Fiscal Policy and the Balance Sheet of the Private Sector*

Hans Gersbach† Jean-Charles Rochet‡
Ernst-Ludwig von Thadden§

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Abstract

This paper characterizes optimal fiscal policy in a growth model with incomplete markets, heterogeneous agents (households and firms) with conflicting interests, and non-insurable idiosyncratic productivity risk. Firms finance productive and risky investments by issuing bonds. Households invest in corporate and public debt. The government maximizes a weighted sum of the welfare of firms' owners and households, and we establish a version of the second welfare theorem: Any constrained Pareto optimal allocation can be decentralized by issuing an appropriate amount of public debt, combined with suitable wealth taxation and redistribution of initial endowments. The optimal debt-to-equity ratio in the economy is non-monotonic function of the welfare weight of firms. Higher government debt reduces firm leverage, increases the risk-free rate r, and decreases the growth rate g. The welfare weight of firms determines whether $r < g$ or $r > g$ at the optimum, with different dynamics in both regimes.

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†ETH Zürich and KOF Swiss Economic Institute, Email: hgersbach@ethz.ch.
‡SFI at Université de Genève and TSE. Email: jeancharles.rochet@gmail.com.
§Universität Mannheim, CEPR, and ECGI. Email: vthadden@uni-mannheim.de.

1 Introduction

How should governments conduct fiscal policy in the presence of incomplete markets, heterogeneous agents with conflicting interests, and non-insurable idiosyncratic production risk? Why are these markets incomplete in the first place, and what does this imply for firms' capital structure and investment? Which fiscal instruments should the government use and how does their use optimally depend on the weight of different interests in society? And how do interest rates, savings, and growth depend on these choices by private agents and the government?

To answer these questions, we develop an analytically tractable dynamic macroeconomic model along classical lines of Merton (1971), Dumas (1989), and more recently Angeletos (2007), He and Krishnamurty (2012) and Brunnermeier and Sannikov (2014). It features incomplete financial markets and two types of risk-averse agents: households and owners of firms subject to idiosyncratic productivity shocks. Firms want to finance their investments by raising outside funds, in particular from households, but face the problem that their revenue is private information and standard outside equity finance therefore impossible. Putting this corporate finance perspective at center stage allows us to address the classic macroeconomic problems outlined above from a new angle.

A benevolent social planner in our economy faces the task of redistributing unobservable output among firms and households such as to share idiosyncratic production risk and optimize intertemporal production and consumption. Intuitively, she must design a dynamic multi-agent mechanism that rewards firms with high output for sharing some of this output with low performing firms and households, without leaving them so much surplus as to jeopardize the risk-sharing objective. Building on the broader mechanism-design approach in our companion paper Biais et al. (2023), the present paper shows that the constrained optimal solution can be decentralized by letting firms issue short-term debt and giving the government three fiscal instruments: issuance of public debt, linear taxation of wealth of firm owners and households at different rates, and redistribution of initial endowments across households and firms. In fact, firm debt is instantaneous. Since firms' idiosyncratic production shocks follow a Brownian motion, this implies that firm debt is safe, as in much of the classic asset pricing literature following Merton (1971).

Hence, although the basic information asymmetry between firm insiders and outsiders prevents firms from issuing equity, firms can react flexibly to output shocks by issuing or repaying instantaneous debt. We assume that such debt transactions are frictionless. Furthermore, firms can use inside equity as a further margin of adjustment. In fact, as we show, they optimally use both margins simultaneously in order to downsize the balance sheet after negative production shocks and size up after positive shocks. As discussed below, this is a departure from the contingent-claims finance literature building on Leland (1994), which assumes costless outside equity issuance and long-term debt that requires costly adjustments.

The reason why these three instruments achieve optimal decentralization is that the firms' scaling decisions in response to their production shocks follow the same logic as the social planner's redistribution policy: firms with higher instantaneous output can scale up (by buying capital from distressed firms), while firms with unfavorable production shocks must scale down to avoid bankruptcy (which they will, because bankruptcy is less attractive than continuation even at very small size). Since firms' market decisions depend on their net worth, there is a role for public debt in affecting firms' balance sheets. To this end, the government optimally mimics firms and issues safe short-term debt, too, which is a perfect substitute to private debt. These bonds are not necessary to finance public spending (although we include such spending in the model to make this very point). Hence, the only tradeable security that is issued by firms (and the government) is safe debt.

As we discuss in Section 6 and the appendix, the decentralization result can be viewed as a kind of Second Welfare Theorem. Just as in classic complete-market settings à la Arrow-Debreu, the public planner can redistribute resources through appropriate taxes or transfers and affect production decisions by issuing bonds. In the equilibrium of the decentralized economy, fiscal policy affects the aggregate balance sheet of the private sector, and in particular corporate leverage, through bonds and taxes. Issuing public debt and distributing the proceeds to firms and households has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the leverage of firms and increases their incentives to undertake risky investments at a given risk-free interest rate. To clear the market for capital, the risk-free interest rate increases. This buffers the risk that owners of firms are bearing. Finally, issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption, which in turn has a negative impact on output growth. The optimal level of government debt is always positive and balances these different effects, depending on the weight of the preferences of firms' owners and households. In particular, Ricardian equivalence does not hold, because changing the firms' budget constraint has real effects.[1]

Equipped with this version of the Second Welfare Theorem, we can then proceed to address the central questions asked in the first paragraph above. In particular, questions about the optimal public debt to GDP ratio, firm leverage in general equilibrium, the relative size of the equilibrium growth and interest rates, g and r, or the role of optimal taxes versus transfers can be answered simply and explicitly.

[1]This is somewhat remarkable as all the usual assumptions for Ricardian equivalence hold: agents are fully rational and forward looking, everybody can borrow and lend at the same safe interest rate, and the path of government expenditures is fixed. But as Barro (1974) himself points out in his classical paper, a further assumption needed is that "the marginal net-wealth effect of government bonds is close to zero." The whole point of the policy considered here is to violate this assumption.

Interestingly, changing the weight of the firm sector in the social welfare function fundamentally changes the structure of public finances and the answers to the above questions.

When this weight is small, it is optimal to issue a small amount of public debt, the interest rate is low, and growth is high. In fact, in the limit, when only household interests matter, the growth rate approaches the Modified Golden Rule rate, in line with the benchmark result by Aiyagari (1994). The reason is that buffering the uninsurable productivity shocks of firms is of little direct importance for welfare, and thus public debt is low, firm equity is relatively small, and corporate leverage is large. This implies low investment demand by firms and thus low interest rates. The two effects combined yield a regime in which $g > r$.

When the welfare weight of firms is greater, buffering their shocks becomes more important. Thus, more public debt is optimally issued, firm equity increases, and corporate leverage declines. Higher firm equity causes interest to rise. Greater wealth in the economy triggers more consumption and thus growth declines. Thus, r increases and g decreases. As we show, for sufficiently large firm welfare weights, the economy switches, in fact, to a regime $r > g$.

However, at some level of the weight of the corporate sector, the economy reaches a level of growth that is so small and a level of debt to GDP that is so large that the wealth effect from issuing public debt becomes less important than the reduction of growth. If the welfare weight of the corporate sector increases beyond that level, losses of firms in case of negative productivity shocks are less and less severe for firms, since their leverage is low and their equity is high. Hence, it is optimal to operate with lower public debt as this reduces current consumption and stimulates growth. As according to the logic described above, the safe interest rate continues to rise, the economy remains in the regime $r > g$, and the government runs an eternal primary surplus. Overall, the optimal debt-to-GDP ratio is a single-peaked function of the welfare weight of firms, and in the limit, if this weight is close to 1, the ratio is larger than when firms have little welfare weight.

The influence of capital and the corporate sector on macroeconomic performance has intrigued the social sciences for decades. Numerous studies in economics and political science document corporate influence on policy-making by showing how lobbying can yield short-run benefits (see for instance Drutman (2015) and the survey of Bombardini and Trebbi (2020)). While this literature mostly addresses the problem of how special interests can be in conflict with the interest of the general public or to what extent business expertise can be of use to policy, our analysis shows that even simple differences in the weight of firms relative to households in the welfare function of a benevolent government have markedly different long-run welfare consequences because of changing fiscal policies and imply different macroeconomic regimes.

Our work builds on and contributes to different strands of the macroeconomic literature on fiscal policy with agent heterogeneity, which we review in the next section. However, it is worth emphasizing one issue of more than recent interest. As our model yields explicit analytic expressions for the equilibrium growth and interest rates, the analysis can directly address the question of the determinants of the difference $r - g$.[2] As noted above, we find that the interest rate may exceed the growth rate of GDP when corporate influence is high, and the opposite occurs when corporate influence is low.[3] In the first case ($g < r$), the intertemporal budget constraint of the government binds, and the value of outstanding debt at each date is equal to the net present value of all future primary surpluses. In the second case, $r < g$ and there is a permanent and growing primary deficit. In this case, the government's intertemporal budget constraint does not bind and, perhaps surprisingly, the public debt-to-GDP ratio is small. The government optimally runs a Ponzi scheme: it eternally repays old debt by issuing new one.

The remainder of this paper is organized as follows. In the next section, we provide a more detailed discussion of the literature. The model is set out in Section 3. In Section 4, we characterize the individually optimal decisions. The equilibrium analysis is presented in Section 5, while Section 6 develops the welfare analysis. In Section 7, we discuss the welfare improvement that can be generated by public debt issuance and its implications for redistribution through taxes. The implications of fiscal policy for optimal growth, interest rates, and the government budget are explored in Section 8. Section 9 presents a brief outlook on further research. Appendix A shows how to implement aggregate consumption profiles as the general equilibrium of our model through the appropriate choice of fiscal policy. In Appendix B we sketch how the outcome of an optimal direct mechanism by a social planner can be decentralized by wealth taxes and government bonds. For completeness, we provide the detailed calculations for the results of Section 4 in Appendix C.

2 Relation to the literature

Our paper is related to several strands of the literature.

First, since the early overlapping generations models dating back to Diamond (1965), a sizable literature has examined how fiscal policy influences the relations between three crucial macroeconomic variables: the real interest rate (r), the growth rate of output (g) and the marginal product of capital (μ).[4] A recent

[2]Some of the important recent contributions to this debate are Barro (2023), Blanchard (2019), Brunnermeier et al. (2021), Cochrane (2019), Cochrane (2022), Dumas et al. (2022), Reis (2021).

[3]The full comparative statics of the determinants of this tradeoff is given in Proposition 8.

[4]In the context of the overlapping generations framework, a recent debate about the sustainability of fiscal policy has focused on when and why governments can run prolonged deficits without being forced

strand of this literature re-examines this question in settings with infinitely-lived agents, using continuous-time methods from asset pricing. It also provides ways of endogenizing r, g and μ. Building on the seminal contributions by He and Krishnamurty (2012), Brunnermeier and Sannikov (2014), and Di Tella (2017), this literature considers economies with aggregate risk and studies the emergence and amplifications of financial crises, as well as the role of intermediaries in this dynamic. Our work is complementary, as we focus on the long-rung behavior of the economy and optimal fiscal policy when fiscal policy is limited by informational frictions between firms and outside investors and there are conflicting interests between households and firms. Unlike Basak and Cuoco (1998) and Brunnermeier and Sannikov (2014, 2016), we follow the more traditional approach in macroeconomics and assume that firm production, rather than capital accumulation, is risky (both approaches are in general largely equivalent). This is because this approach lends itself better to the information-based view of corporate finance at the heart of our model.

Regarding the welfare-enhancing role of public debt, our work builds on Brunnermeier et al. (2021), who focus on how to integrate a bubble term representing government expenditures – without ever raising taxes for them – into the fiscal theory of the price level in the presence of idiosyncratic risks and incomplete markets. They determine what they call the optimal "bubble mining rate", which is the optimal rate of issuing government debt. Brunnermeier et al. (2022) extend this approach and resolve the "public debt valuation puzzle", by showing that the price of debt is procyclical, since the bubble term rises in bad times. Reis (2021) considers a model in which households are hit by idiosyncratic depreciation shocks to their capital and face borrowing constraints. This creates a misallocation of resources and, together with non-insurable idiosyncratic risks, a demand for public debt as a safe asset and as an alternative form of savings. Reis (2021) identifies the determinants of the upper limit of spending that can be financed by debt. We also use a model with uninsurable idiosyncratic productivity risk, but there are no borrowing constraints and debt markets are frictionless. Different from Reis (2021), public debt in our model is not a new asset, it is a perfect substitute for safe corporate debt. Yet, it boosts the amount of corporate wealth and raises r.

This body of work builds on the broad strand of macroeconomic theory analyzing uninsurable idiosyncratic income shocks and incomplete financial markets.

to rely on taxation when $r < g$ – so-called "Ponzi schemes" – in the presence of uncertain production returns (Blanchard and Weil (2001), Blanchard (2019), Jiang et al. (2019) Dumas et al. (2022)). Abel et al. (1989) and Hellwig (2021) examine whether the conditions for dynamic inefficiency have to be based on the returns of all assets or only on the return of the safe asset. Dumas et al. (2022) endogenize the structural deficit in the form of an underfunded social security scheme and characterizes debt capacity limits, and Brumm and Hussman (2023) provide a quantitative theory of optimal and maximal debt to GDP.

Prominent theoretical references are Bewley (1983), Imrohoroğlu (1989), Huggett (1993) and in particular Aiyagari (1994). On the empirical side, Dyrda and Pedroni (2022) provide a calibration of a corresponding Ramsey problem to US data and evaluate redistribution and welfare through public debt, capital, and labour taxes numerically. The seminal paper of Aiyagari (1994), in which households self-insure against idiosyncratic income fluctuations by buying shares of aggregated capital, is widely used to examine the impact of household heterogeneity when markets are incomplete. This literature is large and was surveyed by Heathcote et al. (2009) and Krueger et al. (2016). We follow Angeletos (2007), who has enriched the neoclassical growth model by uninsured idiosyncratic investment risk and characterized the macroeconomic effects of this feature. In our model, only firms are subject to uninsurable idiosyncratic productivity risks and we study optimal fiscal policy. If there is no public debt, the leverage of firms and the equilibrium risk-free interest rate are entirely determined by the relative wealth of firms and households. By issuing public debt, the government can modify the aggregate balance sheet of the private sector and change the portfolio problem of firms, such that firm owners face less risk.

