(m\beta+\varphi)\hat{T}}, \\ X\frac{1-\mu}{\mu} &= 1 + \frac{m(1-\beta)}{m\beta+\varphi}(1 - e^{(m\beta+\varphi)\hat{T}}). \end{aligned} \tag{59}$$

Given that $C(T^+, T^-)$ increases with T^+ and decreases with T^-, for the high-trade stationary equilibrium to exist we must have $T^+ < T^*$ and $T^- \geq \hat{T}$. Clearly, if $T^* < 0$ the stationary equilibrium does not exists because memory T^+ can't be negative and we must have $T^* \geq 0$. Since $A(T^*)$ decreases with T^*, condition $T^* \geq 0$ is equivalent to $A(0) \geq 1$, which can be written as (9):

$$\theta^H(\mu\varphi + m) \geq \varphi + m.$$

Note that $T^+ \leq T^*$ implies the first necessary condition (41), indeed,

$$\theta^H(\mu + (1-\mu)e^{-\varphi T^+}) > A(T^+) \geq 1$$

for any $T^+ \leq T^*$. Therefore we have proven that if (9) holds, $T^+ < T^*$ and $T^- \geq \hat{T}$, then the average quality of sellers in the pool with an N record is high enough that all buyers are ready to buy from them and a high-trade stationary equilibrium exists, that is, these conditions are sufficient. These conditions are also necessary, because if any of them is violated there is no stationary distribution of sellers to records such that the average quality of sellers in the pool with an N record is high enough to sustain trade with all buyers, and condition (51) is violated. QED.

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