A classic part of the literature relevant for our work examines the role of government debt as an asset that can help overcome financial frictions. Woodford (1990) shows how issuing highly liquid public debt can increase the flexibility of liquidity-constrained agents to respond to variations in income and spending opportunities, thereby increasing economic efficiency. Aiyagari and McGrattan (1998) develop a model in which households face a borrowing constraint, which generates a precautionary savings motive. Government debt loosens borrowing constraints and enhances the liquidity of households, which improves consumption smoothing. The authors also stress the cost of higher government debt via adverse wealth distribution, incentive effects, and crowding-out effects on investment. The benefits and costs of public debt determine the optimal quantity of debt. In the presence of moral hazard for firms and of optimal contracts between firms and outside investors, Holmström and Tirole (1998) show that the public supply of liquidity is not necessary in an economy with no aggregate uncertainty and in which intermediaries coordinate the use of scarce private liquidity. Our work challenges this conclusion. Angeletos et al. (2016) explore how public debt can be used as collateral or a liquidity buffer in order to ease financial frictions. Since public debt lowers the liquidity premium but increases the cost of borrowing for the government, there exists a long-run optimal level of public debt. In our paper, there are neither borrowing constraints nor liquidity constraints. Public debt allows firms to buffer their losses, but this raises interest rates, and firms have to pay a higher interest rate on their own debt.

Last but not least, our work has a natural connection to the literature on continuous-time corporate finance. Building on Leland (1994), the contingent claims literature evaluating the tradeoff between bankruptcy costs and tax benefits of debt typically assumes costless equity issuance and long-term debt with positive adjustment costs, exactly opposite to our assumptions in the present paper.[5] As pointed out, e.g., by Abel (2018) and Bolton et al. (2021), this structure is difficult to reconcile with some of the empirical evidence on leverage dynamics, which is one reason why these authors model capital structure with instantaneous, frictionless debt, as in the asset-pricing-based approach of our paper. Different from all these papers we take a strict agency view, which rules out outside equity as a source of funding. Although this makes our model unsuitable for standard tax-bankruptcy cost analyses, in terms of capital structure theory it makes the case for having riskless instantaneous debt in a model of risky corporate earnings. And in fact, when replacing "outside equity" with "inside equity", models of costless equity funding such as the one developped in the paper by Bolton et al. (2021) become similar to our model in several respects. The main difference in terms of corporate finance is that outside equity in our model is infinitely costly, and more broadly that our work explores the macroeconomic consequences of these financial choices. This is a useful new perspective, as corporate finance models typically are partial equilibrium models and assume $r > g$, an assumption that is not always warranted, as our analysis shows.

3 The Model

3.1 The Macroeconomic Environment

The economy features a mass 1 continuum of competitive firms, owned and controlled by their shareholders, a mass 1 continuum of households who do not own any shares, and a government. Time is continuous: $t \in [0, \infty)$. There is only one physical good that can be consumed or invested, and it is taken as a numéraire. There is one financial asset, namely risk-free debt, that can be issued by firms and the government. This debt is real and its unit price is normalized to one: one unit of it can always be exchanged for one unit of the good, i.e. debt can be issued and retired without frictions or costs. The equilibrium between supply and demand of debt at each date t determines the interest rate at date t, denoted r_t.

Firms are run by managers who act in the interest of their shareholders. Firms are individually risky in the sense that they produce random output at each point in time. We assume that these random outputs are not publicly observable, which implies that firms cannot insure their risk away and that their equity cannot be

[5] Two prominent papers among many others in this tradition are Goldstein et al. (2001) or Strebulaev (2007).

traded.[6] Firms can only operate with inside equity and issue debt, which turns out to be risk-free: in equilibrium, default never occurs in our model.

The physical good is initially held by households and firms, but only firms can invest in productive technologies. Households cannot, and so they pass their initial endowments to firms in exchange for debt. They receive interest payments on their savings and decide continuously how much to consume. Households are identical, and are not subject to individual shocks. Without loss of generality, we can therefore aggregate them into a single representative household (the "household sector"), denoted by the superscript H. Firms have more complex decision problems: they can continuously adjust their investments and debt levels, and decide how much dividends to pay their owners for consumption.

The government has to finance an exogenous level of public expenditures and can redistribute wealth between the two sectors, households and firms, by means of taxes and subsidies[7]. These fiscal instruments are choice variables of the government. The dynamics of government debt is determined by the difference between interest payments on outstanding debt and primary surpluses (total tax revenues minus government expenditures).

3.2 The Formal Set-up

There is a continuum of firms $i \in [0, 1]$, with initial endowments (equity) $\tilde{e}^i$. Aggregate equity is denoted by

$$\tilde{E} = \int_0^1 \tilde{e}^i di.$$

The representative household has initial endowment $\tilde{H}$.

At each date t, firm i chooses its volume of productive assets k_t^i, financed by equity e_t^i and debt d_t^i. Equity is inside equity, and debt can be freely issued and traded. The balance sheet equation of firm i at time t is

$$k_t^i = d_t^i + e_t^i. \tag{1}$$

We allow debt d_t^i to be negative, in which case the firm has no debt but invests in bonds issued by other firms or the government.

The firm's instantaneous output net of depreciation is

$$k_t^i[\mu dt + \sigma dz_t^i], \tag{2}$$

where $\mu > 0$ is the average instantaneous return net of depreciation, identical for all firms, $\sigma \geq 0$ is the volatility of the instantaneous return, and the z_t^i are firm-specific

[6] This is as in Basak and Cuoco (1998) and much of the subsequent literature. Brunnermeier and Sannikov (2016) generalize this by assuming that firms can sell equity, but must hold an exogenous minimum fraction of it. In our companion paper, Biais et al. (2023), we microfound such assumptions. In the present paper, we discuss the underlying agency structures in Section 6 and Appendix B.

[7] For simplicity of exposition, we will usually use the word "taxes", with the interpretation that negative taxes are subsidies.

i.i.d. Brownian motions. As the only input in production is capital, (2) represents the firm's earnings before interest and taxes (EBIT).

Since production shocks are independent, they wash out in the aggregate, and aggregate production net of depreciation at time t is

$$Y_t = \mu K_t, \tag{3}$$

where K_t is aggregate capital at date t.

At each time t, the government must finance public goods that cost γK_t, where γ is an exogenous fraction of the aggregate capital stock. Government expenditures as a share of GDP are thus an exogenous constant γ/μ.[8]

The government must finance these expenditures by raising taxes or issuing debt. Furthermore, since markets are incomplete (there is no equity market), there may be more reasons for the government to intervene. For this, it can in principle employ the entire set of fiscal instruments, i.e. any conceivable taxes and debt instruments. In Appendix B, we sketch how the outcome of an optimal direct mechanism in an economy where firms' output is privately observable and firms' owners can divert revenues can be implemented in a market context by using straight debt, one round of initial redistribution, and linear wealth taxes.[9] In the present paper, we therefore directly work with linear wealth taxes. As one benchmark, we discuss the laisser-faire case of no-debt-no-taxes in Section 6.4.

At time $t = 0$, the government can issue debt B_0, which is distributed to households and firms, and further redistribute initial endowments. After redistribution, net wealth is H_0 for the representative household, e_0^i for firm i, and aggregate equity is

$$E_0 = \int_0^1 e_0^i di.$$

The aggregate wealth of the private sector is

$$E_0 + H_0 = K_0 + B_0,$$

where $K_0 = \tilde{E} + \tilde{H}$ is the initial stock of physical capital. Thus the government can modify the balance sheet of the private sector, and increase its net wealth by issuing debt. However, the government cannot produce any output, so the aggregate capital stock of the economy is still K_0. Just as in the famous article of Barro (1974), we will examine whether government debt, as a financial asset, can increase overall welfare.[10]

[8]We do not model the social utility generated by these expenditures explicitly and, therefore, say nothing about their optimal level.

[9]The full analysis of the dynamic multi-agent problem is significantly more complicated and provided in our companion paper, Biais et al. (2023).

[10]The government chooses its initial debt level B_0 such as to support an optimal allocation, consistent with the social objective function introduced in Section 6. In our model, there is no need for one-time lump-sum spending at date 0, which is implicit in the literature on maximum public deficit capacity (see, e.g., Reis (2021) or Brumm and Hussman (2023)).

Government debt evolves according to

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t, \tag{4}$$

where the dot represents the time derivative, r_t is the instantaneous risk-free interest rate, and T_t is net aggregate tax revenue (tax revenue minus subsidies) at time $t > 0$.

Given e_t^i, at each date t, firm i chooses its investment k_t^i and its dividend payout c_t^i to be consumed by its shareholders. At each date, the firm pays a linear tax $\tau_t^E e_t^i$ on its equity. The representative household chooses its consumption flow c_t^H and pays a linear tax $\tau_t^H H_t$ on its wealth. Recall that tax rates can be negative, in which case they represent subsidies. Households and shareholders maximize the expected discounted utility of consumption

$$\int_t^\infty e^{-\rho s} \log c_s^k ds, \quad k = i, H,$$

where $\rho > 0$ is the discount rate, the same for households and firm owners.[11]

4 Individual Decisions

We first characterize the solutions of the household's and the firms' decision problems. These are standard and yield well-known solutions going back to Merton (1971). For completeness, we add the detailed calculations in Appendix C.

4.1 Households

Net of initial lump sum taxes, the representative household has initial net worth $H_0 > 0$ at time $t = 0$, no further income later, and saves via firm and government bonds, which are perfect substitutes. There is no other form of savings, since the good cannot be stored.[12] Hence the household chooses a consumption path $c^H = \left(c_t^H\right)_{t \geq 0}$ that solves

$$\max_{c^H} \int_0^\infty e^{-\rho t} \log c_t^H dt,$$

subject to the equation of motion of wealth

$$\dot{H}_t = (r_t - \tau_t^H) H_t - c_t^H. \tag{5}$$

This is a standard problem. At the optimum,

$$c_t^H = \rho H_t \tag{6}$$

[11]The model can be fully solved for suitable CRRA utilities. The analysis becomes significantly more complex, but the main result regarding the welfare improvement from issuing public debt continues to hold.

[12]Our results continue to hold when the good can be stored and the real interest rate is larger than the return from storage.

for all $t \in [0, \infty)$, and the value function of the household's problem is

$$\rho V^H(t, H_t) = e^{-\rho t} \log(\rho H_t) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^H - \rho \right) ds. \tag{7}$$

Note that equations (5) and (6) imply that H_t is always positive.[13]

4.2 Firms

To simplify the exposition, we first assume $r_t < \mu$ and then verify that this is always the case in equilibrium. With initial equity $e_0^i > 0$ at $t = 0$, the firm's flow of funds is given by

$$k_t^i[\mu dt + \sigma dz_t^i] = [r_t d_t^i + \tau_t^E e_t^i + c_t^i]dt + de_t^i, \tag{8}$$

where the left-hand side represents earnings before interest and taxes and the right-hand side is the sum of interest payments, taxes, consumption of equity holders (dividends), and the change in equity as a residual. (8) reflects the simple corporate accounting identity:

$$\text{EBIT} = \text{interest} + \text{taxes} + \text{dividends} + \text{retained earnings}.$$

The flow of funds equation assumes that the firm is always able and willing to pay the interest on its debt. Below we shall see that the former is always true and that the firm's continuation value is strictly positive if it honors its debt. Hence, strategic default is not an issue, as long as there are bankruptcy costs and no debt renegotiation, which we assume.[14]

The firm then chooses a path k_t^i, d_t^i, c_t^i, $t \geq 0$ that solves

$$\max_{k^i, d^i, c^i} \mathbb{E} \int_0^\infty e^{-\rho s} \log c_s^i ds,$$

subject to the balance sheet constraint (1) and the law of motion (8) for each $t \geq 0$. The Bellman Equation yields the standard solution

$$\begin{aligned} c_t^i &= \rho e_t^i && (9) \\ k_t^i &= \frac{\mu - r_t}{\sigma^2} e_t^i && (10) \end{aligned}$$

and the stochastic law of motion for firm equity

$$de_t^i = \left[\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right] e_t^i dt + \frac{\mu - r_t}{\sigma} e_t^i dz_t^i. \tag{11}$$

[13] Here, we have implicitly assumed that the government sets taxes $\tau_t^H < r_t - \rho$. In Section 5 we show that this is possible. Other choices are obvious nonsense.

[14] There is a large literature on strategic default, which we do not need to discuss here. See Hart and Moore (2001) or Bolton and Scharfstein (1996) for foundational work and Fan and Sundaresan (2000) for an early classic in continuous time.

Condition (10) implies that the capital-to-equity ratio k_t^i/e_t^i is identical across firms. Firms continuously adjust their debt levels, but they all keep the capital-to-equity ratio at the same value

$$x_t \equiv \frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2}. \tag{12}$$

Condition (10) also implies that if we had $k_t^i \leq 0$ for one i, this would hold for all i and therefore yield $K_t \leq 0$ in the aggregate, which justifies our initial assumption $r_t < \mu$. Further, note that liquidity default never occurs. Indeed, the stochastic differential equation (11) describes a Geometric Brownian Motion, with the well-known solution[15]

$$e_t^i = e_0^i exp\left(\int_0^t (r_s - \tau_s^E - \rho + \frac{\sigma^2 x_s^2}{2})ds + \sigma \int_0^t x_s dz_s^i\right) > 0. \tag{13}$$

While its capital-to-equity ratio remains constant, the firm adjusts its debt continuously in response to its earnings shocks. After a high shock, it invests more and issues more debt; after a low shock, it does the opposite. Hence, Brownian productivity shocks are not enough to drive the firm into bankruptcy.[16]

The value function then is the same for all firms (because they all face the same tax rate), strictly positive, and equal to

$$\rho V^E(t,e) = e^{-\rho t}\log(\rho e) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E - \rho + \frac{\sigma^2}{2}x_s^2\right)ds. \tag{14}$$

Note the similarity with the value function for households, the difference being the last term in the integral, which is due to the production risk in the optimization problem.

5 Macroeconomic Equilibrium

5.1 Aggregates

By (5) and (6), households' aggregate wealth H_t follows the law of motion

$$\dot{H}_t = \left(r_t - \tau_t^H - \rho\right) H_t. \tag{15}$$

This wealth is entirely invested in risk-free debt, and the household is indifferent between public debt and corporate debt. Let D_t^H and B_t^H denote the households' holdings of private and public debt, respectively. The households' balance sheet then is

$$H_t = D_t^H + B_t^H. \tag{16}$$

[15]See, e.g., Shreve (2004), p. 147-8.

[16]This is why Abel (2018) assumes discrete earnings shocks and Bolton et al. (2021) a jump-diffusion process.

Individual balance sheets of firms follow random trajectories but, thanks to the Law of Large Numbers, the aggregate balance sheet of the firm sector is deterministic. Denoting by B_t^E the firms' aggregate holdings of public debt (which may be negative), it is simply given by:

$$\begin{array}{c|c} \text{Assets} & \text{Liabilities} \\ \hline K_t & D_t^H \\ B_t^E & E_t \end{array} \tag{17}$$

Aggregating individual investment rules (10) yields aggregate capital as

$$K_t = \frac{\mu - r_t}{\sigma^2} E_t, \tag{18}$$

which produces net gross domestic product Y_t, as defined in (3). Note that $K_t > 0$ at all times.

By the individual laws of motion of firm equity (11) and the Law of Large Numbers, the equation of motion of aggregate firm equity is

$$\begin{aligned} \dot{E}_t &= \int_0^1 \left(\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right) e_t^i di \\ &= \left(\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right) E_t & (19) \\ &= \left(r_t - \tau_t^E - \rho \right) E_t + (\mu - r_t) K_t, & (20) \end{aligned}$$

where the last equality follows from (18).

Government debt is $B_t = B_t^H + B_t^E$ and evolves according to (4), $\dot{B}_t = \gamma K_t + r_t B_t - T_t$, where aggregate tax receipts (or subsidy expenditures if negative) at date $t > 0$ are given by

$$T_t = \tau_t^H H_t + \tau_t^E E_t. \tag{21}$$

Note that we allow B_t to be negative, but this will never be optimal.

5.2 Fiscal Policy and Equilibrium

Equilibrium requires markets to clear at all times, given the fiscal policy in place. Here, fiscal policy consists of two parts:

- at date 0, the government issues debt B_0 and distributes lump-sum subsidies L^H to households and L^E to firms,
- at all further dates $t > 0$ the government collects instantaneous wealth taxes at rates τ_t^H for households and τ_t^E for firms.

The government cannot create real goods, but it can boost the private sector's balance sheet by creating paper assets. Correspondingly, the government's balance sheet identity at $t = 0$ is $B_0 = L^H + L^E$.

In equilibrium, the interest rate path r_t makes the aggregate balance sheet constraint of the economy hold at each point in time t. Consolidating the aggregate firm balance sheet (17) with the households' balance sheet equation (16) yields the private sector's balance sheet

$$\begin{array}{c|c} \text{Assets} & \text{Liabilities} \\ \hline K_t & H_t \\ B_t & E_t \end{array} \tag{22}$$

Note that the aggregate balance sheet is deterministic – there is no aggregate risk in our economy. In equilibrium, all changes must be consistent:

$$\dot{K}_t + \dot{B}_t = \dot{H}_t + \dot{E}_t \tag{23}$$

for all t. Using (15), (18), (20), (4), and (21), condition (23) can be written as

$$\begin{aligned} \dot{K}_t &= \dot{H}_t + \dot{E}_t - \dot{B}_t \\ &= \left(r_t - \tau_t^H - \rho\right) H_t + \left(r_t - \tau_t^E - \rho\right) E_t + (\mu - r_t)K_t - \gamma K_t - r_t B_t + T_t \\ &= (\mu - \gamma)K_t - \rho(H_t + E_t), \end{aligned} \tag{24}$$

which is the economy's IS equation (equality of investment and net savings).

At each date t, the four aggregate variables K_t, B_t, E_t, H_t are linked by the balance sheet identity (22). In fact, by the homogeneity of the firms' investment problem, only two state variables are sufficient: the capital-equity ratio x_t as defined in (12), and $h_t = \frac{H_t}{E_t}$, the ratio of household wealth over firm equity. Note that $x_t > 1$ if and only if $D_t^H - B_t^E > 0$, i.e. if firms are net borrowers. In this case, $x_t - 1$ is the firms' debt to equity ratio. For simplicity of exposition, we will often refer to x_t as "firm leverage". If $x_t < 1$, firms have zero leverage and are net lenders.[17]

The trajectories of the two state variables (x_t, h_t) completely determine all aggregate variables (output, consumption, and investment) in equilibrium.[18] In fact, by (24), the equilibrium growth rate g_t of capital (and thus GDP) is

$$\begin{aligned} g_t = \frac{\dot{K}_t}{K_t} &= \mu - \gamma - \rho\frac{H_t + E_t}{K_t} \\ &= \mu - \gamma - \rho\frac{h_t + 1}{x_t}. \end{aligned} \tag{25}$$

By (15), aggregate household wealth grows according to

$$\frac{\dot{H}_t}{H_t} = \mu - \rho - \tau_t^H - \sigma^2 x_t \tag{26}$$

and aggregate equity, by (19), according to

$$\frac{\dot{E}_t}{E_t} = \mu - \rho - \tau_t^E - \sigma^2 x_t(1 - x_t).$$

[17] This can only happen if $B_t > H_t$, which means that public debt exceeds the total wealth of households.

[18] Equation (13) shows the dynamics of individual equity positions, which are stochastic.

Finally, by (4) and (21), the evolution of government debt B_t is given by

$$\frac{\dot{B}_t}{B_t} = \mu - \sigma^2 x_t + \frac{\gamma x_t - \tau_t^H h_t - \tau_t^E}{1 + h_t - x_t}$$

as long as $B_t \neq 0$, i.e. as long as $x_t \neq h_t + 1$, and by (4) and (21) for all points (x_t, h_t) with $x_t = h_t + 1$. Given this direct relation between equilibria and the $x_t - h_t$ - trajectories, it is useful to characterize the dynamic system (x_t, h_t) in more detail.

The initial values of the state variables are given by the government lump sum transfers at date 0:

$$h_0 = \frac{H_0}{E_0} = \frac{\tilde{H} + L^H}{\tilde{E} + L^E}, \tag{27}$$

$$x_0 = \frac{K_0}{E_0} = \frac{\tilde{H} + \tilde{E}}{\tilde{E} + L^E}. \tag{28}$$

The dynamics of the state variables for $t > 0$ are then determined by the instantaneous tax rates. Indeed, the definitions of x_t and h_t imply:

$$\dot{h}_t = \left(\frac{\dot{H}_t}{H_t} - \frac{\dot{E}_t}{E_t}\right) h_t$$

$$\dot{x}_t = \left(\frac{\dot{K}_t}{K_t} - \frac{\dot{E}_t}{E_t}\right) x_t.$$

By (15), (19), (18), and the definition of x_t, we have

$$\dot{h}_t = (\tau_t^E - \tau_t^H - \sigma^2 x_t^2) h_t. \tag{29}$$

Similarly, using (24),

$$\dot{x}_t = (\sigma^2 x_t^2 - \rho)(1 - x_t) + (\tau_t^E - \gamma) x_t - \rho h_t. \tag{30}$$

If the system (29) and (30) has a solution that stays in the interior of the positive (x, h) quadrant, then this solution yields an equilibrium of our economy, as shown above. Conversely, any equilibrium of our economy yields a solution of (29) and (30) in the interior of the positive quadrant.[19] Going one step further, an inspection of (27)–(28) and (29)–(30) shows that any differentiable trajectory of (29) and (30) in the interior of the positive (x, h) quadrant can be obtained by an appropriate fiscal policy:

- The lump sum transfers L^E and L^H are determined by initial values (x_0, h_0):

$$L^E = \frac{1}{x_0}(\tilde{H} + \tilde{E}) - \tilde{E}, L^H = \frac{h_0}{x_0}(\tilde{H} + \tilde{E}) - \tilde{H}. \tag{31}$$

- Instantaneous tax rates are given by

$$\tau_t^E = \gamma + \sigma^2 x_t (x_t - 1) + \rho(\frac{1 + h_t}{x_t} - 1) + \frac{\dot{x}_t}{x_t}, \tag{32}$$

$$\tau_t^H = \tau_t^E - \sigma^2 x_t^2 - \frac{\dot{h}_t}{h_t}. \tag{33}$$

[19]Note that by (29), the trajectory never hits the x_t-axis. Constellations in which it hits the h_t-axis in finite time are uninteresting.

Thus we have established:

Proposition 1. *For any differentiable trajectory* (x_t, h_t) *in* $\mathbb{R}^2_{++}$ *there is a choice of fiscal policy* (L^E, L^H) *and* (τ^E_t, τ^H_t) *such that the general equilibrium under this policy exists, is unique, and generates* (x_t, h_t)*.*

Note that the converse of Proposition 1 is not true: not every choice of fiscal policy (L^E, L^H) and (τ^E_t, τ^H_t) is sustainable, in the sense that it yields a dynamic system whose solution stays in $\mathbb{R}^2_{++}$ forever.

The simple characterization of equilibrium through trajectories $(x_t, h_t) \in \mathbb{R}^2_{++}$ makes it possible to describe some key policy variables and relations succinctly. In fact, by (22), public debt at time t is positive iff

$$1 + h_t - x_t > 0.$$

We will refer to the locus of points $(x, h) \in \mathbb{R}^2_+$ with $1 + h - x = 0$ as the "Zero-Debt-Line" (ZDL).

The government debt-to-GDP ratio at date t can be expressed as

$$\delta_t \equiv \frac{B_t}{Y_t} = \frac{1 + h_t - x_t}{\mu x_t}. \tag{34}$$

Simple calculations then yield explicit relations between the main aggregate variables of our economy, which we collect in the following proposition.

Proposition 2. *In equilibrium, at any date* t*:*

1. *The interest rate is a linearly decreasing function of firm leverage:*

$$r_t = \mu - \sigma^2 x_t. \tag{35}$$

2. *Output growth is a linearly decreasing function of the debt-to-GDP ratio:*

$$g_t = \mu - \gamma - \rho - \rho\mu\delta_t. \tag{36}$$

3. *The interest rate is smaller than the growth rate,* $r_t < g_t$*, if and only if*

$$\gamma x_t + \rho(h_t + 1) < \sigma^2 x_t^2. \tag{37}$$

As an illustration, Figure 1 shows a particular trajectory of the state variables (29)–(30) (in blue) in a diagram showing the interest-growth boundary (37) (in magenta). In this equilibrium, the economy starts out with zero private debt ($x_0 = 1$), positive public debt, and $r_t > g_t$. Private debt then increases and public debt decreases until the economy crosses the Zero-Debt-Line (in black) when public debt becomes negative, and finally reaches the region where $r_t < g_t$. The equilibrium corresponds to stationary tax rates $\tau^E = 0.22$ and $\tau^H = 0.12$. Note that in this equilibrium, the share of household wealth in total private wealth, $H_t/(H_t + E_t)$, is initially increasing and then converges monotonically to 0. This does not necessarily mean, however, that household wealth decreases in absolute terms, i.e. that households become worse off over time. In fact, an inspection of (26) shows that this depends on the productivity of capital μ, and occurs if and only if μ is sufficiently small.

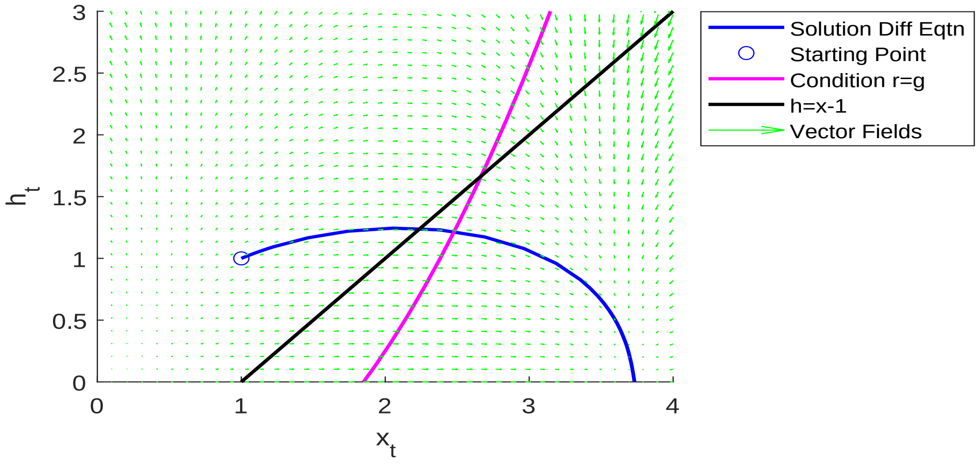

Figure 1: A (x_t, h_t) trajectory for $\rho = 0.04, \sigma = 0.15, \gamma = 0.02$.

6 Welfare

In this section, we first consider the general problem faced by a social planner who can redistribute endowments and resources subject to the assumed informational constraints on firms' output. We then characterize the resulting optimal allocation and policy tools.

6.1 The Planner's Problem

Consider a social planner in the economy of Section 3 without the specific policy instruments of linear taxes and public debt. Finding optimal private consumption and investment trajectories is difficult, because the individual output shocks dz_t^i are private information of the firms, who can divert some of their output and consume it secretly. This general mechanism design problem is solved in our companion paper Biais et al. (2023); we show how to apply this result to the present problem in Appendix B below. A key insight of Biais et al. (2023) is that for any optimal mechanism, the ratio of instantaneous equityholder consumption c_t^i to a firm's capital stock k_t^i is independent of the history of output shocks $\{dz_\tau^i\}_{\tau=0}^t$ and differentiable w.r.t. t. Furthermore, if all firms are treated equally (which we assume), this ratio is independent of i. Hence, as we show in Appendix B.2, if an optimal mechanism identifies $\psi_t = k_t^i/c_t^i$ and optimal household consumption c_t^H, one can define strictly positive and differentiable trajectories $(x_t, h_t) \in \mathbb{R}^2$ that by Proposition 1 generate the private consumption and investment paths from the optimal mechanism as the unique equilibrium outcome of the economy defined in Section 3.2. A second-best welfare analysis can therefore restrict attention to linear taxes and public debt.

6.2 Indirect Utilities

In order to express private consumption utilities in terms of trajectories $(x_t, h_t) \in \mathbb{R}^2_{++}$, note first that under the optimal private consumption rule of households (6), their value function as of time 0 for a given level of initial wealth H_0 is

$$\begin{aligned} \rho V^H(0, H_0) &= \rho \int_0^\infty e^{-\rho t} \log(\rho H_t) dt \\ &= \rho \int_0^\infty e^{-\rho t} (\log(\rho E_t) + \log h_t) dt \end{aligned} \tag{38}$$

by the definition of h_t.

Equityholders' utilities depend on the way initial capital is shared between them. When it is shared equally, they all have the same expected utility at date 0, namely, using equation (19),

$$\begin{aligned} \rho V^E(0, E_0) &= \log(\rho E_0) + \int_0^\infty e^{-\rho t} \left[\frac{\dot{E}_t}{E_t} - \frac{\sigma^2}{2} x_t^2 \right] dt \\ &= \rho \int_0^\infty e^{-\rho t} \left[\log(\rho E_t) - \frac{\sigma^2}{2\rho} x_t^2 \right] dt. \end{aligned} \tag{39}$$

By the definition of K_t and its law of motion (25), we have

$$\begin{aligned} \rho \int_0^\infty e^{-\rho t} \log E_t dt &= \rho \int_0^\infty e^{-\rho t} \left[\log K_t - \log x_t \right] dt \\ &= \log K_0 + \frac{1}{\rho}(\mu - \gamma) - \rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t \right] dt. \end{aligned} \tag{40}$$

Inserting (40) into (38) and (39),

$$\begin{aligned} \rho V^H &= \log(\rho K_0) + \frac{\mu - \gamma}{\rho} \\ &\quad - \rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t - \log h_t \right] dt \\ \rho V^E &= \log(\rho K_0) + \frac{\mu - \gamma}{\rho} \\ &\quad - \rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t + \frac{\sigma^2}{2\rho} x_t^2 \right] dt. \end{aligned}$$

This shows that households and firms have very different preferences over the trajectories of (x_t, h_t). Households' utility is maximum when x_t, h_t go to infinity, such that $h_t < x_t < h_t + 1$. Equityholders, on the other hand, achieve maximum utility when $h_t = 0$ and x_t equals the unique positive solution x_{min} of the equation

$$\sigma^2 x^3 + \rho x - \rho = 0. \tag{41}$$

6.3 Welfare Optima and the Pareto Frontier

The social optimum must take these diverging preferences into account. In this vein, we assume that social welfare is a weighted average of firms' and households' utilities:

$$W = \alpha V^E + (1 - \alpha) V^H,$$

where $0 < \alpha < 1$ is the weight put by the government on the corporate sector. Using the above expressions of V^H and V^E, we have

$$W = \log(\rho K_0) + \frac{\mu - \gamma}{\rho} - \int_0^\infty e^{-\rho t} \left[\frac{1 + h_t}{x_t} + \log x_t - (1 - \alpha) \log h_t + \alpha \frac{\sigma^2 x_t^2}{2\rho} \right] dt.$$

The expression under the integral is bounded and can be maximized pointwise.[20] Hence, W is maximum for constant values of x_t and h_t, namely x^* and h^*, which are uniquely determined by the first-order conditions[21]

$$(1 - \alpha) x^* = h^* \tag{42}$$

$$\frac{\sigma^2}{\rho} x^{*3} + x^* - \frac{1}{\alpha} = 0. \tag{43}$$

For any $0 < \alpha \leq 1$, equation (43) has a unique positive solution x^*. Equation (42) then determines h^*. Hence, there is a unique welfare maximum that corresponds to a stationary point of the (x_t, h_t)-dynamics.[22] Furthermore,

Proposition 3. *When $\sigma > 0$, optimal government debt is strictly positive:*

$$1 + h^* > x^*.$$

Proof. $1 + h^* - x^* = 1 - \alpha x^*$ by (42). The polynomial in (43) is increasing, and strictly positive for all $x \geq 1/\alpha$. Hence, $x^* < 1/\alpha$ and thus $1 + h^* - x^* > 0$. □

Hence, the welfare maximum is not compatible with balanced budgets, and a government wishing to implement this maximum through fiscal policy must issue

[20] This is because of the stationary nature of all decision problems.

[21] It is straightforward to verify that the first-order conditions determine the unique global maximum.

[22] Note that W is well defined for any bounded and piecewise continuous trajectory (x_t, h_t) which would be implemented by more general fiscal policies involving multiple lump sum transfers. Since W is maximum for a constant (x_t, h_t), our restriction to a single episode of lump sum transfers is without loss of generality.

a positive amount of safe debt. Specifically, Proposition 1 implies that the welfare optimum (42)–(43) can be implemented by some combination of initial lump sum transfers (L^{E*}, L^{H*}) and instantaneous tax rates (τ^{E*}, τ^{H*}), which follow from (32) and (33):

$$\tau^{E*} = \gamma + \sigma^2 x^*(x^* - 1) + \rho(\frac{1 + h^*}{x^*} - 1) \tag{44}$$

$$\tau^{H*} = \tau^{E*} - \sigma^2 (x^*)^2 = \gamma - \sigma^2 x^* + \rho(\frac{1 + h^*}{x^*} - 1). \tag{45}$$

Note that the welfare optimum is independent of the government expenditure coefficient γ, while the taxes needed to implement it are not.

By eliminating α between (42) and (43) we obtain a representation of the (constrained) Pareto Frontier in the (h, x) plane:

$$h(x) = x - \frac{\rho}{\rho + \sigma^2 x^2} \tag{46}$$

for $x \geq x_{min}$, where x_{min} is the lower bound given by (41). By (46), each $x \geq x_{min}$ defines a constrained Pareto allocation, which is a steady state with constant interest rate $r = \mu - \sigma^2 x$.

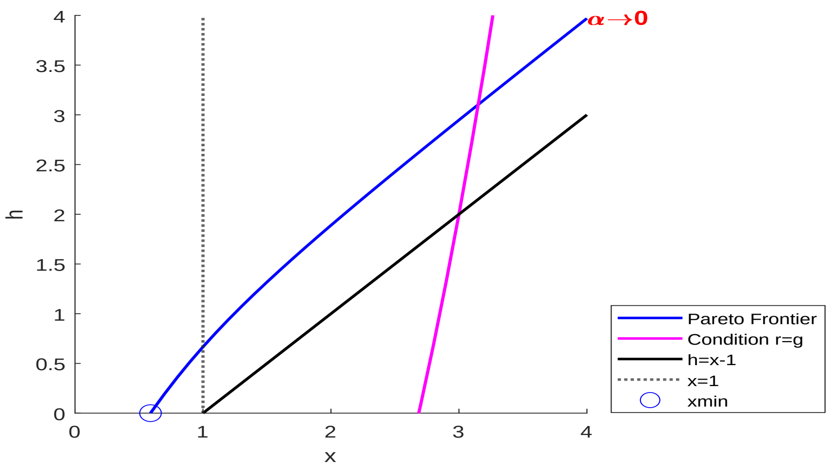

Figure 2: The Pareto Frontier in (x, h) space for $\rho = 0.01, \sigma = 0.2, \gamma = 0.1$.

Figure 2 shows the Pareto Frontier in the (x, h) plane, for the same values of ρ, σ, and γ as Figure 1.

When $\sigma > 0$, the Pareto Frontier lies entirely above the Zero-Debt-Line $h + 1 - x = 0$, and it converges to the diagonal $h = x$ for $x \to \infty$. The Zero-Debt-Line corresponds to the unconstrained Pareto Frontier: when there are no frictions, idiosyncratic risks can be eliminated, which is equivalent to taking $\sigma = 0$. In this case, optimal public debt is zero. In the general case, by (32), (33), (42) and (43), the instantaneous tax/subsidy rates that implement the second best allocations are

$$\tau^E = \gamma + \sigma^2 x^2 - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} \tag{47}$$

$$\tau^H = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2}. \tag{48}$$

A simple inspection shows that the lower bound for the Pareto Frontier satisfies $x_{min} < 1$. Hence, there are Pareto Optima with $K^* < E^*$, i.e. in which firms are net lenders. This means that the situation mentioned in footnote 17 cannot only arise, but can even be optimal. This is the case if α is large, i.e. if fiscal policy caters strongly to firms' interests.

6.4 Laisser-Faire

A passive government does not engage in fiscal policy or redistribution. We therefore define a Laisser-Faire (LF) policy by three features: (i) $B_t = 0$ for all t (balanced budget), (ii) $L^H = L^E = 0$ (no lump-sum redistribution), and (iii) $\tau_t^H = \tau_t^E = \tau_t$ (equal taxation).

Laisser-Faire therefore implies $T_t = \tau_t(H_t + E_t) = \tau_t K_t$. Together with the balanced-budget constraint $T_t = \gamma K_t$, this implies that taxes are constant,

$$\tau_t = \gamma.$$

The trajectory (x_t, h_t) under LF is entirely contained in the Zero-Debt-Line $x = h + 1$ and starts at $x_0 > 1$. Therefore we can focus on the variable x_t alone. Its equation of motion (43) simplifies to

$$\dot{x}_t = -\sigma^2 x_t^2 (x_t - 1).$$

Thus x_t converges monotonically to 1 and h_t to 0.

Interestingly, Pareto Optima are not necessarily Pareto improvements over the Laisser-Faire. This is depicted in the generic Figure 3, which displays the Pareto Frontier in utility space, i.e. the $V^H - V^E$ - plane. By construction, the LF is not Pareto optimal: it is represented by the two values LF^E and LF^H for V^E and V^H, respectively.

Figure 3 also shows the following two properties, which follow formally from an inspection of (38) and (39) together with (42) and (43):

$$\lim_{\alpha \to 0} V^E(x^*, h^*) = \lim_{\alpha \to 1} V^H(x^*, h^*) = -\infty.$$

Hence, the allocation that maximizes W is a Pareto improvement over Laisser-Faire when α is intermediary. As the figure illustrates, when α is large, households strictly prefer Laisser-Faire to the welfare optimum, while firms strictly prefer Laisser-Faire when α is small.

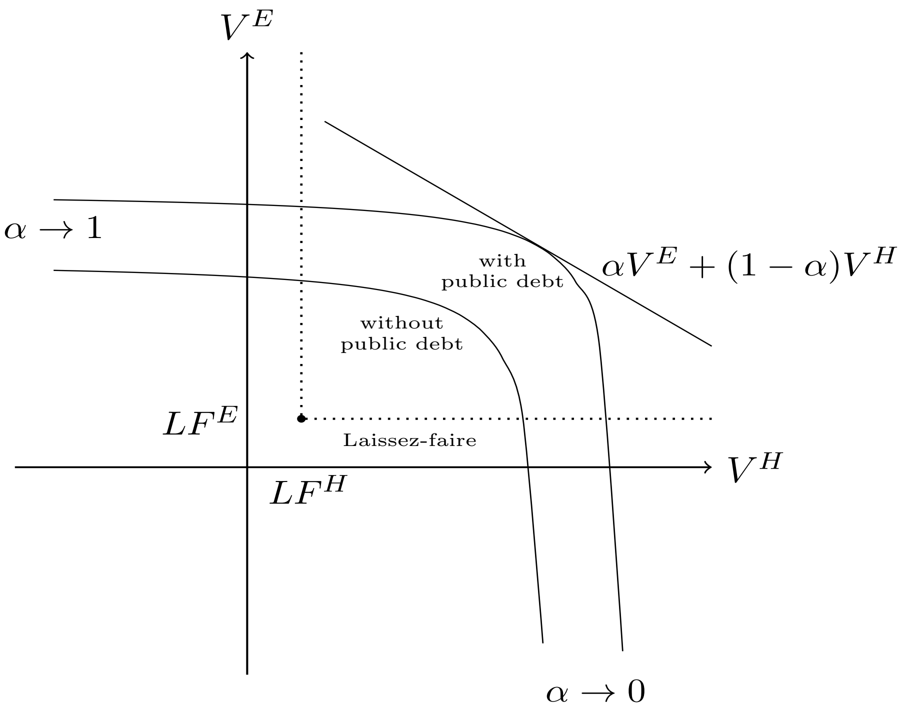

Figure 3: The Pareto Frontiers and Laisser-Faire in utility space.

Note that there exist Pareto improvements over the Laisser-Faire even if fiscal policy is constrained by a balanced budget at each point in time, i.e. if no public debt is issued. This constrained Pareto Frontier is also displayed in Figure 3 and will be discussed next.

7 Public Debt, Taxes, and Redistribution

7.1 Pareto Improving Debt

By Proposition 3, Pareto optimal public debt must be strictly positive at all times. The following thought experiment illustrates this basic result, by explicitly constructing the Pareto improvement that is possible in a situation of balanced government budgets.

As discussed in the previous section, under balanced budgets, $h_t = x_t - 1$ and the welfare function becomes, up to a constant,

$$W_{BB} = \int_0^\infty e^{-\rho t} \left[(1-\alpha)\log(x_t - 1) - 1 - \log x_t - \alpha \frac{\sigma^2 x_t^2}{2\rho} \right] dt.$$

Maximizing W_{BB} yields the maximal welfare that can be achieved without issuing public debt. Again, the expression is maximized at a steady state allocation, characterized by the unique solution $x_{BB} > 1$ of the first order condition

$$\frac{\sigma^2}{\rho}(x^3 - x^2) + x = \frac{1}{\alpha}.$$

Note the similarity with the equation defining x^*, (43), and that private leverage is higher here: $x_{BB} > x^*$. x_{BB} corresponds to the following allocation of initial wealth: $E_0 = \frac{K_0}{x_{BB}}$ and $H_0 = K_0 - E_0$. The growth rate of output is $g = \mu - \gamma - \rho$, and up to additive constants, the expected continuation utilities can be written as follows:

$$\begin{aligned} V_{BB}^H &= \frac{1}{\rho}[\log H_0 - H_0 - E_0] \\ V_{BB}^E &= \frac{1}{\rho}[\log E_0 - H_0 - E_0 - \frac{\sigma^2}{2\rho E_0^2}]. \end{aligned}$$

To understand how this allocation can be Pareto improved, suppose that the government issues a small amount of debt and distributes it to the two categories of agents, so that both ΔE_0 and ΔH_0 are positive. The government also adjusts the tax rates, so that the economy remains in the new steady state. The first order change in households' utility is such that

$$\rho \Delta V_{BB}^H = \frac{\Delta H_0}{H_0} - \Delta(H_0 + E_0),$$

corresponding to the difference between the relative wealth increase and the total wealth increase (equal to new government debt), which reduces growth. The equivalent term for firm equity is

$$\rho \Delta V_{BB}^E = \frac{\Delta E_0}{E_0} - \Delta(H_0 + E_0) + \frac{\sigma^2}{\rho E_0^3} \Delta E_0,$$

where a new term appears, corresponding to the reduction in the risk premium that follows the decrease in private leverage. Hence, it is possible to distribute the additional wealth created by the government in such a way that both types of agents benefit, as long as the following two conditions hold:

$$h_0 < \frac{\Delta H_0}{\Delta E_0} < h_0 + \frac{\sigma^2 x_0^3}{\rho}.$$

This is only possible in an economy with frictions, where idiosyncratic risk cannot be eliminated and $\sigma > 0$. In a frictionless economy, the welfare enhancing role of government debt disappears. Moreover, in the frictionless economy, the welfare optimum is achieved by initial lump sum redistribution, and no further redistribution takes place. Indeed, by (47) and (48), $\tau^E = \tau^H = \gamma$ when $\sigma = 0$. Since firms and households earn the same return on their investments ($\mu = r$) in the frictionless case, they face the same tax rate to finance public expenditures.

Hence, issuing public debt and distributing it to the private sector in adequate proportions has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the leverage of firms and increases the incentives of firms to undertake risky investments at a given risk-free interest rate. To clear the market for capital, the risk-free interest rate thus increases. This buffers the portfolio risk that owners of firms are bearing.[23] Finally, the higher wealth created by the new asset increases consumption of all agents and thus has a negative impact on output growth. All these effects occur jointly and feed back into each other.[24]

However, since these wealth increases must accrue to firms in order to trigger the balance sheet effect, it is necessary to balance them by continuously redistributing wealth from firms to households to maintain steady state growth. In fact, as we show in the next subsection, in the optimum, the initial asset injection must entirely accrue to firms, which in turn has a strong redistributionary consequence in terms of ongoing taxation.

7.2 Optimal Fiscal Policy

To clarify the role of government debt and taxes in our economy further, consider first the case without financial frictions, in which idiosyncratic risks can be diversified away, so that we can effectively take $\sigma = 0$. As we already saw, the optimal allocation is then implemented by redistributing initial wealth, so that $E_0 = \alpha K_0$ and $H_0 = (1-\alpha)K_0$, and having zero government debt at all times. To keep the economy in steady state, taxes should be equal across firms and households: $\tau^E = \tau^H = \gamma$.

Suppose now that as of date 0, frictions appear, such that it is not possible to eliminate idiosyncratic risks anymore and $\sigma > 0$.[25] The optimal response of the government to this shock is to issue debt for an amount $B_0(\sigma) = (\frac{1}{x^*} - \alpha)K_0$ and to distribute it exclusively to the firms. Indeed, the optimality conditions (42)–(43) imply

$$H_0^*(\sigma) = (1-\alpha)K_0.$$

Together with the aggregate balance sheet identity (22), this implies

$$E_0^*(\sigma) = \alpha K_0^*(\sigma) + B_0^*(\sigma).$$

[23] Note that these comparative statics refer to a change from a constrained optimal stationary allocation (x_{BB}, h_{BB}) on the Zero-Debt-Line to the welfare optimum (x^*, h^*). An arbitrary change from any initial allocation $(\tilde{x}, \tilde{h})$ on the ZDL to the welfare optimum, of course, cannot be signed, as there must be redistribution according to the weight α.

[24] Although public debt increases private consumption and thus decreases private investment, public debt does not "crowd out" private investment in the traditional sense (see Blanchard (2008)). "Crowding out" usually refers to the substitution of private spending by public spending, which is impossible in our model, where government expenditure is exogenous.

[25] This thought experiment corresponds to the traditional experiments in macroeconomic classics, such as Bernanke et al. (1996) and Kiyotaki and Moore (1997), where a stationary equilibrium is shocked unexpectedly. The specific shock analyzed in this paper is the same as in Di Tella (2017). In fact, citing from his paper, introducing "an aggregate uncertainty shock that increases idiosyncratic risk in the economy ... can create balance sheet recessions." Different from Di Tella (2017), we are interested in the long-run consequences of market imperfections rather than in booms and recessions.

Thus the firms are initially the only direct beneficiaries of government intervention. The following result shows that in any optimal allocation, households are subsidized afterwards through ongoing taxation.

Proposition 4. *In any optimal allocation, households are subsidized, in the sense that they contribute less to public expenditures than their share in the social welfare function:* $\tau^{H*} H_t < (1-\alpha)\gamma K_t$ *for all* t.

Proof. By (42), the claimed inequality is equivalent to $\tau^{H*} < \gamma$. Equation (45) implies that

$$\tau^{H*} - \gamma = -\sigma^2 x^* + \rho(\frac{1}{x^*} - \alpha),$$

where we have again used (42). Replacing α by $\frac{\rho}{\rho x^* + \sigma^2 (x^*)^3}$, the above equation can be rewritten as

$$\tau^{H*} - \gamma = -\frac{\sigma^4 (x^*)^3}{\rho + \sigma^2 (x^*)^2} < 0.$$

□

Proposition 4 states that households contribute less than their "fair" share of public expenditures.[26]

7.3 Debt

In order to illustrate the different regimes of debt in the welfare optimum in more conventional terms, it is useful to consider the steady state debt-to-GDP ratio. Evaluating (34) at the welfare optimum, by (42), the optimal debt-to-GDP ratio is

$$\delta^* = \frac{1 - \alpha x^*}{\mu x^*} = \frac{\sigma^2 x^*}{\mu(\rho + \sigma^2 x^{*2})}, \tag{49}$$

which is strictly positive by Proposition 3. Our analysis identifies the determinants of the debt-to-GDP ratio and shows how it depends on the political influence of firm interests (captured by parameter α). Differentiating (43) shows that x^* is decreasing in α. From the second equation of (49), it is straightforward to see that δ^* is a quasiconcave function of x^*. Hence, the optimal debt-to-GDP ratio is also single-peaked in α, which is summarized in the following proposition.

Proposition 5. *The optimal debt-to-GDP ratio is a strictly quasiconcave function of the political weight of firm interests, with maximum at* $\widehat{\alpha} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$. *It converges to* 0 *for* $\alpha \to 0$.

Proof. Differentiating (49) shows that δ^* as a function of x is strictly quasiconcave, with maximum at $x = \sqrt{\rho}/\sigma$. An inspection of (41) shows that $x_{\min} \geq \sqrt{\rho}/\sigma$ if and only if $\sqrt{\rho}/\sigma \leq \frac{1}{2}$. Since $x^* \in [x_{\min}, \infty)$, this shows that δ^* as given by (49)

[26] It does not say that $\tau^{H*} < 0$. However, the proposition implies that this is the case if γ is sufficiently small.

is strictly decreasing in x^* if $\sqrt{\rho} \leq \frac{\sigma}{2}$ and strictly quasiconcave with maximum at $\sqrt{\rho}/\sigma$ otherwise. The rest of the proposition follows because $\frac{dx^*}{d\alpha} < 0$ and by inserting $x^* = \sqrt{\rho}/\sigma$ into equation (43). $\square$

Figure 4 illustrates Proposition 5 by plotting the optimal debt-to-GDP ratio as a function of the political weight of firm equity. The figure uses values for ρ and μ that are in the standard range of the literature, and shows how sensitive the optimal debt-to-GDP ratio is to different values of the volatility of idiosyncratic productivity risk. Of course, it is not easy to calibrate σ in the present model. Nevertheless, calibrations for idiosyncratic productivity shocks have been the subject of various studies, and recent work, for instance, by Bloom et al. (2018) or Arellano et al. (2019), has provided estimates for such shocks. Bloom et al. (2018) report that the yearly variance of plant-establishment-level TFP shocks in the US in a non-recession time was 0.198. In order to use these estimates for a numerical illustration, one needs additional information about how much of the volatility is not insurable, which is hard to assess. But the value can serve as an upper bound.

When $\sigma < 2\sqrt{\rho}$, we have $\widehat{\alpha} < 1$ in Proposition 5. Given the preceding discussion, this seems to be the empirically relevant range in our framework.[27] Figure 4 therefore displays the inverse U-shape to be expected according to Proposition 5. The debt-to-GDP ratio is largest if the interests in the economy are relatively balanced, and decreases if one group becomes more and more dominant.

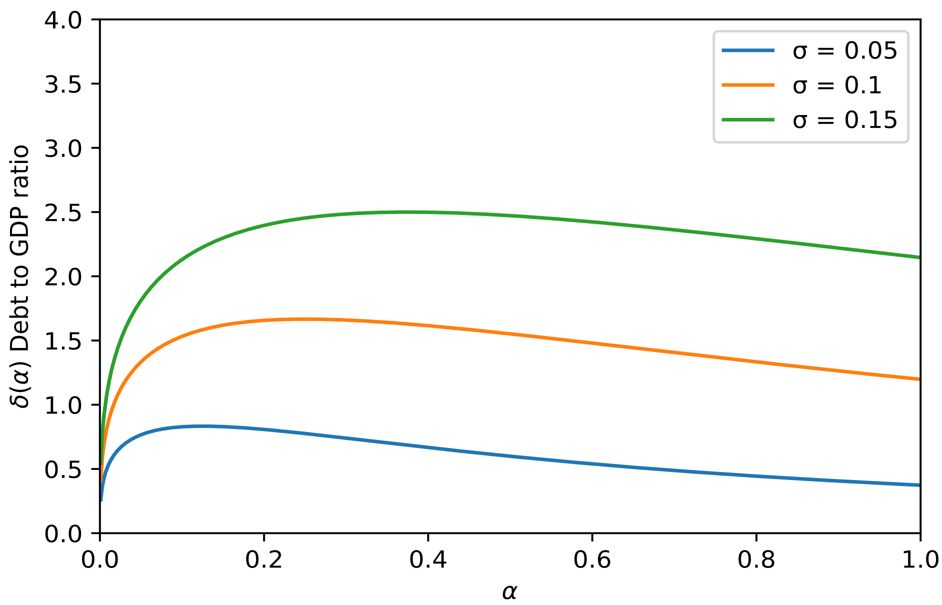

Figure 4: Debt-to-GDP ratio for $\rho = 0.04, \mu = 0.15$ and different values for σ.

[27]For example, it comprises all combinations $\rho \geq 0.02$ and $\sigma \leq 0.28$.

In our model, corporate debt is safe because steady state equity follows a geometric Brownian motion and therefore never reaches zero: firms do not default. Hence, when the government issues public debt, it does *not* create a new type of (safe) asset: government debt is exactly as good as existing corporate debt. However, public debt is valuable because it allows firms to reduce their risk exposure. One interpretation is that firms can buffer some of their losses by holding public debt on the asset side of their balance sheet. Another, equivalent interpretation is that firms reduce their leverage by buying back some of their equity. A necessary requirement for our analysis is the credibility of the government's promise to never default, of course. But since the government is assumed to maximize social welfare, which is achieved in the steady state with sustainable debt issuance, there is neither a reason for the government to default nor for the private sector to refuse buying new government debt. Not defaulting is time-consistent for our benevolent government.

Extending our model, though, in the spirit of the seminal papers of Calvo (1988) and Cole and Kehoe (2000), one can ask whether default can be a problem. Suppose for example that for whatever reason—for instance, coordination failures in debt issuance auctions—, there is a chance at a particular point in time t that the private sector refuses to roll over public debt, since it anticipates default of the government in the future. But since the government relies on taxation of wealth, even this would not cause default. By the basic balance sheet identity, $B_t = H_t + E_t - K_t$, which is strictly smaller than $H_t + E_t$. Hence, off the equilibrium the government can confiscate sufficient private wealth in emergency taxation to stop such a debt run in the first place.[28]

8 Interest, Growth, and the Dynamics of the Government Budget

8.1 Interest

It is straightforward to apply the steady state conditions (35) and (43) to the determinants of interest rates in Proposition 2.

Proposition 6. *The optimal interest rate $r^* = \mu - \sigma^2 x^*$ is an increasing function of μ and α and a decreasing function of ρ. It is negative if μ or α are sufficiently small.*

Proposition 6 sheds some light on the recent debate about the observation that real interest rates have indeed fallen over the last decades and have reached negative territory in a variety of industrialized countries. At the center of most explanations

[28] Things would be slightly more complicated if government debt constituted a fully liquid real promise. But even then, one can show that there is sufficient tax backing out of equilibrium if σ is not too large.

for this phenomenon is the observation that the amount of savings, relative to investment demand, has changed. While some explanations put emphasis on the origin of changes in savings, others put more emphasis on changes in productivity or put emphasis on both. One prominent voice is Rachel and Summers (2019), who stress that these secular movements are for a larger part a reflection of changes in saving and investment propensities. They argue that the industrialized world will probably face a longer period of secular stagnation, with sluggish growth and low real interest rates.[29]

Our results point to structural factors that might contribute to low real interest rates. For instance, and consistent with Proposition 6, permanent shifts in the objectives of policy-making with respect to risk-bearing versus non-risk-bearing agents can induce a secular decline and even negative values of real interest rates. Proposition 6 is also consistent with the suggested link between aggregate productivity and interest rates. Moreover, our results qualify the standard logic that higher savings rates lead to lower real interest rates. If ρ declines and thus the saving rate increases, the real interest rate increases. This occurs since the risk-bearing corporate sector operates with a larger share of wealth in the form of equity and is thus willing to absorb savings by households at a higher interest rate. Simply focusing on household savings may therefore not suffice to address the secular stagnation problem.

8.2 Growth

We now turn to the determinants of the optimal growth rate g^*, obtained by evaluating (25) at the optimal stationary levels (x^*, h^*).

Proposition 7. *(i) At the optimum, the growth rate g^* and private leverage x^* are related by*

$$g^* = \mu - \gamma - \rho - \frac{\rho\sigma^2 x^*}{\rho + \sigma^2 x^{*2}}. \tag{50}$$

(ii) As a function of α, g^* is strictly quasiconvex with minimum at $\widehat{\alpha} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$.
(iii) When $\alpha \to 0$, $x^* \to \infty$ and the optimal growth rate converges to the Modified Golden Rule rate $\mu - \gamma - \rho$.

Proof. (50) follows from substituting h^* from (46) into the expression for growth, (25). The rest follows the proof of Proposition 5. □

Proposition 7 is the mirror image of Proposition 5. It shows that the political weights in the welfare function may have a non-monotonic impact on growth and

[29] For discussions (and evidence) how to differentiate whether rising income inequality or an aging of the population can have contributed to an increase in savings see e.g. Mian et al. (2021), and see also von Weizsäcker and Krämer (2019) on how technological progress and demography may have jointly contributed to a secular decline in real interest rates.

this impact is moderated by impatience and risk, ρ and σ. As argued after Proposition 5, plausible parameter values imply that $\widehat{\alpha} < 1$, i.e. that growth is minimized at interim values of α. But by (50), growth is unambiguously maximized for $\alpha \to 0$, i.e. if corporate interests become irrelevant.

While taxation and redistribution ensure that the wealth of firms and of households increase at the same rate on average, there is growing inequality among firms. Indeed, by the standard theory of Brownian motion, if all firms start out with equity $e_0^i = e_0$ at time 0, then, in any optimum (x, h), equity e_t^i at time t as given by (13) is log-normally distributed with mean and variance

$$\begin{aligned} E[e_t] &= e_0 \exp g^* t \\ var(e_t) &= e_0^2 \left[\exp 2g^* t\right] \left[\exp\left(\sigma^2 x^2 t\right) - 1\right]. \end{aligned}$$

Thus the coefficient of dispersion of firms wealth grows over time:

$$\frac{\sqrt{var(e_t)}}{E[e_t]} = \sqrt{\exp(\sigma^2 x^2 t) - 1}.$$

Firms' heterogeneity is endogenous in our economy: even if the initial redistribution of capital equalizes initial wealth among firms, the impossibility to tax individual profits implies that the coefficient of dispersion of the distribution of firms' wealth necessarily grows over time.

The preceding results and the description of welfare optima in Section 6.3 now make it possible to fully characterize the optimal relation between the growth and the interest rate in our model.

Proposition 8. *(i) At the welfare optimum, $g^* > r^*$ if and only if*

$$2\alpha\left(\rho + \gamma\right) + \left(\rho + \gamma + \alpha\right)\sqrt{\alpha\left(1 + \frac{\gamma}{\rho}\right)} < \sigma^2. \tag{51}$$

(ii) The left hand side of formula (51) being increasing in α, the growth rate is more likely to be higher than the interest rate when the political weight of the corporate sector α is small.

Proof. From (35), (25), and (42) we have

$$r^* - g^* = \frac{\rho}{x^*} - \sigma^2 x^* + \rho(1 - \alpha) + \gamma.$$

Using (43), this implies

$$\frac{x^{*2}}{\rho}(r^* - g^*) = \left(1 - \alpha + \frac{\gamma}{\rho}\right) x^{*2} + 2x^* - \frac{1}{\alpha}.$$

Hence, we have $r^* < g^*$ iff $x^* < \widetilde{x}$, where $\widetilde{x}$ is the unique positive solution to

$$x^2 + \frac{2}{y}x - \frac{1}{\alpha y} = 0, \tag{52}$$

i.e.

$$\widetilde{x} = \frac{1}{y}\left[\sqrt{1+\frac{y}{\alpha}} - 1\right],$$

where $y \equiv 1-\alpha+\frac{\gamma}{\rho}$. Again, using the definition of x^* in (43), which can be written as

$$f(x) \equiv x^3 + \frac{\rho}{\sigma^2}x - \frac{\rho}{\alpha\sigma^2} = 0,$$

the condition $x^* < \widetilde{x}$ is equivalent to $f(\widetilde{x}) > 0$. Substituting and using (52) twice shows that this is the case if and only if

$$\left(4\alpha + y + \frac{\rho\alpha}{\sigma^2}y^2\right)\left[\sqrt{1+\frac{y}{\alpha}} - 1\right] > 2y + \frac{\rho}{\sigma^2}y^3.$$

In a number of straightforward steps, this inequality can be re-written as (51).

□

Proposition 8 provides precise information about the determinants of the difference between the interest and the growth rate at the welfare optimum. As discussed in the introduction, historically, the case $g > r$ seems to be more relevant than the opposite case. This has important consequences for the sustainability of government deficits, as we discuss below. In particular, the prediction of Proposition 8 is that the growth rate will optimally exceed the interest rate when the private propensity to consume ρ, the size of the public sector γ, and the political weight of corporate interests α are low, and when idiosyncratic production risk σ is large. These predictions are independent of the productivity of capital, μ.

As noted above, x^* decreases monotonically in α and becomes large when $\alpha \to 0$. Hence, the comparative statics variation of α allows us to plot the optimal debt-to-GDP ratio δ against x^*, the optimal corporate leverage. Figure 5, which mirrors Figure 4, plots this relation, which is independent of γ, under the assumption $\sigma < 2\sqrt{\rho}$, which implies $\widehat{\alpha} < 1$ in Propositions 5 and 7 and thus the inverse U-shape of the curve. The figure shows that at the welfare optimum, corporate leverage and the public debt-to-GDP ratio are not comonotonic. In fact, there is an interior maximum of δ, corresponding to an interior value of α. Public debt-to-GDP is first relatively low, for low levels of corporate leverage, then it increases, and later declines. It is monotonically decreasing for sufficiently high levels of corporate leverage.

Figure 5 also illustrates the insight of Proposition 8 that depending on the welfare weight of firms, the economy can be in different regimes $r - g > 0$ or $r - g < 0$, a question to which we turn now.

8.3 The Sustainability of Fiscal Policy

In many OECD countries, real rates of return on safe assets have been *below* growth rates for some time now. Yet mean rates of return on risky assets have been

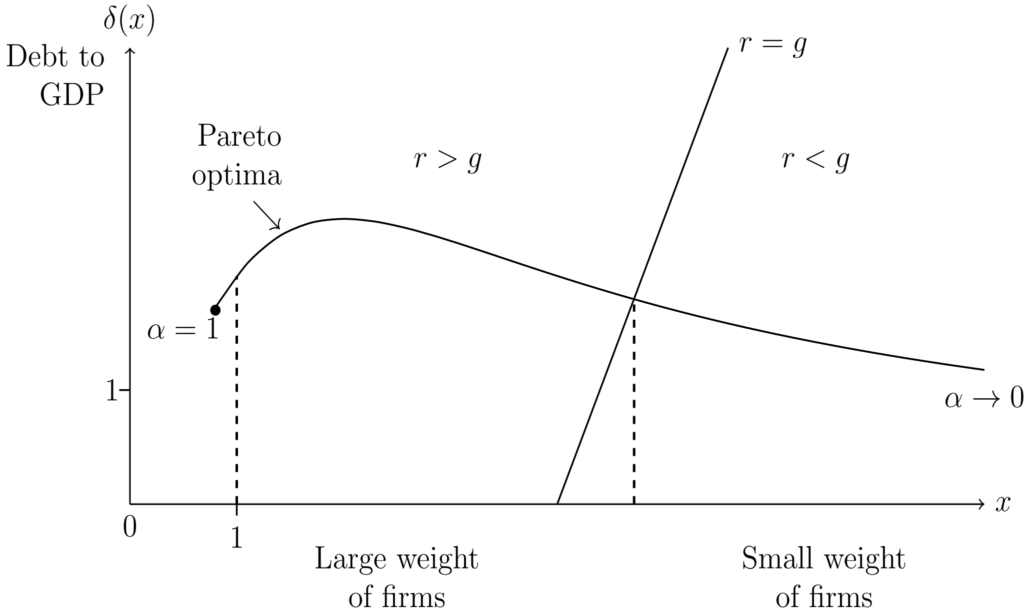

Figure 5: Regimes for parameter values $\rho = 0.04, \sigma = 0.1, \mu = 0.1$.

above growth rates. Whether this represents an instance of dynamic inefficiency in overlapping generation frameworks has been addressed in a series of important contributions and we refer to Hellwig (2021) and Dumas et al. (2022) for recent discussions and assessments.

Whether $r < g$ or not is a central question in current debates about the sustainability of the US' and other countries' fiscal policy. From an asset pricing perspective, Cochrane (2019) describes the limits of public deficits by noting that in models with infinitely-lived agents, "[t]he market value of government debt equals the present discounted value of primary surpluses." In conformity with our results, Cochrane (2022) argues that under complete financial markets ($\sigma = 0$ in our model), a permanent relationship $r < g$ is theoretically implausible, and empirically unlikely when r and g are measured correctly.[30] On the other hand, Blanchard (2019) adopts a more positive view on the theoretical possibility of $r < g$ and investigates the potential and limitations of a large fiscal expansion at little or no fiscal cost.

In our model of an economy with idiosyncratic production risk and imperfect macroeconomic risk-sharing, the return on safe debt r can fall below g. If buffering losses of firm owners has less weight in the welfare function, public debt issuance and reduction of corporate leverage are less important. As a consequence, firms are only willing to invest in risky production if the real interest rate is sufficiently low. Hence, there is a role for government policy, to actively reduce r in such cases.

[30] Cochrane (2022) provides a comprehensive account how the $r < g$ debate is connected to the fiscal theory of the price level.

Figure 5 summarizes one our main insight, that the relationship between r and g is a consequence of the weight of firm owners in the welfare function and the associated optimal debt issuance and taxation.

Our analysis is consistent with both views about the dynamics of the government budget and shows how to reconcile them. The government's flow budget constraint at date t, (4), can be written as

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t = r B_t - S_t, \tag{53}$$

where S_t is the primary surplus. Consider an arbitrary steady state (not necessarily optimal) and let r and g be the associated interest and growth rates, respectively. Since in steady state, all endogenous quantities evolve at the same rate, we have

$$B_t = e^{gt} B_0 \quad \text{and} \quad S_t = e^{gt} S_0. \tag{54}$$

Discounting and integrating (53) between dates 0 and some later date T yields:[31]

$$B_0 = \int_0^T S_t e^{-rt} dt + B_T e^{-rT}. \tag{55}$$

This relation can be viewed as the balance sheet identity for the public sector, with liabilities B_0 and two types of assets as follows:

Assets	Liabilities
$X_0 = \int_0^T S_t e^{-rt} dt$	B_0
$Y_0 = B_T e^{-rT}$	

where we let $T \to \infty$.

As in our previous discussion, we can distinguish two cases. The first case is $r > g$. Then Y_0 tends to zero when T tends to ∞, and we thus obtain the standard relationship that the value of debt equals the net present value of future primary surpluses, as argued by Cochrane (2019). The second case is $r < g$. Then Y_0 tends to $+\infty$ and X_0 tends to $-\infty$. Hence, in the limit the balance sheet identity $X_0 + Y_0 = B_0$ is not well defined. However, we can interpret $B_T e^{-rT}$ as a form of intangible asset for the government, which can be attributed to its capacity to borrow again in the future and may be called government "goodwill". In fact, our analysis shows that it is rather the government's "eternal power to issue safe debt"—rather than to tax—that creates this intangible asset. As long as the government can convince investors of its capacity to sustain a high enough level of growth, this intangible asset has a positive value.

To illustrate this point, we can consider the following simple example where the government does not raise any taxes but can still sustain a positive debt level

[31] Which discount rate should be used for the government budget constraint has been the subject of recent work. Brunnermeier et al. (2021), and Reis (2021) offer particular rationales for using discount rates different from r.

and positive public expenditures. Take $H_t \equiv 0$ (no households), for simplicity, and assume that $0 < \gamma - \rho < \frac{1}{3}\sigma^2$. In this economy, in the absence of public debt, the private sector balance sheet is $K_t = E_t$. Consider the dynamics of $x_t = 1 - \frac{B_t}{E_t}$. By equation (30), this quantity evolves as $\dot{x_t} = \sigma^2 \phi(x_t)$, where

$$\phi(x) = -x^3 + x^2 - \frac{\gamma - \rho}{\sigma^2} x - \frac{\rho}{\sigma^2}.$$

It is easy to show that the equation $\phi(x) = 0$ has two solutions $x_- < x_+$ on (0,1). Moreover ϕ changes sign twice on this interval. Therefore, x_+ is a locally stable equilibrium of x_t: if the economy starts close to it, it converges to it. In this economy, public debt and public expenditures are thus sustainable even if the government never raises taxes.

In light of the results of Sections 7.3 - 8.2 and under the assumption that σ is not too large,[32] we can therefore distinguish two polar cases for the influence of firm interests on the sustainability of government deficits. First, if α is small, $g > r$ in equilibrium, and the government runs increasing budget deficits that it covers by taxes and by rolling over ever-increasing public debt. Nevertheless, by Proposition 5 the public debt-to-GDP ratio is small. Second, if α is large, we have $g < r$ in equilibrium, "[t]he market value of government debt equals the present discounted value of primary surpluses" (Cochrane (2019)), and the public debt-to-GDP ratio is intermediary. For medium values of α, the public debt-to-GDP ratio is large, the growth rate is low, and the sign of $r - g$ depends on γ, σ, and γ as given by (51). Hence, government deficits have a "Cochranian" interpretation or a "Blanchardian" one, depending on α. Perhaps surprisingly, while abandoning strict fiscal discipline by allowing ever increasing public deficits, the latter class of equilibria features smaller public debt-to-GDP ratios than the former class, as shown in Proposition 5.

9 Conclusion

We have presented a simple model in which government debt issuance affects corporate leverage, and thus the investment and growth dynamics of the economy, through changes in the mix of private and public debt. It highlights how the weights of firm owners and households in the government welfare function impacts the relationship between r and g. In this sense, interest, growth, and public debt are a matter of redistributionary political tradeoffs.

Our model also allows many extensions. First, our paper has implications for

[32] We need the inequality in (51) to be reversed for $\alpha = 1$, which implies an upper bound on σ. This is consistent with our discussion of plausible parameter ranges in Section 7.3, and in particular with the assumption $\sigma < 2\sqrt{\rho}$ that ensures $\widehat{\alpha} < 1$ in Proposition 5. If σ is large (which seems implausible empirically), we have $r < g$ for all α.

normative macroeconomic theories in which government debt serves a socially desirable purpose. The often-held view that the amount of government debt, and in particular the rise of government debt over the past decades, is an optimal response to changing fundamentals is strongly debated. For instance Yared (2019) provides a comprehensive account of political economy theories on government debt and discusses how these theories may explain a substantial part of the long-term trend in government debt accumulation. Adding political factors, e.g. political turnover between households and equity holders when embedding our model in a simple election framework could shed light on the welfare increasing role of government debt in a democracy.

Second, one can embed our model in a monetary version of the model, (Gersbach et al. (2023)), in which central bank reserves play a safety role for commercial banks, as government debt in the current model. Since the Great Financial Crisis of 2007-2009, the reserves of commercial banks in the US, the UK, Japan, and in the Euro Area have strongly increased, albeit to different degrees. Our preliminary results support the argument that banks' holding large amounts of central bank reserves is desirable from a welfare perspective when banks face significant uninsurable idiosyncratic risks.

Third, the model can be embedded in a small open-economy context. Then, the interest rate is exogenous to our economy, but the amount of physical capital available for risky investments can be increased by borrowing in international capital markets. In this framework, public debt issuance reduces corporate leverage, and may spur higher investments and growth as long as repayment of international borrowing is ensured.

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Appendix

In this appendix, we first characterize preferences over aggregate consumption streams in the framework of the decentralized equilibrium model of Sections 3 - 5. Then, in Appendix B, we use mechanism design theory to characterize the second-best allocation from a general social planning perspective and show that this outcome can be implemented by the linear tax cum debt framework analyzed in Sections 3 - 5.

Appendix A: Implementation of Aggregate Consumption Profiles

Let $\mathbf{C}^k = (C_t^k)_{t=0}^{\infty}$, $k = H, E$ denote strictly positive, differentiable aggregate consumption profiles of households and equityholders, respectively.[33] Let $\tilde{H}$ be the initial endowment of the representative household and $\tilde{E}$ the (identical) endowments of equityholders, respectively. We show under what conditions and how one can construct a fiscal policy with linear wealth taxes and government debt such that these aggregate consumption profiles arise in the corresponding general equilibrium.

Let $\mathbf{C} = \mathbf{C}^E + \mathbf{C}^H$ denote total aggregate consumption. Clearly, for these consumption profiles to be feasible, it must be possible to produce them. Aggregate capital evolves according to the IS equation

$$\dot{K}_t = (\mu - \gamma)K_t - C_t. \tag{A1}$$

Since any efficient consumption plan must use all initial endowments, $K_0 = \tilde{H} + \tilde{E}$. Hence, integrating (A1) by standard methods,

$$K_t = (\tilde{H} + \tilde{E})e^{(\mu-\gamma)t} - \int_0^t e^{(\mu-\gamma)(t-s)} C_s ds. \tag{A2}$$

For $(\mathbf{C}^E, \mathbf{C}^H)$ to be feasible, it is necessary that $K_t > 0$ for all t, which is equivalent to

$$\int_0^t e^{(\mu-\gamma)(t-s)} C_s ds < (\tilde{H} + \tilde{E})e^{(\mu-\gamma)t} \text{ for all } t \geq 0,$$

which, in turn,is equivalent to

$$\int_0^\infty e^{-(\mu-\gamma)s} C_s ds \leq \tilde{H} + \tilde{E}. \tag{A3}$$

[33]Of course, equityholders' individual consumption streams are risky. We will identify individual consumption streams by equityholders that aggregate to $\mathbf{C}^E$. Note that this distinction is not needed for households, whose consumption stream is certain.

Definition 1. *An aggregate consumption profile* $(\mathbf{C}^E, \mathbf{C}^H)$ *is called "admissible" if both components are strictly positive, differentiable, and aggregate consumption satisfies (A3).*

Condition (A3) is a modified transversality condition; consumption profiles that do not satisfy it cannot be sustained by the economy's productive capacity given in (2). The conditions of positivity and differentiability are needed to define the law of motions, in line with the preceding analysis.[34]

As the following proposition shows, admissibility is not only necessary, but also sufficient to implement a consumption profile as an equilibrium outcome.

Proposition 9. *Suppose* $(\mathbf{C}^E, \mathbf{C}^H)$ *is admissible. Then there is a unique set of policy parameters,*

$$L^H = \tilde{H} - \frac{1}{\rho} C_0^H \tag{A4}$$

$$L^E = \tilde{E} - \frac{1}{\rho} C_0^E \tag{A5}$$

$$\tau_t^H = \mu - \rho - \sigma^2 x_t - \frac{\dot{C}_t^H}{C_t^H} \tag{A6}$$

$$\tau_t^E = \mu - \rho - \sigma^2 x_t + \sigma^2 x_t^2 - \frac{\dot{C}_t^E}{C_t^E} \tag{A7}$$

where

$$x_t = \rho \frac{K_t}{C_t^E} \tag{A8}$$

and K_t *is given by (A2), such that* $(\mathbf{C}^E, \mathbf{C}^H)$ *are the aggregate consumption profiles arising in the unique general equilibrium with these policy parameters.*

Proof. If $(\mathbf{C}^E, \mathbf{C}^H)$ can be decentralized, individually optimal consumption (6) and (9) imply that aggregate household net worth and firm equity are

$$H_t = \frac{1}{\rho} C_t^H, \; E_t = \frac{1}{\rho} C_t^E \tag{A9}$$

and (18) implies $r_t = \mu - \sigma^2 x_t$.

By (A9), x_t as defined in (A8) then is the aggregate capital-equity ratio. By (A2), x_t is fully determined by $(\mathbf{C}^E, \mathbf{C}^H)$.

Again by (A9), $\dot{H}_t / H_t = \dot{C}_t^H / C_t^H$, and (15) implies that if $(\mathbf{C}^E, \mathbf{C}^H)$ can be decentralized, τ_t^H must be given by (A6). (A7) follows similarly. Because of (A9), (A4) follows from

$$\frac{1}{\rho} C_0^H = H_0 = \tilde{H} - L^H$$

[34] It is possible to work with piecewise differentiable profiles.

and (A5) by a similar argument. Equilibrium public debt then is

$$\begin{aligned} B_t &= H_t + E_t - K_t \\ &= \frac{1}{\rho} C_t - K_t \end{aligned} \tag{A10}$$

Under the fiscal policy defined by (A4)–(A8), the aggregate quantities thus defined are consistent with the individual decision rules (6), (9), and (10) derived in Section 4, evaluated at the interest rate $r_t = \mu - \sigma^2 x_t$. Market clearing is implied at all times by (A10). We thus have identified the unique general equilibrium that implements the aggregate consumption profiles $(\mathbf{C}^E, \mathbf{C}^H)$. $\square$

Appendix B: Second-Best Allocation

Appendix A has shown how to obtain aggregate consumption profiles as general equilibrium outcomes with linear taxes and debt. Within this class of outcomes, this provides a natural preference ordering over such profiles. However, this does not allow us to characterize and rank individual (stochastic) consumption profiles, nor does it address the problem whether linear taxes and debt are the best way to implement consumption profiles. In this appendix, we sketch how to deal with these two questions by explicitly addressing the underlying friction of the allocation problem and characterizing the optimal direct mechanism designed by a social planner who takes this friction into account. The full analysis is beyond the scope of the present paper, we present it in a broader context in our companion paper, Biais et al. (2023).

B.1 The Mechanism Design Problem

We simplify the exposition by normalizing exogenous government expenditures to 0: $\gamma = 0$. As noted in section 3, at each time t the random shocks dz_t^i and thus each firm i's instantaneous output at time t

$$dy_t^i = k_t^i[\mu dt + \sigma dz_t^i]$$

are private information of the firms' owners. Only total instantaneous output Y_t being certain, is commonly known. Hence, an equityholder can divert some of his/her firm's output, consume this amount secretly, and claim to have incurred a negative productivity shock.

By the Revelation Principle, we can focus on direct mechanisms where each firm reports its production shock to the social planner. We denote the report of owner i by $d\hat{z}_t^i$. The planner must determine, for each time $t \geq 0$ and firm $i \in [0, 1]$, instantaneous consumption c_t^i and the new capital stock k_t^i, as well as household consumption C_t^H. As in Appendix A, let C_t^E denote aggregate consumption by equityholders. Note that household consumption (individually or in the aggregate) must be non-random, because there is no aggregate risk in the economy to share.

In order to formulate the optimization problem, we follow Sannikov (2008) and express instantaneous choices as a function of firms' continuation utilities ω_t^i, rather than of their realized output shocks.[35] For any consumption process $(c_t^i)_{t\geq 0}$, this continuation utility of firm i at time t is

$$\omega_t^i = \mathbb{E}_t[\int_t^\infty e^{-\rho(s-t)} \log c_s^i ds \mid \mathcal{F}_t], \tag{B1}$$

where $\mathcal{F}_t$ is the filtration induced by the stochastic process $(c_t^i)_{t\geq 0}$, which is also the augmented filtration generated by the Brownian motion z_t^i. Letting $c_s^i = c_s(\omega_t^i)$, using the Martingale Representation Theorem to express total payoffs as an Itô-integral, and differentiating (B1) with respect to t yields the following stochastic differential equation:

$$d\omega_t = [\rho\omega_t - \log c_t(\omega_t)]dt + \psi_t(\omega_t)\,[dy_t - \mu k_t dt] \tag{B2}$$

$$= [\rho\omega_t - \log c_t(\omega_t)]dt + \psi_t(\omega_t)\sigma dz_t^i \tag{B3}$$

We have dropped the superscript i because all firms are identical ex ante, and ex post, at each time t, they are fully characterized by their past history, as summarized by ω_t. The function ψ_t is a crucial element of the planner's mechanism. By (B2), $\psi_t(\omega_t)$ is the sensitivity of the continuation utility to output and thus a measure of the firm's risk exposure under truth-telling. But by (B3) $\sigma\psi_t$ is also the sensitivity to current performance reports and therefore a measure of the firm's incentives to lie.

This potential lying entails a series of incentive constraints to which we turn now. Suppose that at time t, firm i is in state ω_t^i, holds capital k_t^i, and has an output shock dz_t^i. It should report the output truthfully, which gives an instantaneous payoff[36]

$$u(c_t^i)dt + \sigma\psi_t(\omega_t)dz_t^i.$$

Instead, the firm can divert an amount $\beta k_t^i \sigma dt$, $\beta > 0$, by reporting a less favorable output shock $d\hat{z}_t^i = dz_t^i - \beta dt$ and consuming the diverted output privately. To prevent this deviation, the mechanism must therefore satisfy the following incentive constraint:

$$\underbrace{u(c_t^i)dt + \sigma\psi_t(\omega_t)dz_t^i}_{\text{truth-telling}} \quad \geq \quad \underbrace{u(c_t^i + \beta\sigma k_t^i)dt + \sigma\psi_t(\omega_t)d\widehat{z}_t^i}_{\text{diversion}} \tag{B4}$$

[35]Instead of an agency model with diversion for private benefit, Sannikov (2008) analyzes a model of effort provision under risk-aversion. The logic is the same.

[36]We formulate the payoff for a general utility function to show the mechanics more clearly.

for all $\beta > 0$. This is equivalent to

$$\sigma \psi_t(\omega_t) \geq \frac{1}{\beta} \left[u(c_t^i + \beta \sigma k_t^i) - u(c_t^i) \right]. \tag{B5}$$

By concavity, the right-hand side of (B5) is decreasing in β. Since (B4) must hold for all $\beta > 0$, it therefore holds iff (B5) holds for $\beta \to 0$, i.e.

$$\psi_t(\omega_t) \geq k_t^i u'(c_t^i). \tag{B6}$$

Hence, to avoid private benefit-taking, the reward from increasing capital k_t^i based on the performance sensitivity ψ_t must be sufficiently large, relative to the marginal utility of consumption weighed by the current capital stock. As noted above, $\psi_t(\omega_t)$ is a measure of the firm's optimal risk exposure and thus imposes a real cost on firms. Hence, at the optimum, (B6) must bind for almost all ω. In our log utility framework, this implies the incentive constraint

$$\psi_t(\omega_t) = \frac{k_t(\omega_t)}{c_t(\omega_t)}. \tag{B7}$$

Hence, the (stochastic) law of motion of ω_t is given by

$$d\omega_t = (\rho \omega_t - \log c_t(\omega_t))\, dt + \sigma \frac{k_t(\omega_t)}{c_t(\omega_t)} dz_t \tag{B8}$$

with initial value $\omega_0 > 0$ (the same for all firms).

The planner's problem now is to choose consumption, capital, and sensitivity policies $\{c_t^i(\cdot)\}_{t=0}^{\infty}$, $\{k_t^i(\cdot)\}_{t=0}^{\infty}$, $\{\psi_t^i(\cdot)\}_{t=0}^{\infty}$, respectively, for each firm i and a (deterministic) consumption path $\{C_t^H\}_{t=0}^{\infty}$ for the representative household such as to maximize the weighted expected utilities of all agents. Furthermore, the planner must set initial values ω_0^E, ω_0^H for firms and households, respectively, where we assume that s/he treats all firms equally. The control variables $\{c_t^i(\cdot)\}_{t=0}^{\infty}$, $\{k_t^i(\cdot)\}_{t=0}^{\infty}$, $\{\psi_t^i(\cdot)\}_{t=0}^{\infty}$ all depend on the single firms' continuation utilities ω_t^i, $t \geq 0$, $i \in [0, 1]$ subject to the aggregate constraints that we describe next.

The preceding description reveals a conceptual difficulty that also renders the problem technically more difficult than the single-agent problem of Sannikov (2008). The reason is that the variables ω_t^i have two different roles in this problem. First, for each firm i, $(\omega_t^i)_{t=0}^{\infty}$ is a random process that describes the evolution of the firm's payoffs. Second, at each time t, $(\omega_t^i)_{i=0}^{1}$ is a continuum of random variables that (fully) describes the firm population at time t. Since the identity of the individual firm does not matter by assumption, we can describe these continua by distributions $d\nu_t$ over $\mathbb{R}$.[37] The actual state variable is therefore $\{d\nu_t\}$, describing the evolution of the continuum of firms indexed by $\omega \in \mathbb{R}$, together with the aggregate capital

[37]Note that in principle, the $d\nu_t$'s are measures. The fact that their total mass is 1 (i.e. that they are "distributions") is due to the assumption that the firms are a continuum of size 1.

available at each time for redistribution. Hence, the planner's problem is one of mean-field control theory (see, e.g., Carmona (2020)).

Under suitable assumptions, we can assume that the continua $(\omega_t^i)_{i=0}^1$ satisfy the Law of Large Numbers and therefore that aggregate capital, defined as

$$K_t = \int k_t(\omega) d\nu_t(\omega), \tag{B9}$$

is well defined and non-random. The planner's problem therefore is, at any time $t \geq 0$ and for given initial conditions K_t and $d\nu_t$, to

$$\max \alpha \int \omega_t d\nu_t(\omega_t) + (1-\alpha) \int_t^\infty e^{-\rho(s-t)} \log C_s^H ds, \tag{B10}$$

subject to the constraints

$$d\omega_s = [\rho\omega_s - \log c_s(\omega_s)]ds + \psi_s(\omega_s)\sigma dz_s^i \tag{B11}$$

$$\psi_s(\omega_s) = \frac{k_s(\omega_s)}{c_s(\omega_s)} \tag{B12}$$

$$\dot{K}_s = \mu K_s - C_s^H - C_s^E \tag{B13}$$

$$c_s(\omega_s) > 0, k_s(\omega_s) \geq 0 \text{ for all } \omega_s \tag{B14}$$

$$C_s^H > 0 \tag{B15}$$

for all $s \geq t$, where we have denoted aggregate equityholder consumption by

$$C_s^E = \int c_s(\omega) d\nu_s(\omega) \tag{B16}$$

which, again, is non-random by the Law of Large Numbers. Note that while there are no individual "budget constraints", the planner's aggregate resource constraint is (B13). (B13) links individual behavior to the aggregate (the "mean field").[38]

Problem (B10)–(B15) is defined for general starting times $t \geq 0$. Denote its value, if it exists, by $V(t, K_t, d\nu_t)$. For the problem starting at time 0, we have $t = 0$, $K_0 = \tilde{H} + \tilde{E}$, and $d\nu_0 = \delta_{\omega_0}$, the Dirac measure concentrated at ω_0 (as assumed, the planner treats all firms equally ex ante). For this reason, we also have $k_0(\omega_0) = K$.

In parallel work, Biais et al. (2023), we show that problem (B10)–(B15) has a solution and provide a characterization. In particular, if the controls $\{c_t^i(\cdot)\}_{t=0}^\infty$, $\{k_t^i(\cdot)\}_{t=0}^\infty$, $\{\psi_t^i(\cdot)\}_{t=0}^\infty$, $i \in [0,1]$, and $\{c_t^H\}_{t=0}^\infty$, as well as the initial states ω_0^E, ω_0^H are optimal, then the performance sensitivities must be deterministic (and therefore identical across firms) and differentiable (w.r.t. t), and we can write $\{\psi_t\}_{t=0}^\infty$. Also, C_t^E, C_t^H, and K_t must be differentiable (with respect to t).

[38](B13) assumes that the right-hand side of (B9) is differentiable with respect to t. While not implausible, this needs to be proved. We do so in Biais et al. (2023).

B.2 Decentralized Implementation

We now show that an allocation with the above properties can be decentralized as an equilibrium with linear wealth taxes and debt.

First, define

$$x_t = \rho\psi_t \tag{B17}$$

$$h_t = \psi_t \frac{C_t^H}{K_t} \tag{B18}$$

for all t, where K_t is given by (B9). Note that incentive-compatibility implies that $\psi_t > 0$ for all t. Hence, $x_t > 0$ for all t. (B13) then implies that $h_t > 0$ for all t.

Next, define tax rates $(\tau_t^E, \tau_t^H) \in \mathbb{R}^2$, $t \geq 0$, by (32)–(33) in the main text.

Third, define an index $E_t = K_t/x_t$, $t \geq 0$, and call it "aggregate equity". Then define "household wealth" by $H_t = C_t^H/\rho$ and the value of "government bonds" as $B_t = H_t + E_t - K_t$. These three indices are not quantities of real goods. Next, transfer the households' real endowment $\widetilde{H}$ to the firms, and define the firms' "initial equity" as $e_0^i = (\tilde{H} + \tilde{E})/x_0$ for all i.

Proposition 1 now implies that the decentralized economy with initial endowments and fiscal policy as just defined has a unique equilibrium that generates the same consumption and production allocation as the planning allocation we started out with. It is important to realize that we have only used some of the properties of the optimal mechanism (differentiability and the non-stochastic nature of ψ_t). Hence, we have decentralized non-optimal allocations, too. This is consistent with our approach in Sections 5 and 6, where we have first derived equilibria more generally and then determined the welfare optimum from that set.

Appendix C: The Individual Decision Problems

For Online Publication

For completeness, this appendix provides a detailed solution to the individual optimization problems of Section 4 that were only sketched in the main text.

C.1 Households

Suppose that the representative household has initial net worth n_0^H at time $t = 0$, no further income later, and can only save via safe debt. Consider the variation of the household's decision problem in which the household starts out at time $t \geq 0$ with net worth $n > 0$. It chooses a consumption path c_s^H, $s \geq t$, to solve the standard consumption problem

$$\begin{aligned} &\max_{c^H} \int_t^\infty e^{-\rho s} \log c_s^H ds \\ &dn_s^H = \left((r_s - \tau_s^H)n_s^H - c_s^H\right) ds \qquad \text{(B1)} \\ &n_t^H = n \\ &n_s^H \geq 0. \end{aligned}$$

Denote the optimal consumption path for this problem by $c_s^H(t, n)$.

Remark 1. *The problem is homogeneous and invariant to scaling. Hence, if $c_s^H = c_s^H(t, n)$, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = n$, then αc_s^H, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = \alpha n$, for $\alpha > 0$.*

Hence, any optimal path satisfies

$$c_s^H(t, n) = c_s^H(t, 1)n.$$

Let $V^H(t, n)$ be the value function of the problem. Homogeneity implies

$$\begin{aligned} V^H(t, n) &= \int_t^\infty e^{-\rho s} \log c_s^H(t, n) ds \\ &= \frac{e^{-\rho t}}{\rho} \log n + v^H(t), \qquad \text{(B2)} \end{aligned}$$

where

$$v^H(t) = \int_t^\infty e^{-\rho s} \log c_s^H(t, 1) ds \qquad \text{(B3)}$$

is independent of n.

Ignoring the non-negativity conditions (which will be satisfied at the optimum), the Bellman Equation of the household's problem is

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{\partial V^H}{\partial n} \left((r_t - \tau_t^H) n - c \right) \right] = 0.$$

From (B2), we have

$$\frac{\partial V^H}{\partial n} = \frac{e^{-\rho t}}{\rho n},$$

such that the Bellman Equation becomes

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{e^{-\rho t}}{\rho n} \left((r_t - \tau_t^H) n - c \right) \right] = 0. \tag{B4}$$

It is easy to see that the first-order condition

$$c = \rho n \tag{B5}$$

is necessary and sufficient for the maximization problem in (B4). In particular, (B5) implies that $c > 0$. The Bellman Equation thus is equivalent to

$$-e^{-\rho t} \log n + \dot{v}^H(t) + e^{-\rho t} \left[\log \rho n - 1 + \frac{r_t - \tau_t^H}{\rho} \right] = 0,$$

which is equivalent to

$$\dot{v}^H(t) = \frac{e^{-\rho t}}{\rho} \left[\rho - \rho \log \rho - r_t + \tau_t^H \right].$$

This can be integrated explicitly to yield

$$\rho v^H(t) = (1 - \log \rho) \left(1 - e^{-\rho t} \right) - \int_0^t e^{-\rho s} (r_s - \tau_s^H) ds + \rho v^H(0). \tag{B6}$$

By (B5), if $n_s^H(t,n)$ is on the trajectory generated by $c_s^H(t,n)$, $s \geq t$, the optimal policy is

$$c_s^H(t,n) = \rho n_s^H(t,n). \tag{B7}$$

Hence, inserting (B7) into (B1) yields the law of motion for household savings with initial value 1 at time $t = 0$, $n_s^H(0,1)$, as

$$\frac{d n_s^H(0,1)}{ds} = (r_s - \tau_s^H - \rho) n_s^H(0,1).$$

Integrating yields

$$\log n_s^H(0,1) = \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau, \tag{B8}$$

where the constant of integration in (B8) is $\log n_0^H(0,1) = \log 1 = 0$, by the construction of v.

Inserting (B7) and (B8) into (B3) yields, for $t = 0$,

$$\begin{aligned} v^H(0) &= \int_0^\infty e^{-\rho s}(\log\rho + \log n_s^H(0,1))ds \\ &= \frac{\log\rho}{\rho} + \int_0^\infty e^{-\rho s}\int_0^s (r_\tau - \tau_\tau^H - \rho)d\tau ds \\ &= \frac{\log\rho}{\rho} - \frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho\tau}(r_\tau - \tau_\tau^H)d\tau. \end{aligned}$$

Combining this with (B6) yields

$$\rho v^H(t) = -(1 - \log\rho)e^{-\rho t} + \int_t^\infty e^{-\rho s}(r_s - \tau_s^H)ds,$$

which together with (B2) yields the households' value function as

$$\rho V^H(t,n) = e^{-\rho t}\left(\log(\rho n) - 1\right) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^H\right)ds,$$

which is (7) in the main text.

C.2 Firms

Net of initial lump sum taxes, at time $t = 0$ firm i has an initial equity position $e_0^i > 0$. Consider the variation where a firm starts at time t with equity $e^i > 0$. It chooses a path k_s^i, e_s^i, c_s^i, $s \geq t$ such as to

$$\begin{aligned} &\max_{k^i,e^i,c^i} \mathbb{E}\int_t^\infty e^{-\rho s}\log c_s^i ds \\ &de_s^i = \left[(\mu - r_s)k_s^i + (r_s - \tau_s^E)e_s^i - c_s^i\right]ds + \sigma k_s^i dz_s^i \qquad \text{(B9)} \\ &e_t^i = e^i \\ &e_s^i \geq 0, \end{aligned}$$

where equation (B9) is the flow of funds equation (8) in the main text, after substituting out $d_s^i = k_s^i - e_s^i$ from the balance sheet equation (1). Denote the value function of the problem by $V^E(t, e^i)$.

Since, as in the household problem, the feasible set is homogeneous, any solution is invariant to scaling, and we must have, at the optimum,

$$(k_s^i(t,e^i), c_s^i(t,e^i)) = (k_s^i(t,1)e^i, c_s^i(t,1)e^i).$$

Therefore,

$$V^E(t,e^i) = \frac{e^{-\rho t}}{\rho}\log e^i + v^E(t), \tag{B10}$$

where

$$v^E(t) = \mathbb{E}\int_t^\infty e^{-\rho s}\log c_s^i(t,1)ds \tag{B11}$$

is independent of e^i.

We first solve the unconstrained problem, in which we ignore the non-negativity constraint on e^i_s. In this case, the Bellman Equation is

$$\frac{\partial V^E}{\partial t} + \max_{k,c}\left[e^{-\rho t}\log c + \frac{\partial V^E}{\partial e}\left((\mu - r_t)k + (r_t - \tau^E_t)e^i - c\right) + \frac{\partial^2 V^E}{\partial e^2}\frac{\sigma^2}{2}k^2\right] = 0.$$

From (B10), we have

$$\begin{aligned}\frac{\partial V^E}{\partial e} &= \frac{e^{-\rho t}}{\rho e^i}\\ \frac{\partial^2 V^E}{\partial e^2} &= -\frac{e^{-\rho t}}{\rho (e^i)^2}.\end{aligned}$$

The Bellman Equation therefore becomes

$$\frac{\partial V^E}{\partial t} + max_{k,c}e^{-\rho t}\left[\log c + \frac{1}{\rho e^i}\left((\mu - r_t)k + (r_t - \tau^E_t)e^i - c\right) - \frac{1}{2\rho (e^i)^2}\sigma^2 k^2\right] = 0 \tag{B12}$$

and the first-order conditions

$$c = \rho e^i \tag{B13}$$

$$k = \frac{\mu - r_t}{\sigma^2}e^i \tag{B14}$$

are necessary and sufficient for the maximum in (B12). In particular, (B13) implies that $c > 0$.[39] The Bellman Equation therefore is equivalent to

$$\begin{aligned}-e^{-\rho t}\log e^i + \dot{v}^E(t) + e^{-\rho t}\left[\log \rho e^i - 1 + \frac{r_t - \tau^E_t}{\rho} + \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right] &= 0\\ \Leftrightarrow \dot{v}^E(t) = e^{-\rho t}\left[1 - \log\rho - \frac{r_t - \tau^E_t}{\rho} - \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right].\end{aligned}$$

This is a deterministic ODE that can be integrated explicitly to yield

$$\rho v^E(t) = (1 - \log\rho)\left(1 - e^{-\rho t}\right) - \int_0^t e^{-\rho s}\left(r_s - \tau^E_s + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds + \rho v^E(0) \tag{B15}$$

From (B13)–(B14), if $e^i_s = e^i_s(t, e^i)$ is on a trajectory generated by $c^i_s(t, e^i)$ and $k^i_s(t, e^i)$, $s \geq t$, the optimal policy is

$$c^i_s(t, e^i) = \rho e^i_s \tag{B16}$$

$$k^i_s(t, e^i) = \frac{\mu - r_s}{\sigma^2}e^i_s. \tag{B17}$$

Hence, inserting (B16) and (B17) into the equation of motion (B9) yields the (random) law of motion for firm equity, with $s \geq t$ and $e^i_t = e^i_t(t, e^i) = e^i$, as

$$de^i_s = \left[\left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau^E_s - \rho\right]e^i_s ds + \frac{\mu - r_s}{\sigma}e^i_s dz^i_s \tag{B18}$$

$$\equiv (\beta_s - \rho)e^i_s ds + \gamma_s e^i_s dz^i_s, \tag{B19}$$

[39]Note that for the argument to work, there is no need to impose the condition $r_t < \mu$ at this stage.

where we have set, for simplicity,

$$\beta_s = \left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E \tag{B20}$$

$$\gamma_s = \frac{\mu - r_s}{\sigma}. \tag{B21}$$

We must determine $v^E(0)$. From (B11), using (B17), we have

$$\begin{aligned} v^E(0) &= \mathbb{E}\int_0^\infty e^{-\rho s}\log \rho e_s^i(0,1)ds \\ &= \frac{\log\rho}{\rho} + \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1)ds. \end{aligned} \tag{B22}$$

Applying the Itô-Doeblin formula (Shreve (2004), p. 187) to (B19) yields

$$\begin{aligned} d\log e_s^i &= \frac{1}{e_s^i}de_s^i - \frac{1}{2\left(e_s^i\right)^2}\gamma_s^2\left(e_s^i\right)^2 ds \\ &= \left(\beta_s - \rho - \frac{1}{2}\gamma_s^2\right)ds + \gamma_s dz_s^i \end{aligned}$$

For $e_s^i = e_s^i(0,1)$, where by definition $e_0^i = 1$, this means that with probability 1,

$$\log e_s^i(0,1) = \int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau + \int_0^s \gamma_\tau dz_\tau^i.$$

By the definition of the stochastic integral, under standard integrability assumptions for r_s,

$$\mathbb{E}\int_0^s \gamma_\tau dz_\tau^i = 0$$

for every s. The expectation in (B22) therefore is

$$\begin{aligned} \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1)ds &= \int_0^\infty e^{-\rho s}\int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \int_0^\infty e^{-\rho s}\int_0^s \left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho\tau}\left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau. \end{aligned}$$

Inserting this into (B22) and using (B20)–(B21),

$$\rho v^E(0) = \log\rho - 1 + \int_0^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds.$$

Combining this with (B15) yields

$$\rho v^E(t) = -e^{-\rho t}(1 - \log\rho) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds. \tag{B23}$$

Finally, inserting (B23) into the value function (B10), yields

$$\rho V^E(t, e^i) = e^{-\rho t}(\log \rho e^i - 1) + \int_t^{\infty} e^{-\rho s} \left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2} \right) ds,$$

which is (14) in the main text.